SPECTRUM OF CONVOLUTION DILATION OPERATORS ON WEIGHTED L^P SPACES

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We consider the convolution dilation operator

$$W_{c,\alpha}f(x) = \alpha \int_{\mathbf{R}} c(\alpha x - y) f(y) dy, \quad f \in L^p(\mathbf{R}),$$

where α is a real number strictly larger than 1, and c is a compactly supported integrable kernel with $\int_{\mathbf{R}} c(x)dx = 1$. For any sufficiently large number K the space $L^p([-K, K])$ of all L^p-functions with support in the interval $[-K, K]$ is an invariant space of $W_{c,\alpha}$. It is known that $W_{c,\alpha}$ restricted to $L^p([-K, K])$ is a compact operator with eigenvalues α^{-k}, $k = 0, 1, \ldots$, and spectrum $\{\alpha^{-k} : k = 1, 2, \ldots\} \cup \{0\}$, which are independent of c and K. This result is better understood in the context of weighted L^p space, $L^p_w(\mathbf{R})$ that comprises functions f for which fw belong to $L^p(\mathbf{R})$. We prove that under an oscillation condition on w, $W_{c,\alpha}$ is a compact operator on $L^p_w(\mathbf{R})$ if and only if $\lim_{|x|\to\infty} w(x)/w(\alpha x) = 0$. Further, $W_{c,\alpha}$ has exactly the same eigenvalues and spectrum as its restriction to $L^p([-K, K])$. We also prove that if $\lim_{|x|\to\infty} w(x)/w(\alpha x) = r$ for some positive constant r, then the spectrum of $W_{c,\alpha}$ on the space $L^p_w(\mathbf{R})$ is the closed disc $D_s := \{\lambda \in \mathbf{C} : |\lambda| \le r\alpha^{1-1/p}\}$ in addition to the set $\{\alpha^{-k} : k = 1, 2, \ldots\}$, and that all nonzero complex numbers with absolute value strictly less than r are eigenvalues of the operator $W_{c,\alpha}$ on $L^p_w(\mathbf{R})$. In particular, for $w = 1$ the results say that the spectrum of $W_{c,\alpha}$ on $L^p(\mathbf{R})$ is the closed disc with centre at the origin and radius $\alpha^{1-1/p}$, and that all nonzero complex numbers with absolute value strictly less than 1 are its eigenvalues.

1 Introduction

Take a real number α strictly larger than one, and a compactly supported function c in $L^1(\mathbf{R})$ with $\int_{\mathbf{R}} c(x)dx = 1$. Define a convolution dilation operator $W_{c,\alpha} : L^p(\mathbf{R}) \to L^p(\mathbf{R})$ by

$$W_{c,\alpha}f(x) := \alpha \int_{\mathbf{R}} c(\alpha x - y) f(y) dy, \quad f \in L^p(\mathbf{R}). \tag{1.1}$$

The operator $W_{c,\alpha}$ is a continuous analogy of the *transfer operator* (also known as *Ruelle operator*) that arises in a number of different context, such as wavelet analysis[3,15,20], stationary subdivision[4,8,12,21], and dynamical systems[16,17,18]. It is easy to check that $W_{c,\alpha}$ is a bounded operator on $L^p(\mathbf{R})$ for any $1 \le$

$p \leq \infty$. For any $K > 0$, denote by $L^p([-K,K])$ the space of all L^p-functions with support in the interval $[-K,K]$, and by $L^p_0([-K,K])$ the space of all functions $f \in L^p([-K,K])$ with $\int_{\mathbf{R}} f(x)dx = 0$. Let K_0 be the smallest positive number that satisfies $\mathrm{supp}(c) \subset [-(\alpha-1)K_0, (\alpha-1)K_0]$. Then it can be checked that for any $K \geq K_0$, the spaces $L^p([-K,K])$ and $L^p_0([-K,K])$ are invariant subspaces of $W_{c,\alpha}$.

