STABILITY CRITERION FOR CONVOLUTION-DOMINATED INFINITE MATRICES

QIYU SUN

ABSTRACT. Let ℓ^p be the space of all p-summable sequences on $\mathbb{Z}$. An infinite matrix is said to have ℓ^p-stability if it is bounded and has bounded inverse on ℓ^p. In this paper, a practical criterion is established for the ℓ^p-stability of convolution-dominated infinite matrices.

1. INTRODUCTION

Let $\mathcal{C}$ be the set of all infinite matrices $A := (a(j,j'))_{j,j'\in\mathbb{Z}}$ with

$$\|A\|_{\mathcal{C}} = \sum_{k\in\mathbb{Z}} \sup_{j-j'=k} |a(j,j')| < \infty.$$

Let $\ell^p := \ell^p(\mathbb{Z})$ be the set of all p-summable sequences on $\mathbb{Z}$ with the standard norm $\|\cdot\|_p$. An infinite matrix $A := (a(j,j'))_{j,j'\in\mathbb{Z}} \in \mathcal{C}$ defines a bounded linear operator on $\ell^p, 1 \le p \le \infty$, in the sense that

$$Ac = \Big(\sum_{j'\in\mathbb{Z}} a(j,j')c(j')\Big)_{j\in\mathbb{Z}} \tag{1.1}$$

where $c = (c(j))_{j\in\mathbb{Z}} \in \ell^p$. Given a summable sequence $h = (h(j))_{j\in\mathbb{Z}} \in \ell^1$, define the *convolution operator* C_h on $\ell^p, 1 \le p \le \infty$, by

$$C_h:\ \ell^p \ni \big(b(j)\big)_{j\in\mathbb{Z}} \longmapsto \Big(\sum_{k\in\mathbb{Z}} h(j-k)b(k)\Big)_{j\in\mathbb{Z}} \in \ell^p. \tag{1.2}$$

Observe that the linear operator associated with an infinite matrix $A \in \mathcal{C}$ is dominated by a convolution operator in the sense that

$$|(Ac)(j)| \le (C_h|c|)(j) := \sum_{j'\in\mathbb{Z}} h(j-j')|c(j')|, \quad j \in \mathbb{Z} \tag{1.3}$$

for any sequence $c = (c(j))_{j\in\mathbb{Z}} \in \ell^p, 1 \le p \le \infty$, where $|c| = (|c(j)|)_{j\in\mathbb{Z}}$ and the sequence $(\sup_{j-j'=k} |a(j,j')|)_{k\in\mathbb{Z}}$ can be chosen to be the sequence $h = (h(j))_{j\in\mathbb{Z}}$ in (1.3). So infinite matrices in the set $\mathcal{C}$ are said to be *convolution-dominated.*

Convolution-dominated infinite matrices were introduced by Gohberg, Kaashoek, and Woerdeman [12] as a generalization of Toeplitz matrices. They showed that the class $\mathcal{C}$ equipped with the standard matrix multiplication and the above norm $\|\cdot\|_{\mathcal{C}}$ is an inverse-closed Banach subalgebra of $\mathcal{B}(\ell^p)$ for $p = 2$. Here $\mathcal{B}(\ell^p), 1 \le p \le \infty$, is the space of all bounded linear operators on ℓ^p with the standard operator norm, and a subalgebra $\mathcal{A}$ of a Banach algebra $\mathcal{B}$ is said to be *inverse-closed* if an operator $T \in \mathcal{A}$ has an inverse T^{-1} in $\mathcal{B}$ then $T^{-1} \in \mathcal{A}$ ([7, 11, 21]). The inverse-closed property for convolution-dominated infinite matrices was rediscovered by Sjöstrand [25]

2000 *Mathematics Subject Classification.* 47B35, 40E05, 65F05, 42C40, 47G30, 94A20.

with a completely different proof and an application to a deep theorem about pseudodifferential operators. Recently Shin and Sun [23] generalized Gohberg, Kaashoek and Woerdeman's result and proved that the class $\mathcal{C}$ is an inverse-closed Banach subalgebra of $\mathcal{B}(\ell^p)$ for any $1 \le p \le \infty$. The readers may refer to [5, 10, 20, 23, 25, 27] and the references therein for related results and various generalizations on the inverse-closed property for convolution-dominated infinite matrices.

Convolution-dominated infinite matrices arise and have been used in the study of spline approximation ([8, 9]), wavelets and affine frames ([6, 18]), Gabor frames and non-uniform sampling ([3, 14, 15, 26]), and pseudo-differential operators ([13, 16, 24, 25] and the references therein). Examples of convolution-dominated infinite matrices include the infinite matrix $\big(a(j-j')\big)_{j,j'\in\mathbb{Z}}$ associated with convolution operators, and the infinite matrix $\big(a(j-j')e^{-2\pi\sqrt{-1}\theta j'(j-j')}\big)_{i,j\in\mathbb{Z}}$ associated with twisted convolution operators, where $\theta \in \mathbb{R}$ and the sequence $a = (a(j))_{j\in\mathbb{Z}}$ satisfies $\sum_{j\in\mathbb{Z}} |a(j)| < \infty$ ([1, 14, 19, 27, 29]).

A convolution-dominated infinite matrix A is said to have *ℓ^p-stability* if there are two positive constants C_1 and C_2 such that

$$C_1\|c\|_p \le \|Ac\|_p \le C_2\|c\|_p \quad \text{for all } c \in \ell^p. \tag{1.4}$$

The ℓ^p-stability is one of basic assumptions for infinite matrices arising in the study of spline approximation, Gabor time-frequency analysis, nonuniform sampling, and algebra of pseudo-differential operators, see [1, 3, 6, 8, 9, 10, 14, 15, 16, 18, 19, 23, 24, 25, 26, 27, 29] and the references therein. **Practical criteria** for the ℓ^p-stability of a convolution-dominated infinite matrix will play important roles in the further study of those topics.

