M Band Scaling Functions with Minimal Support Are Asymmetric

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In this note, it is proved that all the M band scaling functions with minimal support are asymmetric except the Haar functions.

1. INTRODUCTION AND RESULT

Fix an integer $M \geq 2$. In this note, a *multiresolution* with dilation M means a sequence of nested subspaces $V_j, j \in \mathbf{Z}$, of $L^2 := L^2(\mathbf{R})$

$$\cdots \subset V_{-2} \subset V_{-1} \subset V_0 \subset V_1 \subset V_2 \subset \cdots$$

such that $M^{j/2}\phi(M^j \cdot -k), k \in \mathbf{Z}$, is an orthonormal bases of V_j for some compactly supported L^2 function ϕ, and such that $\cup_{j\in\mathbf{Z}} V_j$ is dense in L^2. The function ϕ in the above definition of a multiresolution is called an M *band scaling function*, or a *scaling function* for short. Then the scaling function ϕ in this note has *orthonormal integer shifts*, i.e.,

$$\int_{\mathbf{R}} |\phi(x)|^2 dx = 1 \quad and \quad \int_{\mathbf{R}} \phi(x)\phi(x-k)dx = 0 \quad \forall\ k \in \mathbf{Z}\backslash\{0\}.$$

We say that an L^2 function f is *symmetric* if

$$f(x_0 + \cdot) = f(x_0 - \cdot)$$

for some $x_0 \in \mathbf{R}$. The symmetry of a scaling function is nice, and asymmetry can be a nuisance in some applications. For $M = 2$, it is well known that there is not any compactly supported symmetric scaling function except the Haar function([6]), but for $M \geq 3$, some compactly supported symmetric non-Haar scaling functions have been constructed ([1, 4, 7, 9]).

We say that a compactly supported L^2 function ϕ satisfies the *moment conditions of order* N if $\hat{\phi}(0) = 1$ and $D^\alpha \hat{\phi}(2k\pi) = 0$ for all $k \in \mathbf{Z}\backslash\{0\}$ and $0 \leq \alpha \leq N-1$, where N is a positive integer. Here and hereafter, the Fourier transform $\hat{f}$ of an integrable function f is defined by $\hat{f}(\xi) = \int_{\mathbf{R}} e^{-ix\xi} f(x)dx$. A multiresolution with its scaling function satisfying moment conditions of higher order is nice since functions in that multiresolution would provide better approximation to smooth functions. For $M = 2$ and $N \geq 1$, in the class of scaling functions satisfying the moment conditions of order N, a scaling function with minimal support was constructed in [6]. Moreover, that scaling function is asymmetric except $N = 1$, and has its asymptotic Hölder regularity proportionally to $(1 - \ln 3/\ln 4)N$ as N tends to infinity ([5, 6, 10, 15]). For $M \geq 3$ and $N \geq 1$, similar scaling function with minimal support, which we denoted by $\phi_{M,N}$, was constructed in [8]. It was shown that $\phi_{M,N}$ is just the Haar function for $N = 1$ (hence symmetric for $N = 1$), has asymptotic Hölder regularity proportionally to $\ln(\sin M\pi/(2M+2))^{-1}(\ln M)^{-1}N$ for even M, and to $(4\ln M)^{-1}\ln N$ for odd M as N tends to infinity ([2, 11, 13]), and has explicit expression if $N \leq M - 1$ ([3, 14]). In this note, we show that the M band scaling function $\phi_{M,N}$ with minimal support is asymmetric for $N \geq 2$.

THEOREM 1.1. *Let $N \geq 2$ and $\phi_{M,N}$ be the minimal supported M band scaling function satisfying moment conditions of order N. Then $\phi_{M,N}$ is asymmetric.*

2. PROOF OF THEOREM 1.1

A compactly supported L^2 function f is said to be *refinable* if $\int_{\mathbf{R}} f(x) = 1$ and

$$f = \sum_{k\in\mathbf{Z}} c(k) f(M\cdot - k) \tag{1}$$

for some sequence $\{c(k)\}_{k\in\mathbf{Z}}$ having finite support. The equation (1) is called a *refinement equation*, and the function $H(\xi) = \frac{1}{M}\sum_{k\in\mathbf{Z}} c(k)e^{-ik\xi}$ is known as the *symbol* of that refinement equation (1), or of the refinable function f. For symmetric refinable functions f, we have

LEMMA 2.2. *Let f be a compactly supported refinable function and $H(\xi)$ be corresponding symbol. If f is symmetric, then*

$$|H(\xi)|^2 = \Big(g\Big(\sin^2\frac{\xi}{2}\Big)\Big)^2 \quad \text{or} \quad \cos^2\frac{\xi}{2}\Big(g\Big(\sin^2\frac{\xi}{2}\Big)\Big)^2$$

for some polynomial g with $g(0) = 1$.