An eigenfunction ϕ of $W_{c,\alpha}$ with eigenvalue 1 is a solution of the convolution dilation equation

$$\phi = \alpha \int_{\mathbf{R}} c(\alpha \cdot -y)\phi(y)dy. \tag{1.2}$$

The simplest convolution dilation equation is one with kernel $c = \frac{1}{2}\chi_{(-1,1]}$ and dilation $\alpha = 2$, and it was studied by Kebaya and Iri[14] and Rvachev[19] independently. Recent interests in convolution dilation equations are associated with nonstationary multiresolution and wavelets[5,9], nonstationary subdivision processes[6,7], and invariant densities for model sets and quasicrystals[1,2]. It is known that (1.2) has a unique compactly supported solution ϕ normalized so that $\int_{\mathbf{R}} \phi(x)dx = 1$ and the solution ϕ is infinitely differentiable and supported in $[-K_0, K_0]$[13].

For a Banach space X and an operator T on X, we shall denote the resolvent set, spectrum, the set of all eigenvalues, and the spectral radius of T on X by $P(T,X)$, $\sigma(T,X), E(T,X)$ and $\rho(T,X)$ respectively. Then

$$E(T,X) \subset \sigma(T,X) = \mathbf{C} \backslash P(T,X) \tag{1.3}$$

and

$$\rho(T,X) = \sup\{|\lambda| : \ \lambda \in \sigma(T,X)\} = \lim_{n\to\infty} \|T^n\|^{1/n}. \tag{1.4}$$

Note that if $\phi \in L^p([-K,K])$ is the solution of (1.2), then $\phi^{(k)}$, the k-th derivative of ϕ, is the eigenfunction of the restricted operator $W_{c,\alpha}|_{L^p([-K,K])}$ with eigenvalue α^{-k} for any $K \geq K_0$. This follows by taking derivatives of both sides of (1.2). Set

$$\Sigma_0 := \{\alpha^{-k} : \ k = 0, 1, \ldots\}.$$

Then any $\lambda \in \Sigma_0$ is an eigenvalue of the operator $W_{c,\alpha}$ on the Banach space $L^p([-K,K])$, and any $\lambda \in \Sigma_0 \backslash \{1\}$ is an eigenvalue of the operator $W_{c,\alpha}$ on $L^p_0([-K,K])$. Moreover, the operator $W_{c,\alpha}$ is a compact operator on $L^p([-K,K])$ and on $L^p_0([-K,K])$ for any $K \geq K_0$[13]. Therefore, the following result about spectrum of the restricted operator $W_{c,\alpha}|_{L^p([-K,K])}$ and $W_{c,\alpha}|_{L^p_0([-K,K])}$ follows[13].

Theorem 1.1 *Let $\alpha > 1$, $1 \le p \le \infty$, c be a compactly supported function in $L^1(\mathbf{R})$ with $\int_{\mathbf{R}} c(x)dx = 1$, and let $K \ge K_0$. Then $W_{c,\alpha}$ is a compact operator on $L^p([-K,K])$ and $L^p_0([-K,K])$. Moreover*