However, up to the knowledge of the author, little is known about practical criteria for the ℓ^p-stability of an infinite matrix. For an infinite matrix $A = (a(j-j'))_{j,j'\in\mathbb{Z}}$ associated with convolution operators, there is a very useful criterion for its ℓ^p-stability. It states that A has ℓ^p-stability if and only if the Fourier series $\hat a(\xi) := \sum_{j\in\mathbb{Z}} a(j)e^{-ij\xi}$ of the generating sequence $a = (a(j))_{j\in\mathbb{Z}} \in \ell^1$ does not vanish on the real line, i.e.,

$$\hat a(\xi) \ne 0 \quad \text{for all} \ \ \xi \in \mathbb{R}. \tag{1.5}$$

Applying this criterion for the ℓ^p-stability, one concludes that the spectrum $\sigma_p(C_a)$ of the convolution operator C_a as an operator on ℓ^p is independent of $1 \le p \le \infty$, i.e.,

$$\sigma_p(C_a) = \sigma_q(C_a) \quad \text{for all } 1 \le p, q \le \infty \tag{1.6}$$

see [4, 17, 22, 23] and the references therein for the discussion on spectrum of various convolution operators. Applying the above criterion again, together with the classical Wiener's lemma ([29]), it follows that the inverse of an ℓ^p-stable convolution operator C_a is a convolution operator C_b associated with another summable sequence b.

For a convolution-dominated infinite matrix $A = (a(j,j'))_{j,j'\in\mathbb{Z}}$, a popular sufficient condition for its ℓ^1-stability and ℓ^∞-stability is that A is *diagonal-dominated*, i.e.,

$$\inf_{j\in\mathbb{Z}}\Big(|a(j,j)|-\max\Big(\sum_{j'\neq j}|a(j,j')|,\sum_{j'\neq j}|a(j',j)|\Big)\Big)>0. \tag{1.7}$$

In this paper, we provide a practical criterion for the ℓ^p-stability of convolution-dominated infinite matrices. We show that a convolution-dominated infinite matrix A has ℓ^p-stability if and only if it has certain "diagonal-blocks-dominated" property (see Theorem 2.1 for the precise statement).

2. Main Theorem

To state our criterion for the ℓ^p-stability of convolution-dominated infinite matrices, we introduce two concepts. Given an infinite matrix A, define the *truncation matrices* A_s, $s\geq 0$, by

$$A_s=(a(i,j)\chi_{(-s,s)}(i-j))_{i,j\in\mathbb{Z}}$$

where χ_E is the characteristic function on a set E. Given $y\in\mathbb{R}$ and $1\leq N\in\mathbb{Z}$, define the operator χ_y^N on ℓ^p by

$$\chi_y^N:\ell^p\ni\big(c(j)\big)_{j\in\mathbb{Z}}\longmapsto\big(c(j)\chi_{(-N,N)}(j-y)\big)_{j\in\mathbb{Z}}\in\ell^p.$$

The operator χ_y^N is a diagonal matrix $\mathrm{diag}(\chi_{(-N,N)}(j-y))_{j\in\mathbb{Z}}$.

Theorem 2.1. *Let $1\leq p\leq\infty$, and A be a convolution-dominated infinite matrix in the class $\mathcal{C}$. Then the following statements are equivalent.*

(i) *The infinite matrix A has ℓ^p-stability.*

(ii) *There exist a positive constant C_0 and a positive integer N_0 such that*

$$\|\chi_n^{2N}A\chi_n^N c\|_p\geq C_0\|\chi_n^N c\|_p,\qquad c\in\ell^p, \tag{2.1}$$

hold for all integers $N\geq N_0$ and $n\in N\mathbb{Z}$.

(iii) *There exist a positive integer N_0 and a positive constant α satisfying*

$$\alpha>2(5+2^{1-p})^{1/p}\inf_{0\leq s\leq N_0}\big(\|A-A_s\|_{\mathcal{C}}+\frac{s}{N_0}\|A\|_{\mathcal{C}}\big) \tag{2.2}$$

such that

$$\|\chi_n^{2N_0}A\chi_n^{N_0}c\|_p\geq\alpha\|\chi_n^{N_0}c\|_p,\qquad c\in\ell^p, \tag{2.3}$$

hold for all $n\in N_0\mathbb{Z}$.

Taking $N_0=1$ in (2.2) and (2.3), we obtain a sufficient condition (2.4), which is a strong version of the diagonal-domination condition (1.7), for the ℓ^∞-stability of a convolution-dominated infinite matrix.

Corollary 2.2. *Let $A=(a(j,j'))_{j,j'\in\mathbb{Z}}$ be a convolution-dominated infinite matrix in the class $\mathcal{C}$. If*

$$\inf_{j\in\mathbb{Z}}|a(j,j)|-2\sum_{0\neq k\in\mathbb{Z}}\sup_{j-j'=k}|a(j,j')|>0, \tag{2.4}$$

then A has ℓ^∞-stability.

We say that an infinite matrix $A = (a(i,j))_{i,j\in\mathbb{Z}}$ is a *band matrix* if $a(i,j) = 0$ for all $i,j \in \mathbb{Z}$ satisfying $j > i+k$ or $j < i-k$. The quantity $2k+1$ is the *bandwidth* of the matrix A. For a band matrix A with bandwidth $2k+1$, $A - A_s$ is the zero matrix if $s > k$. Therefore for $N > k$,

$$\inf_{0\le s\le N}\Big(\|A-A_s\|_{\mathcal{C}} + \frac{s}{N}\|A\|_{\mathcal{C}}\Big) \le \frac{k}{N}\|A\|_{\mathcal{C}}.$$

This, together with Theorem 2.1, gives the following sufficient condition for a band matrix to have ℓ^p-stability.