A complete proof of Lemma 2.2 was provided in [1]. We omit the detail here. In particular, for any compactly supported refinable function f and and corresponding symbol $H(\xi)$, f is symmetric if and only if $H(\xi) = e^{ik\xi}H(-\xi)$ for some integer k ([1, 12]). Then Lemma 2.2 is an easy consequence of the above claim.

From the nest condition of a multiresolution, any scaling function ϕ can be written as linear combination of $\phi(2\cdot - k), k \in \mathbf{Z}$, using some square summable sequence $\{c(k)\}_{k\in\mathbf{Z}}$. On the other hand, $c(k) = M\int_{\mathbf{R}} \phi(x)\phi(Mx - k)dx$ for any $k \in \mathbf{Z}$ by the orthonormal condition for the scaling function in the definition of a multiresolution, which leads to the finite support of the sequence $\{c(k)\}_{k\in\mathbf{Z}}$. Thus any scaling function is refinable.

To study the asymmetry of the minimal supported scaling function $\phi_{M,N}$, we need some knowledge about corresponding symbol $H_{M,N}$. For $N \geq 1$, let

$$P_{M,N}(\xi) = \sum_{n=0}^{N-1} \sum_{n_1+\cdots+n_{M-1}=n} \prod_{s=1}^{M-1} \binom{N-1+n_s}{n_s} \left(\sin \frac{s\pi}{M}\right)^{-2n_s} \sin^{2n} \frac{\xi}{2}. \quad (2)$$

Then $P_{M,N}(\xi) \geq P_{M,N}(0) = 1$, and

$$\sum_{s=0}^{M-1} \left(\frac{\sin^2 M\xi/2}{M^2 \sin^2(\xi/2 + s\pi/M)} \right)^N P_{M,N}\left(\xi + \frac{2s\pi}{M}\right) = 1 \quad \text{for any} \quad \xi \in [-\pi, \pi]. (3)$$

Moreover, $P_{M,N}(\xi)$ has minimal degree of $\sin^2 \xi/2$ in the class of polynomials of $\sin^2 \xi/2$ satisfying (3) (see for instance [1, 2, 8]). By Riesz Lemma([6]), there exist trigonometric polynomials $Q_{M,N}(\xi)$ with real coefficients such that

$$Q_{M,N}(0) = 1 \quad \text{and} \quad Q_{M,N}(\xi) Q_{M,N}(-\xi) = P_{M,N}(\xi). \quad (4)$$

Define

$$H_{M,N}(\xi) = \left(\frac{1 - e^{-iM\xi}}{M - Me^{-i\xi}} \right)^N Q_{M,N}(\xi). \quad (5)$$

Then we have

Lemma 2.3. *Let $\phi_{M,N}$ be the minimal supported M band scaling function satisfying moment conditions of order N, and let $H_{M,N}$ be defined by (5). Then $\phi_{M,N}$ is the refinable function with corresponding symbol $e^{ik\xi} H_{M,N}(\xi)$ for some integer k.*

The proof of Lemma 2.3 can be found in [8], see also [1], we omit the detail here.
Now we reach the stage to start the proof of Theorem 1.1.

Proof of Theorem 1.1 On the contrary, $\phi_{M,N}$ is symmetric for some $M \geq 3$ and $N \geq 2$. Then by Lemmas 2.2 and 2.3,

$$|H_{M,N}(\xi)|^2 = (g(\sin^2 \xi/2))^2 \quad \text{or} \quad \cos^2 \xi/2 \, (g(\sin^2 \xi/2))^2 \quad (6)$$

for some polynomial g with $g(0) = 1$. Note that

$$\frac{\sin^2 M\xi}{M^2 \sin^2 \xi} = \prod_{s=1}^{M-1} \left(1 - \frac{\sin^2 \xi}{\sin^2 s\pi/M}\right), \quad (7)$$

which follows from comparing the roots and values at the origin of both sides of the above equation. Then combining (4), (5), (6) and (7) leads to

$$P_{M,N}(\xi) = (h(\sin^2 \xi/2))^2 \quad \text{or} \quad \cos^2 \xi/2 \, (h(\sin^2 \xi/2))^2 \quad (8)$$

for some polynomial h with $h(0) = 1$. This together with the fact that $P_{M,N}(\xi)$ is a polynomial of degree $N-1$ about $\sin^2 \xi/2$ leads to

$$P_{M,N}(\xi) = \cos^2 \xi/2 \, (h(\sin^2 \xi/2))^2 \quad (9)$$

for even N, and

$$P_{M,N}(\xi) = (h(\sin^2 \xi/2))^2 \tag{10}$$

for odd N. Moreover, the polynomials h in (9) and (10) have degree k_0 and satisfy $h(0) = 1$, where k_0 is the largest integer smaller than or equal to $(N-1)/2$.