$$E(W_{c,\alpha}, L^p([-K,K])) = \Sigma_0, \tag{1.5}$$

$$\sigma(W_{c,\alpha}, L^p([-K,K])) = \{0\} \cup \Sigma_0, \tag{1.6}$$

$$\rho(W_{c,\alpha}, L^p([-K,K])) = 1, \tag{1.7}$$

and

$$\rho(W_{c,\alpha}, L^p_0([-K,K])) = \alpha^{-1}. \tag{1.8}$$

Observe that the spectrum of $W_{c,\alpha}$ restricted to $L^p([-K,K])$ depends only on α. In particular, it is independent of K as long as $K \ge K_0$. Therefore, one would expect that the spectrum of the operator $W_{c,\alpha}$ on the entire space $L^p(\mathbf{R})$ to be the same as in (1.6). However, this is not the case, and it turns out that the spectrum of the operator $W_{c,\alpha}$ on $L^p(\mathbf{R})$ is the closed disc with radius $\alpha^{1-1/p}$, and that all the nonzero complex numbers with absolute value strictly less than one are eigenvalues of $W_{c,\alpha}$ (see Corollary 1.5 for detail). This big difference in the spectra of $W_{c,\alpha}$ on $L^p([-K,K])$ and $L^p(\mathbf{R})$ can be better understood if we consider $W_{c,\alpha}$ as an operator on the weighted spaces $L^p_w(\mathbf{R}), 1 \le p \le \infty$, which comprise all functions f with $fw \in L^p(\mathbf{R})$. Here and hereafter, a weight w means a positive measurable function on $\mathbf{R}$, and the norm $\|\cdot\|_{p,w}$ of a function f on $L^p_w(\mathbf{R})$ is the usual L^p norm of fw.

In order to study the spectrum of $W_{c,\alpha}$ on the space $L^p_w(\mathbf{R})$, it must at least be a bounded operator. This imposes the following restrictions on the weight w.

(i) There is a positive constant C_0 such that

$$C_0^{-1} w(x) \le w(y) \le C_0 w(x) \quad a.e. \text{ for all } |x-y| \le 1. \tag{1.9}$$

(ii) There is a positive constant C_1 such that

$$w(x) \le C_1 w(\alpha x) \qquad a.e. \text{ for all } x \in \mathbf{R}. \tag{1.10}$$

We shall assume throughout the paper that (1.9) is satisfied. If w satisfies both (1.9) and (1.10), then $W_{c,\alpha}$ is a bounded operator on $L^p_w(\mathbf{R})$ for any $1 \le p \le \infty$. We remark that if w satisfies (1.9) then (1.10) is a necessary and sufficient condition for $W_{c,\alpha}$ to be bounded on $L^p_w(\mathbf{R})$ for any $1 \le p \le \infty$. We state this as

Theorem 1.2 *Let $1 \le p \le \infty$, $\alpha > 1$, c be a compactly supported function in $L^1(\mathbf{R})$ with $\int_{\mathbf{R}} c(x)dx = 1$, and w be a weight function satisfying* (1.9). *Then $W_{c,\alpha}$ is bounded on $L^p_w(\mathbf{R})$ if and only if w satisfies* (1.10).

For any $K > 0$, let $L^p_w([-K, K])$ be the space of all $L^p_w(\mathbf{R})$ functions with support in the interval $[-K, K]$. Observe that for any weight w that satisfies (1.9), the norm $\|\cdot\|_{p,w}$ in $L^p_w([-K, K])$ is equivalent to $\|\cdot\|_p$ in $L^p([-K, K])$. Therefore, part of the results of Theorem 1.1 can be stated as follows.

Theorem 1.3 *Let α, p, c, K and $W_{c,\alpha}$ be as in Theorem 1.1, and suppose that w satisfies* (1.9). *Then $W_{c,\alpha}$ is a compact operator on $L^p_w([-K, K])$. Moreover*

$$E(W_{c,\alpha}, L^p_w([-K, K])) = \Sigma_0, \tag{1.11}$$

$$\sigma(W_{c,\alpha}, L^p_w([-K, K])) = \{0\} \cup \Sigma_0, \tag{1.12}$$

and

$$\rho(W_{c,\alpha}, L^p_w([-K, K])) = 1. \tag{1.13}$$

If

$$\lim_{|x|\to\infty} \frac{w(x)}{w(\alpha x)} = r > 0, \tag{1.14}$$

then w satisfies *(1.10)*, and we have

Theorem 1.4 *Let $1 \le p \le \infty, \alpha > 1$, c be a compactly supported function in $L^1(\mathbf{R})$ with $\int_{\mathbf{R}} c(x)dx = 1$, and w be a weight function that satisfies* (1.9) *and* (1.14) *for some $r > 0$. Then*