Corollary 2.3. *Let $1 \le p \le \infty$ and A be a convolution-dominated band matrix in the class $\mathcal{C}$ with bandwidth $2k+1$. If there exists an integer $N_0 > k$ such that*

$$\|A\chi_n^{N_0}c\|_p \ge \alpha\|\chi_n^{N_0}c\|_p, \quad c\in\ell^p, \tag{2.5}$$

holds for some constant α strictly larger than $2(5+2^{1-p})^{1/p}k\|A\|_{\mathcal{C}}/N_0$, then A has ℓ^p-stability.

If we further assume that the infinite matrix A in Corollary 2.3 has the form $A = (a(j-j'))_{j,j'\in\mathbb{Z}}$ for some finite sequence $a = (a(j))_{j\in\mathbb{Z}}$ satisfying $a(j) = 0$ for $|j| > k$, then $\|A\|_{\mathcal{C}} = \sum_{|j|\le k}|a(j)|$ and the condition (2.5) can reformulated as follows:

$$\|\tilde{A}_{N_0}c\|_p \ge \frac{\gamma k}{N_0}\Big(\sum_{|j|\le k}|a(j)|\Big)\|c\|_p, \quad c\in\mathbb{R}^{2N_0+1}, \tag{2.6}$$

holds for some $\gamma > 2(5+2^{1-p})^{1/p}$, where

$$\tilde{A}_{N_0} = (a(j-j'))_{-N_0-k\le j\le N_0+k,\,-N_0\le j'\le N_0} \tag{2.7}$$

and

$$\|c\|_p = \begin{cases} (\sum_{j=-k_1}^{k_2}|c(j)|^p)^{1/p} & \text{if } 1\le p<\infty \\ \sup_{-k_1\le j\le k_2}|c(j)| & \text{if } p=\infty, \end{cases}$$

for $c = (c(-k_1),\cdots,c(0),\ldots,c(k_2))^T \in \mathbb{R}^{k_1+k_2+1}$. As a conclusion from (2.6) and (2.7), we see that if $A = (a(j-j'))_{j,j'\in\mathbb{Z}}$ does not have ℓ^p-stability, then for any large integer N,

$$\inf_{0\ne c\in\mathbb{R}^{2N+1}}\frac{\|\tilde{A}_N c\|_p}{\|c\|_p} \le \frac{2(5+2^{1-p})^{1/p}k}{N}\Big(\sum_{|j|\le k}|a(j)|\Big). \tag{2.8}$$

For the special case $p = 2$, the above inequality (2.8) can be interpreted as the minimal eigenvalue of $(\tilde{A}_N)^T\tilde{A}_N$ is less than or equal to $\frac{\sqrt{22}k^2}{N^2}\big(\sum_{|j|\le k}|a(j)|\big)^2$, and it can also be rewritten as

$$\inf_{0\ne P_N\in\Pi_N}\frac{\Big(\int_{-\pi}^{\pi}|\hat{a}(\xi)|^2|P_N(\xi)|^2d\xi\Big)^{1/2}}{\Big(\int_{-\pi}^{\pi}|P_N(\xi)|^2d\xi\Big)^{1/2}} \le \frac{\sqrt{22}k}{N}\Big(\sum_{|j|\le k}|a(j)|\Big), \tag{2.9}$$

where $\hat{a}(\xi) = \sum_{j\in\mathbb{Z}}a(j)e^{-ij\xi}$ and Π_N is the set of all trigonometrical polynomial of degree at most N.

If the sequence $a = (a(j))_{j\in\mathbb{Z}}$ satisfies $a(0) = 1, a(-1) = -1$, and $a(j) = 0$ otherwise, then the bandwidth of the infinite matrix $A = (a(j-j'))_{j,j'\in\mathbb{Z}}$ is equal to 3, the norm $\|A\|_{\mathcal{C}}$ of the associated infinite matrix A is equal to 2,

$$\tilde{A}_N = \begin{pmatrix} -1 & 0 & 0 & \cdots & 0 & 0 \\ 1 & -1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & -1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & -1 \\ 0 & 0 & 0 & \cdots & 0 & 1 \end{pmatrix}, \tag{2.10}$$

and

$$\inf_{0\neq c\in\mathbb{R}^{2N+1}} \frac{\|\tilde{A}_N c\|_p}{\|c\|_p} \geq \frac{1}{N+1},$$

where the last inequality holds since the matrix

$$\tilde{B}_N := \begin{pmatrix} -1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ -1 & -1 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ -1 & -1 & -1 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ -1 & -1 & -1 & \cdots & -1 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & 1 & 1 & \cdots & 1 & 1 \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 1 & \cdots & 1 & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 1 & 1 \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 1 \end{pmatrix}.$$

is a left inverse of the matrix $\tilde{A}_N$. Therefore the order N^{-1} in (2.8) can not be improved in general, but the author believes that the bound constant $2(5+2^{1-p})^{1/p}$ in (2.2) and (2.8) is not optimal and could be improved.

3. Proof

We say that a discrete subset Λ of $\mathbb{R}^d$ is *relatively-separated* if

$$R(\Lambda) := \sup_{x\in\mathbb{R}^d} \sum_{\lambda\in\Lambda} \chi_{\lambda+[-1/2,1/2)^d}(x) < \infty \tag{3.1}$$

([1, 23, 27]). Clearly, the set $\mathbb{Z}$ of all integers is a relatively-separated subset of $\mathbb{R}$ with

$$R(\mathbb{Z}) = 1. \tag{3.2}$$

Given a discrete set Λ, let $\ell^p(\Lambda)$ be the set of all p-summable sequences on the set Λ with standard norm $\|\cdot\|_{\ell^p(\Lambda)}$ or $\|\cdot\|_p$ for brevity.