For even N, it follows from (9) that $P_{M,N}(\pi) = 0$, which is a contradiction since $P_{M,N}(\pi) \geq 1$ by (2). Therefore it remains to prove the assertion for odd N. In that case, substituting (10) into (3), and using the fact that for any $1 \leq s \leq M-1$,

$$\frac{\sin^2 M\xi/2}{M^2 \sin^2(\xi + 2s\pi/M)} = O\left(\sin^2 \frac{\xi}{2}\right) \quad \text{as} \quad \xi \to 0,$$

we get

$$\left(\frac{\sin^2 M\xi/2}{M^2 \sin^2 \xi/2}\right)^{2k_0+1} (h(\sin^2 \xi/2))^2 = 1 + O\left(\sin^{4k_0+2} \frac{\xi}{2}\right) \tag{11}$$

as $\xi \to 0$. Here and hereafter, $A(\xi) = O(B(\xi))$ as $\xi \to 0$ means that $A(\xi)/B(\xi)$ is bounded in a small neighborhood of the origin. Then taking square root at both sides of (11), using $h(0) = 1$, and multiplying $(\sin^2 M\xi/2)^{-k_0-1/2}(M^2 \sin^2 \xi/2)^{k_0+1/2}$ lead to

$$h(\sin^2 \xi/2) = \left(\frac{\sin^2 M\xi/2}{M^2 \sin^2 \xi/2}\right)^{-k_0-1/2} + O\left(\sin^{4k_0+2} \frac{\xi}{2}\right), \ \xi \to 0. \tag{12}$$

By (7) and the Taylor expansion of the function $(1+t)^{-1}$ at the origin,

$$(1-t)^{-1} = \sum_{n=0}^{k_0+1} t^n + O(t^{k_0+2}) \quad \text{as} \quad t \to 0,$$

we obtain

$$\begin{aligned}
&\left(\frac{\sin^2 M\xi/2}{M^2 \sin^2 \xi/2}\right)^{-1} \\
&= \prod_{s=1}^{M-1} \left(\sum_{n=0}^{k_0+1} \left(\frac{\sin^2 \xi/2}{\sin^2 s\pi/M}\right)^n\right) + O\left(\sin^{2k_0+4} \frac{\xi}{2}\right) \\
&= \sum_{n=0}^{k_0+1} \beta_{M,N}(n) \sin^{2n} \frac{\xi}{2} + O\left(\sin^{2k_0+4} \frac{\xi}{2}\right) \quad \text{as} \quad \xi \to 0,
\end{aligned} \tag{13}$$

where $\beta_{M,N}(n), 0 \leq n \leq k_0+1$, are real numbers. Hence it follows from (13) that

$$\beta_{M,N}(0) = 1 \quad \text{and} \quad \beta_{M,N}(n) > 0 \quad \forall\ 0 \leq n \leq k_0+1. \tag{14}$$

Substituting (13) in (12), and using $\beta_{M,N}(0) = 1$ from (14) and Taylor expansion for the function $(1+t)^{k_0+1/2}$ at the origin,

$$(1+t)^{k_0+1/2} = 1 + \sum_{n'=1}^{k_0+1} \frac{\prod_{j=0}^{n'-1}(k_0+1/2-j)}{n'!} t^{n'} + O(t^{k_0+2}) \quad \text{as } t \to 0,$$

we get

$$h\Big(\sin^2\frac{\xi}{2}\Big) = 1 + \sum_{n'=1}^{k_0+1} \frac{\prod_{j=0}^{n'-1}(k_0+1/2-j)}{n'!} \times \left(\sum_{n=1}^{k_0+1} \beta_{M,N}(n)\sin^{2n}\frac{\xi}{2}\right)^{n'} + O\left(\sin^{2k_0+4}\frac{\xi}{2}\right), \quad \xi \to 0. \tag{15}$$

Write

$$1 + \sum_{n'=1}^{k_0+1} \frac{\prod_{j=0}^{n'-1}(k_0+1/2-j)}{n'!} \left(\sum_{n=1}^{k_0+1} \beta_{M,N}(n)\sin^{2n}\frac{\xi}{2}\right)^{n'} = \sum_{n=0}^{k_0+1} \gamma_{M,N}(n)\sin^{2n}\frac{\xi}{2} + O\left(\sin^{2k_0+4}\frac{\xi}{2}\right), \quad \xi \to 0. \tag{16}$$

Then it follows from (14) and (16) that

$$\gamma_{M,N}(s) > 0 \quad \forall\, 0 \le s \le k_0+1. \tag{17}$$

Combining (15) and (16), and using the fact that h is a polynomial with degree k_0, we obtain

$$h\Big(\sin^2\frac{\xi}{2}\Big) = \sum_{n=0}^{k_0+1} \gamma_{M,N}(n)\sin^{2n}\frac{\xi}{2}$$

and $\gamma_{M,N}(k_0+1) = 0$, which contradicts (17). This completes the proof of Theorem 1.1.

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