$$\begin{aligned} E(W_{c,\alpha}, L^p_w(\mathbf{R})) &\supset \{\lambda \in \mathbf{C} : 0 < |\lambda| < r\} \cup \Sigma_0, \\ \sigma(W_{c,\alpha}, L^p_w(\mathbf{R})) &= \{\lambda \in \mathbf{C} : |\lambda| \le r\alpha^{1-1/p}\} \cup \Sigma_0, \\ P(W_{c,\alpha}, L^p_w(\mathbf{R})) &= \{\lambda \in \mathbf{C} : |\lambda| > r\alpha^{1-1/p}\} \backslash \Sigma_0, \\ \rho(W_{c,\alpha}, L^p_w(\mathbf{R})) &= \max(1, r\alpha^{1-1/p}). \end{aligned}$$

Consider the weight $w_s(x) = (1 + |x|)^s$, where $s \in \mathbf{R}$. Then $\lim_{|x|\to\infty} w_s(x)/w_s(\alpha x) = \alpha^{-s}$ and $L^p(\mathbf{R}) = L^p_{w_0}(\mathbf{R})$. Therefore, by taking $w = w_s$ in Theorem 1.4, we have

Corollary 1.5 *Let p, α, c and $W_{c,\alpha}$ be as in Theorem 1.4, and let $w_s(x) = (1+|x|)^s$, $s \in \mathbf{R}$. Then*

$$\begin{aligned}
E(W_{c,\alpha}, L^p_{w_s}(\mathbf{R})) &\supset \{\lambda \in \mathbf{C} : 0 < |\lambda| < \alpha^{-s}\} \cup \Sigma_0, \\
\sigma(W_{c,\alpha}, L^p_{w_s}(\mathbf{R})) &= \{\lambda \in \mathbf{C} : |\lambda| \le \alpha^{-s+1-1/p}\} \cup \Sigma_0, \\
P(W_{c,\alpha}, L^p_{w_s}(\mathbf{R})) &= \{\lambda \in \mathbf{C} : |\lambda| > \alpha^{-s+1-1/p}\} \backslash \Sigma_0, \\
\rho(W_{c,\alpha}, L^p_{w_s}(\mathbf{R})) &= \max(1, \alpha^{-s+1-1/p}).
\end{aligned}$$

Next, we shall show that if (1.9) is satisfied then

$$\lim_{|x|\to\infty} \frac{w(x)}{w(\alpha x)} = 0 \tag{1.15}$$

is a necessary and sufficient condition for $W_{c,\alpha}$ to be a compact operator on $L^p_w(\mathbf{R})$. This characterization of $W_{c,\alpha}$ as a compact operator on $L^p_w(\mathbf{R})$ as well as the results on its spectrum (Theorem 1.6 below) put Theorem 1.1 in a proper perspective.

Theorem 1.6 *Let $1 \le p \le \infty$, $\alpha > 1$, and c be a compactly supported function in $L^1(\mathbf{R})$ with $\int_{\mathbf{R}} c(x)dx = 1$, and suppose that w is a weight function that satisfies* (1.9). *Then $W_{c,\alpha}$ is a compact operator on $L^p_w(\mathbf{R})$ if and only if w satisfies* (1.15). *Furthermore, if* (1.15) *holds, then*

$$E(W_{c,\alpha}, L^p_w(\mathbf{R})) = \Sigma_0 \tag{1.16}$$

and

$$\sigma(W_{c,\alpha}, L^p_w(\mathbf{R})) = \{0\} \cup \Sigma_0. \tag{1.17}$$

Now, take positive numbers λ and γ with $\gamma \le 1$. Since $\lim_{|x|\to\infty} e^{\lambda(1-\alpha^\gamma)|x|^\gamma} = 0$, by setting $w(x) = e^{\lambda|x|^\gamma}$ in Theorem 1.6, we obtain the following corollary.