Given two relatively-separated subsets Λ and Λ' of $\mathbb{R}^d$, define

$$\mathcal{C}(\Lambda,\Lambda') = \Big\{A := \big(a(\lambda,\lambda')\big)_{\lambda\in\Lambda,\lambda'\in\Lambda'} \Big| \ \|A\|_{\mathcal{C}(\Lambda,\Lambda')} < \infty\Big\},$$

where

$$\|A\|_{\mathcal{C}(\Lambda,\Lambda')} = \sum_{k\in\mathbb{Z}^d} \sup_{\lambda\in\Lambda,\lambda'\in\Lambda'} |a(\lambda,\lambda')|\chi_{k+[-1/2,1/2]^d}(\lambda-\lambda').$$

It is obvious that

$$\mathcal{C}(\mathbb{Z},\mathbb{Z})=\mathcal{C}. \tag{3.3}$$

Given an infinite matrix $A=(a(\lambda,\lambda'))_{\lambda\in\Lambda,\lambda'\in\Lambda'}$, define its *truncation matrices* $A_s, s\ge 0$, by

$$A_s=\Big(a(\lambda,\lambda')\chi_{(-s,s)^d}(\lambda-\lambda')\Big)_{\lambda\in\Lambda,\lambda'\in\Lambda'}.$$

For any $y\in\mathbb{R}^d$ and a positive integer N, define the operator χ_y^N on $\ell^p(\Lambda)$ by

$$\chi_n^N:\ \ell^p(\Lambda)\ni\big(c(\lambda)\big)_{\lambda\in\Lambda}\longmapsto\big(c(\lambda)\chi_{(-N,N)^d}(\lambda-y)\big)_{\lambda\in\Lambda}\in\ell^p(\Lambda). \tag{3.4}$$

In this section, we establish the following criterion for the ℓ^p-stability of infinite matrices in the class $\mathcal{C}(\Lambda,\Lambda')$, which is a slight generalization of Theorem 2.1 by (3.2) and (3.3).

Theorem 3.1. *Let $1\le p\le\infty$, the subsets Λ,Λ' of $\mathbb{R}^d$ be relatively-separated, and the infinite matrix A belong to $\mathcal{C}(\Lambda,\Lambda')$. Then the following statements are equivalent to each other:*

(i) *The infinite matrix A has ℓ^p-stability, i.e., there exist positive constants C_1 and C_2 such that*

$$C_1\|c\|_{\ell^p(\Lambda')}\le\|Ac\|_{\ell^p(\Lambda)}\le C_2\|c\|_{\ell^p(\Lambda')}\quad\text{for all } c\in\ell^p(\Lambda'). \tag{3.5}$$

(ii) *There exist a positive constant C_0 and a positive integer N_0 such that*

$$\|\chi_n^{2N}A\chi_n^Nc\|_{\ell^p(\Lambda)}\ge C_0\|\chi_n^Nc\|_{\ell^p(\Lambda')}\quad\text{for all}\quad c\in\ell^p(\Lambda'), \tag{3.6}$$

where $N_0\le N\in\mathbb{Z}$ and $n\in N\mathbb{Z}^d$.

(iii) *There exist a positive integer N_0 and a positive constant α satisfying*

$$\alpha>2(5+2^{1-p})^{d/p}R(\Lambda)^{1/p}R(\Lambda')^{1-1/p}\inf_{0\le s\le N_0}\Big(\|A-A_s\|_{\mathcal{C}(\Lambda,\Lambda')}+\frac{ds}{N_0}\|A\|_{\mathcal{C}(\Lambda,\Lambda')}\Big) \tag{3.7}$$

such that

$$\|\chi_n^{2N_0}A\chi_n^{N_0}c\|_{\ell^p(\Lambda)}\ge\alpha\|\chi_n^{N_0}c\|_{\ell^p(\Lambda')} \tag{3.8}$$

hold for all $c\in\ell^p(\Lambda')$ and $n\in N_0\mathbb{Z}$.

Using the above theorem, we obtain the following equivalence of ℓ^p-stability for infinite matrices having certain off-diagonal decay, which is established in [2, 28, 23] for $\gamma>d(d+1),\gamma>0$, and $\gamma\ge0$ respectively.

Corollary 3.2. *Let Λ,Λ' be relatively-separated subsets of $\mathbb{R}^d$, and $A=(a(\lambda,\lambda'))_{\lambda\in\Lambda,\lambda'\in\Lambda'}$ satisfy*

$$\|A\|_{\mathcal{C}_\gamma(\Lambda,\Lambda')}=\sum_{k\in\mathbb{Z}^d}(1+|k|)^\gamma\sup_{\lambda\in\Lambda,\lambda'\in\Lambda'}|a(\lambda,\lambda')|\chi_{k+[-1/2,1/2]^d}(\lambda-\lambda')<\infty$$

where $\gamma>0$. Then the ℓ^p-stability of the infinite matrix A are equivalent to each other for different $1\le p\le\infty$.