Corollary 1.7 *Let $p, \alpha, c, W_{c,\alpha}$ be as in Theorem 1.6, and let $w(x) = e^{\lambda|x|^\gamma}$ for some $\lambda > 0$ and $0 < \gamma \le 1$. Then $W_{c,\alpha}$ is a compact operator on $L^p_w(\mathbf{R})$, and* (1.16) *and* (1.17) *hold.*

We remark that the spectral properties of $W_{c,\alpha}$ are reminiscent of those of the transfer operators and their adjoints, which are the subdivision operators[3,10,11,17,18,21]. However our results in Theorems 1.4 and 1.6 for the continuous case are more precise and complete than those of available in the literature. We thank Professor Zhou Ding-Xuan for pointing out the similarity of our results with those of the transfer operators, and for providing the related references.

This paper is organized as follows. Theorem 1.2 is proved in Section 2, which deals with the question of boundedness of the operator $W_{c,\alpha}$ on $L^p_w(\mathbf{R})$. Spectral properties of $W_{c,\alpha}$ on $L^p_w(\mathbf{R})$ are developed and proved in Section 3. Theorem 1.4 is derived from stronger results proved in that section. The last section is devoted to the characterization in terms of w for $W_{c,\alpha}$ to be a compact operator on $L^p_w(\mathbf{R})$. It contains a proof of Theorem 1.6.

2 Boundedness of $W_{c,\alpha}$ on $L^p_w(\mathbf{R})$

The requirement that $W_{c,\alpha}$ be a bounded linear operator on $L^p_w(\mathbf{R})$ entails constraints on the weight w as given by Theorem 1.2. To develop the proof of Theorem 1.2, we shall first establish a result, which is also essential in setting up the proof of Theorem 1.4.

Theorem 2.1 *Let $1 \le p \le \infty$, $\alpha > 1$, c be a compactly supported function in $L^1(\mathbf{R})$ with $\int_{\mathbf{R}} c(x)dx = 1$, and w be a weight function that satisfies* (1.9). *Then there exists a positive constant C independent of n and f such that*

$$\|W^n_{c,\alpha} f\|_{p,w} \le C\alpha^{n(1-1/p)} \|f\|_{p,w(\alpha^{-n}\cdot)}$$

for all $n \ge 1$ and $f \in L^p_{w(\alpha^{-n}\cdot)}(\mathbf{R})$.

Rewriting (1.1) as

$$W_{c,\alpha} f(x) = \alpha^2 \int_{\mathbf{R}} c(\alpha(x-y)) f(\alpha y) dy, \tag{2.1}$$

and repeated application of (2.1) n times gives

$$W^n_{c,\alpha} f(x) = \int_{\mathbf{R}} K_n(x-y)\alpha^n f(\alpha^n y)dy \quad \text{for all} \quad n \ge 1, \tag{2.2}$$

where

$$K_n(x) = (\alpha c(\alpha\cdot)) * \cdots * (\alpha^n c(\alpha^n\cdot)), \tag{2.3}$$

and $f * g$ denotes the convolution of two integrable functions f and g.

Lemma 2.2 *Let* α *and* c *be as in Theorem 2.1,* K_0 *be chosen so that* c *is supported in* $[-(\alpha-1)K_0,(\alpha-1)K_0]$*,* ϕ *be the solution of* (1.2) *normalized so that* $\int_{\mathbf{R}}\phi(x)dx=1$*, and* $K_n(x), n\ge 1$*, be as in* (2.3)*. Then*

$$\mathit{supp}(K_n(\cdot))\subset[-K_0,K_0] \quad \textit{for all} \;\; n\ge 1, \tag{2.4}$$

and

$$\lim_{n\to\infty}\|K_n-\phi\|_1=0. \tag{2.5}$$

Proof. Note that $\alpha^k c(\alpha^k\cdot)$ is supported in $[-(\alpha-1)\alpha^{-k}K_0,(\alpha-1)\alpha^{-k}K_0]$ for any $k\ge 0$. Therefore $K_n(x), n\ge 1$, are supported in $[-(\alpha-1)K_0\sum_{k=1}^n\alpha^{-k},(\alpha-1)K_0\sum_{k=1}^n\alpha^{-k}]\subset[-K_0,K_0]$. This proves (2.4).