Proof. Let $1\le p\le\infty$ and A have ℓ^p-stability. Then by Theorem 3.1 there exists a positive constant C_0 and a positive integer N_0 such that

$$\|\chi_n^{2N}A\chi_n^Nc\|_{\ell^p(\Lambda)}\ge C_0\|\chi_n^Nc\|_{\ell^p(\Lambda')}\quad\text{for all}\quad c\in\ell^p(\Lambda'), \tag{3.9}$$

where $N_0 \le N \in \mathbb{Z}$ and $n \in N\mathbb{Z}^d$. From the equivalence of different norms on a finite-dimensional space, we have that

$$((2N)^d R(\Lambda))^{\min(1/q-1/p,0)} \|\chi_n^N c\|_{\ell^p(\Lambda)} \le \|\chi_n^N c\|_{\ell^q(\Lambda)} \le ((2N)^d R(\Lambda))^{\max(1/q-1/p,0)} \|\chi_n^N c\|_{\ell^p(\Lambda)} \text{ for all } c \in \ell^p(\Lambda),$$

where $1 \le p, q \le \infty, 1 \le N \in \mathbb{Z}$ and $n \in N\mathbb{Z}^d$ ([2, 23]). Therefore for $1 \le q \le \infty$,

$$\|\chi_n^{2N} A \chi_n^N c\|_{\ell^q(\Lambda)} \ge C_0 (2N)^{-d|1/p-1/q|} R(\Lambda')^{\min(1/p-1/q,0)} \times R(\Lambda)^{-\max(1/p-1/q,0)} \|\chi_n^N c\|_{\ell^q(\Lambda')} \quad \text{for all} \quad c \in \ell^q(\Lambda'), \tag{3.10}$$

where $N_0 \le N \in \mathbb{Z}$ and $n \in N\mathbb{Z}^d$. We notice that

$$\inf_{0 \le s \le N} \|A - A_s\|_{\mathcal{C}(\Lambda,\Lambda')} + \frac{ds}{N} \|A\|_{\mathcal{C}(\Lambda,\Lambda')} \le \|A\|_{\mathcal{C}_\gamma(\Lambda,\Lambda')} \inf_{0 \le s \le N} s^\gamma + \frac{ds}{N} \le (d+1) \|A\|_{\mathcal{C}_\gamma(\Lambda,\Lambda')} N^{-\gamma/(1+\gamma)}. \tag{3.11}$$

Thus for $1 \le q \le \infty$ with $d|1/p - 1/q| < \gamma/(1+\gamma)$, it follows from (3.10) and (3.11) that there exists a sufficiently large integer N_0 such that

$$\|\chi_n^{2N} A \chi_n^N c\|_{\ell^q(\Lambda)} \ge \alpha \|\chi_n^N c\|_{\ell^q(\Lambda')} \tag{3.12}$$

hold for all $c \in \ell^q(\Lambda'), N \ge N_0$ and $n \in N\mathbb{Z}^d$, where α is a positive constant larger than $2(5+2^{1-q})^{d/q} R(\Lambda)^{1/q} R(\Lambda')^{1-1/q} \inf_{0 \le s \le N_0} \big(\|A - A_s\|_{\mathcal{C}(\Lambda,\Lambda')} + \frac{ds}{N_0} \|A\|_{\mathcal{C}(\Lambda,\Lambda')}\big)$. Then by Theorem 3.1, the infinite matrix A has ℓ^q-stability for all $1 \le q \le \infty$ with $d|1/q - 1/p| < \gamma/(1+\gamma)$. Applying the above trick repeatedly, we prove the ℓ^q-stability of the infinite matrix A for any $1 \le q \le \infty$.

□

To prove Theorem 3.1, we first recall some basic properties for infinite matrices A in the class $\mathcal{C}(\Lambda, \Lambda')$ and its truncation matrices $A_s, s \ge 0$.

Lemma 3.3. ([23]) *Let $1 \le p \le \infty$, the subsets Λ, Λ' of $\mathbb{R}^d$ be relatively-separated, A be an infinite matrix in the class $\mathcal{C}(\Lambda, \Lambda')$, and $A_s, s \ge 0$, be the truncation matrices of A. Then*

$$\|Ac\|_{\ell^p(\Lambda)} \le R(\Lambda)^{1/p} R(\Lambda')^{1-1/p} \|A\|_{\mathcal{C}(\Lambda,\Lambda')} \|c\|_{\ell^p(\Lambda')} \quad \text{for all } c \in \ell^p(\Lambda'), \tag{3.13}$$

$$\lim_{s \to +\infty} \|A - A_s\|_{\mathcal{C}(\Lambda,\Lambda')} = 0, \tag{3.14}$$

$$\lim_{N \to +\infty} \inf_{0 \le s \le N} \|A - A_s\|_{\mathcal{C}(\Lambda,\Lambda')} + \frac{ds}{N} \|A\|_{\mathcal{C}(\Lambda,\Lambda')} = 0, \tag{3.15}$$

and

$$\|A_s\|_{\mathcal{C}} \le \|A\|_{\mathcal{C}} \quad \text{for all } s \ge 0. \tag{3.16}$$

Let $\psi_0(x_1, \ldots, x_d) = \prod_{i=1}^d \max(\min(2 - 2|x_i|, 1), 0)$ be a cut-off function on $\mathbb{R}^d$. Then

$$0 \le \chi_{[-1/2,1/2]^d}(x) \le \psi_0(x) \le \chi_{(-1,1)^d}(x) \le 1 \quad \text{for all } x \in \mathbb{R}^d, \tag{3.17}$$

and

$$|\psi_0(x) - \psi_0(y)| \le 2d \|x - y\|_\infty \quad \text{for all } x, y \in \mathbb{R} \tag{3.18}$$

where $\|x\|_\infty = \max_{1\le i\le d} |x_i|$ for $x = (x_1, \dots, x_d)$. Define the multiplication operator Ψ_n^N on $\ell^p(\Lambda)$ by

$$\Psi_n^N:\ \ell^p(\Lambda) \ni (c(\lambda))_{\lambda\in\Lambda} \longmapsto \Big(\psi_0\big(\frac{\lambda-n}{N}\big)c(\lambda)\Big)_{\lambda\in\Lambda} \in \ell^p(\Lambda). \tag{3.19}$$

Applying (3.17) and (3.18) for the cut-off function ψ_0, we obtain the following properties for the multiplication operators $\Psi_n^N, n \in N\mathbb{Z}$.

Lemma 3.4. *Let $1 \le N \in \mathbb{Z}$, Λ be a relatively-separated subset of $\mathbb{R}^d$, and the multiplication operators $\Psi_n^N, n \in N\mathbb{Z}^d$, be as in* (3.19)*. Then*

$$\|\Psi_n^N c\|_{\ell^p(\Lambda)} \le \|\chi_n^N c\|_{\ell^p(\Lambda)} \quad \text{for all } c \in \ell^p(\Lambda) \tag{3.20}$$

where $1 \le p \le \infty$,

$$\|c\|_{\ell^p(\Lambda)} \le \Big(\sum_{n\in N\mathbb{Z}^d} \|\Psi_n^N c\|_{\ell^p(\Lambda)}^p\Big)^{1/p} \le 2^{d/p}\|c\|_{\ell^p(\Lambda)} \quad \text{for all } c \in \ell^p(\Lambda) \tag{3.21}$$

(3.22)
$$4^{d/p}\|c\|_{\ell^p(\Lambda)} \le \Big(\sum_{n\in N\mathbb{Z}^d} \|\Psi_n^{4N} c\|_{\ell^p(\Lambda)}^p\Big)^{1/p} \le (5+2^{1-p})^{d/p}\|c\|_{\ell^p(\Lambda)} \quad \text{for all } c \in \ell^p(\Lambda),$$

where $1 \le p < \infty$, and

$$\|c\|_{\ell^\infty(\Lambda)} = \sup_{n\in N\mathbb{Z}^d} \|\Psi_n^N c\|_{\ell^\infty(\Lambda)} = \sup_{n\in N\mathbb{Z}^d} \|\Psi_n^{4N} c\|_{\ell^\infty(\Lambda)} \quad \text{for all } c \in \ell^\infty(\Lambda). \tag{3.23}$$

To prove Theorem 2.1, we also need the following result.

Lemma 3.5. ([23]) *Let $N \ge 1$, the subsets Λ, Λ' of $\mathbb{R}^d$ be relatively-separated, A be an infinite matrix in the class $\mathcal{C}(\Lambda, \Lambda')$, A_N be the truncation matrix of A, and $\Psi_n^N, n \in N\mathbb{Z}^d$, be the multiplication operators in* (3.19)*. Then*

$$\|\Psi_n^N A_N - A_N \Psi_n^N\|_{\mathcal{C}(\Lambda,\Lambda')} \le \inf_{0\le s\le N} \Big(\|A_N - A_s\|_{\mathcal{C}(\Lambda,\Lambda')} + \frac{2ds}{N}\|A_s\|_{\mathcal{C}(\Lambda,\Lambda')}\Big). \tag{3.24}$$

Now we start to prove Theorem 3.1.

Proof of Theorem 3.1. (i)$\Longrightarrow$(ii): By the ℓ^p-stability of the infinite matrix A, there exists a positive constant C_0 (independent of $n \in N\mathbb{Z}^d$ and $1 \le N \in \mathbb{Z}$) such that

$$\|A\chi_n^N c\|_{\ell^p(\Lambda)} \ge C_0\|\chi_n^N c\|_{\ell^p(\Lambda')} \quad \text{for all } c \in \ell^p(\Lambda'), \tag{3.25}$$

where $n \in N\mathbb{Z}^d$ and $N \ge 1$. Noting

$$\chi_n^{2N} A_N \psi_n^N = A_N \psi_n^N \tag{3.26}$$

and applying (3.13) yield

$$\begin{aligned} & \|A\chi_n^N c - \chi_n^{2N} A\chi_n^N c\|_{\ell^p(\Lambda)} \\ = \ & \|(I-\chi_n^{2N})(A-A_N)\chi_n^N c\|_{\ell^p(\Lambda)} \\ \le \ & R(\Lambda)^{1/p} R(\Lambda')^{1-1/p}\|A-A_N\|_{\mathcal{C}(\Lambda,\Lambda')}\|\chi_n^N c\|_{\ell^p(\Lambda')}, \end{aligned} \tag{3.27}$$

where I is the identity operator. Combining the estimates in (3.25) and (3.27) proves that

$$\|\chi_n^{2N} A\chi_n^N c\|_{\ell^p(\Lambda)} \ge \big(C_0 - R(\Lambda)^{1/p} R(\Lambda')^{1-1/p}\|A-A_N\|_{\mathcal{C}(\Lambda,\Lambda')}\big)\|\chi_n^N c\|_{\ell^p(\Lambda')} \tag{3.28}$$

hold for all $c \in \ell^p(\Lambda')$, where $n \in N\mathbb{Z}^d$ and $N \geq 1$. The conclusion (ii) then follows from (3.14) and (3.28).

(ii)$\Longrightarrow$(iii): The implication follows from (3.15).