To prove (2.5), note that by (1.2), (2.2) and (2.3), we have

$$K_n-\phi=W_{c,\alpha}^{n-1}(c-\phi). \tag{2.6}$$

Since $\int_{\mathbf{R}}(c(x)-\phi(x))dx=0$ and $\operatorname{supp}(c-\phi)\subset[-\alpha K_0,\alpha K_0]$, by (1.8), (2.6) and the definition of spectral radius, there exists a positive constant C independent of n such that

$$\|K_n-\phi\|_1=\|W_{c,\alpha}^{n-1}(c-\phi)\|_1\le C\left(\frac{1+\alpha^{-1}}{2}\right)^{n-1}\|c-\phi\|_1 \quad \text{for all} \quad n\ge 1.$$

This gives (2.5). ♠

Proof of Theorem 2.1. For $1\le p<\infty$, it follows from (1.9), (2.2) and Lemma 2.2 that for any $f\in L^p_{w(\alpha^{-n}\cdot)}(\mathbf{R})$,

$$\begin{aligned}\|W_{c,\alpha}^n f\|_{p,w}^p &= \int_{\mathbf{R}}\left|\int_{\mathbf{R}}K_n(x-y)\alpha^n f(\alpha^n y)dy\right|^p w(x)^p dx\\ &\le C_1\alpha^{np}\int_{\mathbf{R}}\left|\int_{\mathbf{R}}|K_n(x-y)||f(\alpha^n y)|w(y)dy\right|^p dx\\ &\le C_1\alpha^{np}\int_{\mathbf{R}}\left(\int_{\mathbf{R}}|K_n(x-y)||f(\alpha^n y)|^p|w(y)|^p dy\right)\\ &\quad\times\left(\int_{\mathbf{R}}|K_n(x-y)|dy\right)^{p-1}dx\\ &\le C_2\alpha^{n(p-1)}\|f\|_{p,w(\alpha^{-n}\cdot)}^p\ ,\end{aligned}$$

where C_1, C_2 are positive constants independent of f and n. Similarly for $p=\infty$, we have

$$\|W_{c,\alpha}^n f\|_{\infty,w}\le C\alpha^n\|f\|_{\infty,w(\alpha^{-n}\cdot)}, \tag{2.7}$$

where C is independent of f and n. ♠

Proof of Theorem 1.2. If (1.10) holds, the boundedness of $W_{c,\alpha}$ follows from Theorem 2.1. We shall prove the converse by contradiction. The norm of the operator $W_{c,\alpha}$ on $L^p_w(\mathbf{R})$ will be denoted by $\|W_{c,\alpha}\|_{L^p_w(\mathbf{R})}$. Let $N_0 = 2C_0^{\alpha+2K_0+4}\|W_{c,\alpha}\|_{L^p_w(\mathbf{R})}$, where C_0 is the constant in (1.9). Suppose on the contrary that there is a set E with positive measure such that

$$w(x) \geq N_0 w(\alpha x), \quad x \in E. \tag{2.8}$$

Let k be an integer such that $[k, k+1] \cap E$ has a positive measure. It then follows from (1.9) and (2.8) that

$$w(x) \geq C_0^{-\alpha-2} N_0 w(\alpha x) \quad x \in [k, k+1]. \tag{2.9}$$

Let ϕ be the solution of (1.2) normalized so that $\int_{\mathbf{R}} \phi(x)dx = 1$, and set $\phi_k = \phi(\cdot - \alpha k)$, $k \geq 1$. By (1.1), we have

$$W_{c,\alpha} g_t = W_{c,\alpha} g(\ \cdot\ - t/\alpha), \quad g \in L^p_w(\mathbf{R}), \tag{2.10}$$

where $g_t = g(\ \cdot\ - t)$, $t \in \mathbf{R}$. This together with (1.2) lead to