(iii)$\Longrightarrow$(i): Let $1 \leq p < \infty$. Take any $n \in N_0\mathbb{Z}^d$ and $c \in \ell^p(\Lambda')$. By the assumption (iii) for the infinite matrix A,

$$\|\chi_n^{2N_0} A \Psi_n^{N_0} c\|_{\ell^p(\Lambda)} = \|\chi_n^{2N_0} A \chi_n^{N_0} \Psi_n^{N_0} c\|_{\ell^p(\Lambda)} \geq \alpha \|\Psi_n^{N_0} c\|_{\ell^p(\Lambda')}. \tag{3.29}$$

This together with (3.13) and (3.26) implies that

$$\begin{aligned}
& \|A_{N_0} \Psi_n^{N_0} c\|_{\ell^p(\Lambda)} \\
= \ & \|\chi_n^{2N_0} (A_{N_0} - A + A) \Psi_n^{N_0} c\|_{\ell^p(\Lambda)} \\
\geq \ & \|\chi_n^{2N_0} A \chi_n^{N_0} \Psi_n^{N_0} c\|_{\ell^p(\Lambda)} - \|\chi_n^{2N_0} (A_{N_0} - A) \Psi_n^{N_0} c\|_{\ell^p(\Lambda)} \\
\geq \ & \big(\alpha - R(\Lambda)^{1/p} R(\Lambda')^{1-1/p} \|A - A_{N_0}\|_{\mathcal{C}(\Lambda,\Lambda')}\big) \|\Psi_n^{N_0} c\|_{\ell^p(\Lambda')}.
\end{aligned} \tag{3.30}$$

From (3.13) and (3.24) it follows that

$$\begin{aligned}
& \|(\Psi_n^{N_0} A_{N_0} - A_{N_0} \Psi_n^{N_0}) c\|_{\ell^p(\Lambda)} \\
= \ & \|(\Psi_n^{N_0} A_{N_0} - A_{N_0} \Psi_n^{N_0}) \Psi_n^{4N_0} c\|_{\ell^p(\Lambda)} \\
\leq \ & R(\Lambda)^{1/p} R(\Lambda')^{1-1/p} \|\Psi_n^{N_0} A_{N_0} - A_{N_0} \Psi_n^{N_0}\|_{\mathcal{C}(\Lambda,\Lambda')} \|\Psi_n^{4N_0} c\|_{\ell^p(\Lambda')} \\
\leq \ & R(\Lambda)^{1/p} R(\Lambda')^{1-1/p} \\
& \times \inf_{0 \leq s \leq N_0} \Big(\|A_{N_0} - A_s\|_{\mathcal{C}} + \frac{2ds}{N_0} \|A_{N_0}\|_{\mathcal{C}} \Big) \|\Psi_n^{4N_0} c\|_{\ell^p(\Lambda')}.
\end{aligned} \tag{3.31}$$

Combining (3.21), (3.22), (3.30) and (3.31), we get

$$\begin{aligned}
& 2^{d/p} \|A_{N_0} c\|_{\ell^p(\Lambda)} \geq \Big(\sum_{n \in N_0\mathbb{Z}} \|\Psi_n^{N_0} A_{N_0} c\|_{\ell^p(\Lambda)}^p \Big)^{1/p} \\
\geq \ & \Big(\alpha - R(\Lambda)^{1/p} R(\Lambda')^{1-1/p} \|A - A_{N_0}\|_{\mathcal{C}(\Lambda,\Lambda')}\Big) \Big(\sum_{n \in N_0\mathbb{Z}} \|\Psi_n^{N_0} c\|_{\ell^p(\Lambda')}^p \Big)^{1/p} \\
& - R(\Lambda)^{1/p} R(\Lambda')^{1-1/p} \inf_{0 \leq s \leq N_0} \Big(\|A_{N_0} - A_s\|_{\mathcal{C}(\Lambda,\Lambda')} + \frac{2ds}{N_0} \|A_{N_0}\|_{\mathcal{C}(\Lambda,\Lambda')} \Big) \\
& \times \Big(\sum_{n \in N_0\mathbb{Z}} \|\Psi_n^{4N_0} c\|_{\ell^p(\Lambda')}^p \Big)^{1/p} \\
\geq \ & \Big(\alpha - R(\Lambda)^{1/p} R(\Lambda')^{1-1/p} \|A - A_{N_0}\|_{\mathcal{C}(\Lambda,\Lambda')} - (5 + 2^{1-p})^{1/p} R(\Lambda)^{1/p} R(\Lambda')^{1-1/p} \\
& \times \inf_{0 \leq s \leq N_0} \big(\|A_{N_0} - A_s\|_{\mathcal{C}(\Lambda,\Lambda')} + \frac{2ds}{N_0} \|A_{N_0}\|_{\mathcal{C}(\Lambda,\Lambda')} \big) \Big) \|c\|_{\ell^p(\Lambda')}.
\end{aligned}$$

Therefore

$$\begin{aligned}
&\|Ac\|_{\ell^p(\Lambda)} \ge \|A_{N_0}c\|_{\ell^p(\Lambda)} - \|(A-A_{N_0})c\|_{\ell^p(\Lambda)}\\
\ge\ & 2^{-1/p}\Big(\alpha - (1+2^{d/p})R(\Lambda)^{1/p}R(\Lambda')^{1-1/p}\|A-A_{N_0}\|_{\mathcal{C}(\Lambda,\Lambda')}\\
&-(5+2^{1-p})^{d/p}R(\Lambda)^{1/p}R(\Lambda')^{1-1/p}\\
&\times \inf_{0\le s\le N_0}\big(\|A_{N_0}-A_s\|_{\mathcal{C}(\Lambda,\Lambda')} + \frac{2ds}{N_0}\|A_{N_0}\|_{\mathcal{C}(\Lambda,\Lambda')}\big)\Big)\|c\|_{\ell^p(\Lambda')}\\
\ge\ & 2^{-d/p}\Big(\alpha - 2(5+2^{1-p})^{1/p}R(\Lambda)^{1/p}\\
&\times R(\Lambda')^{1-1/p}\inf_{0\le s\le N_0}\big(\|A-A_s\|_{\mathcal{C}(\Lambda,\Lambda')} + \frac{ds}{N_0}\|A\|_{\mathcal{C}(\Lambda,\Lambda')}\big)\Big)\|c\|_{\ell^p(\Lambda')},
\end{aligned}$$

and the conclusion (i) for $1\le p<\infty$ follows.

The conclusion (i) for $p=\infty$ can be proved by similar argument. We omit the details here.

□

The author thanks Professors Deguang Han, Zuhair M. Nashed, Xianliang Shi, and Wai-Shing Tang for their discussion and suggestions in preparing the manuscript.

References

[1] A. Aldroubi and K. Gröchenig, Nonuniform sampling and reconstruction in shift-invariant space, *SIAM Review*, **43**(2001), 585–620.

[2] A. Aldroubi, A. Baskakov and I. Krishtal, Slanted matrices, Banach frames, and sampling, *J. Funct. Anal.*, **255**(2008), 1667–1691.

[3] R. Balan, P. G. Casazza, C. Heil, and Z. Landau, Density, overcompleteness and localization of frames I. Theory; II. Gabor system, *J. Fourier Anal. Appl.*, **12**(2006), 105–143; 309–344.

[4] B. A. Barnes, When is the spectrum of a convolution operator on L^p independent of p? *Proc. Edinburgh Math. Soc.*, **33**(1990), 327–332.

[5] A. G. Baskakov, Wiener's theorem and asymptotic estimates for elements of inverse matrices, *Funktsional. Anal. i Prilozhen*, **24**(1990), 64–65; translation in *Funct. Anal. Appl.*, **24**(1990), 222–224.

[6] C. K. Chui, W. He, and J. Stöckler, Nonstationary tight wavelet frames I: Bounded intervals; II: unbounded intervals, *Appl. Comp. Harmonic Anal.*, **17**(2004), 141–197; **18**(2005), 25–66.

[7] A. Connes, C^* algebres et geometrie differentielle. *C. R. Acad. Sci. Paris Ser. A-B*, **290**(1980), A599–A604.

[8] C. de Boor, A bound on the L_∞-norm of the L_2-approximation by splines in terms of a global mesh ratio, *Math. Comp.*, **30**(1976), 687–694.

[9] S. Demko, Inverse of band matrices and local convergences of spline projections, *SIAM J. Numer. Anal.*, **14**(1977), 616–619.

[10] G. Fendler, K. Gröchenig, and M. Leinert, Convolution-dominated operators on discrete groups, *Integral Equations Operator Theory*, **61**(2008), 493–509.

[11] I. M. Gelfand, D. A. Raikov, and G. E. Silov, *Commutative Normed Rings*, New York: Chelsea 1964.

[12] I. Gohberg, M. A. Kaashoek, and H. J. Woerdeman, The band method for positive and strictly contractive extension problems: an alternative version and new applications, *Integral Equations Operator Theory*, **12**(1989), 343–382.

[13] K. Gröchenig, Time-frequency analysis of Sjöstrand's class, *Rev. Mat. Iberoam.*, **22**(2006), 703–724.

[14] K. Gröchenig and M. Leinert, Wiener's lemma for twisted convolution and Gabor frames, *J. Amer. Math. Soc.*, **17**(2003), 1–18.

[15] K. Gröchenig and M. Leinert, Symmetry of matrix algebras and symbolic calculus for infinite matrices, *Trans, Amer. Math. Soc.*, **358**(2006), 2695–2711.

[16] K. Gröchenig and T. Strohmer, Pseudodifferential operators on locally compact abelian groups and Sjöstrand's symbol class, *Journal für die reine und angewandte Mathematik*, **613**(2007), 121–146.

[17] A. Hulanicki, On the spectrum of convolution operators on groups with polynomial growth, *Invent. Math.*, 17(1972), 135–142.

[18] S. Jaffard, Properiétés des matrices bien localisées prés de leur diagonale et quelques applications, *Ann. Inst. Henri Poincaré*, **7**(1990), 461–476.

[19] R.-Q. Jia and C. A. Micchelli, Using the refinement equations for the construction of prewavelets. II. Powers of two, In *Curves and Surfaces (Chamonix-Mont-Blanc, 1990)*, Academic Press, Boston, MA, 1991, 209–246.

[20] V. G. Kurbatov, Algebras of difference and integral operators, *Funktsional. Anal. I. Prilozhen.*, **24**(1990), 87–88.

[21] M. A. Naimark, *Normed Algebras*, Wolters-Noordhoff Publishing Groningen, 1972.

[22] T. Pytlik, On the spectral radius of elements in group algrebras, *Bull. Acad. Polon. Sci. Ser. Sci. Math.*, **21**(1973), 899–902.

[23] C. E. Shin and Q. Sun, Stability of localized operators, *J. Funct. Anal.*, **256**(2009), 2417–2439.

[24] J. Sjöstrand, An algebra of pseudodifferential operators, *Math. Res. Lett.*, **1**(1994), 185–192.

[25] J. Sjöstrand, Wiener type algebra of pseudodifferential operators, Centre de Mathematiques, Ecole Polytechnique, Palaiseau France, Seminaire 1994–1995, December 1994.

[26] Q. Sun, Non-uniform sampling and reconstruction for signals with finite rate of innovations, *SIAM J. Math. Anal.*, **38**(2006), 1389–1422.

[27] Q. Sun, Wiener's lemma for infinite matrices, *Trans. Amer. Math. Soc.*, **359**(2007), 3099–3123.

[28] R. Tessera, Finding left inverse for operators on $\ell^p(\mathbb{Z}^d)$ with polynomial decay, Preprint 2008.

[29] N. Wiener, Tauberian Theorem, *Ann. Math.*, **33**(1932), 1–100.

Department of Mathematics, University of Central Florida, Orlando, FL 32816, USA

Email: `qsun@mail.ucf.edu`