$$W_{c,\alpha}\phi_k = \phi(\cdot - k). \tag{2.11}$$

Now (2.11), (1.9) and (2.9) give

$$\begin{aligned} \|W_{c,\alpha}\phi_k\|_{p,w} &= \|\phi(\cdot - k)\|_{p,w} \geq C_0^{-K_0-1} w(k)\|\phi\|_p \\ &\geq 2C_0^{K_0+1}\|W_{c,\alpha}\|_{L^p_w(\mathbf{R})} w(\alpha k)\|\phi\|_p \geq 2\|W_{c,\alpha}\|_{L^p_w(\mathbf{R})}\|\phi_k\|_{p,w} \neq 0, \end{aligned}$$

which is a contradiction. ♠

3 Spectrum of $W_{c,\alpha}$ on $L^p_w(\mathbf{R})$

The main object of this section is to prove Theorem 1.4. In particular, we shall prove a slightly stronger result.

Theorem 3.1 *Let $1 \leq p \leq \infty$, $\alpha > 1$, c be a compactly supported function in $L^1(\mathbf{R})$ with $\int_{\mathbf{R}} c(x)dx = 1$, and w be a weight function that satisfies* (1.9) *and* (1.10). *If*

$$\liminf_{|x|\to\infty} \frac{w(x)}{w(\alpha x)} = r_1 > 0, \tag{3.1}$$

then

$$E(W_{c,\alpha}, L^p_w(\mathbf{R})) \supset \{\lambda \in \mathbf{C} : 0 < |\lambda| < r_1\} \cup \Sigma_0 \tag{3.2}$$

and

$$P(W_{c,\alpha}, L^p_w(\mathbf{R})) \subset \{\lambda \in \mathbf{C} : |\lambda| > r_1\alpha^{1-1/p}\}\backslash\Sigma_0. \tag{3.3}$$

Theorem 3.2 *Let* $1 \le p \le \infty$, $\alpha > 1$, c *be a compactly supported function in* $L^1(\mathbf{R})$ *with* $\int_{\mathbf{R}} c(x)dx = 1$, *and* w *be a weight function that satisfies* (1.9) *and* (1.10). *If*

$$\limsup_{|x|\to\infty} \frac{w(x)}{w(\alpha x)} = r_2 > 0, \tag{3.4}$$

then

$$P(W_{c,\alpha}, L^p_w(\mathbf{R})) \supset \{\lambda \in \mathbf{C} : |\lambda| > r_2\alpha^{1-1/p}\}\backslash\Sigma_0. \tag{3.5}$$

It is clear that Theorem 1.4 follows directly from (1.3), (1.4) and Theorems 3.1 and 3.2. To set up the proofs of Theorems 3.1 and 3.2, we need some elementary properties on the support of $W^n_{c,\alpha}f$, the asymptotic behavior of the weights that satisfy (3.1) or (3.4), and the relationship between the norms in $L^p(\mathbf{R})$ and $L^p_w(\mathbf{R})$ for compactly supported functions. These properties follow directly from (1.1), (3.1) and (3.4), and the definition of the weighted space $L^p_w(\mathbf{R})$ respectively. We shall state these results but omit the details of their proofs.

Lemma 3.3 *Let* $\alpha > 1$, c *be an integrable function with* $\int_{\mathbf{R}} c(x) = 1$, K_0 *be the smallest positive number such that supp*$(c) \subset [-(\alpha-1)K_0, (\alpha-1)K_0]$, *and let* $W_{c,\alpha}$ *be defined as in* (1.1). *Then*

$$\begin{aligned} supp(W^n_{c,\alpha}f) &\subset [\alpha^{-n}a - (1-\alpha^{-n})K_0, \alpha^{-n}b + (1-\alpha^{-n})K_0] \\ &\subset [\alpha^{-n}a - K_0, \alpha^{-n}b + K_0] \end{aligned}$$

for any function f *with support in* $[a, b]$ *and for all* $n \ge 1$.

Lemma 3.4 *Let* $w(x)$ *be a weight function that satisfies* (1.9) *and* (1.10).

(i) If w *satisfies* (3.1), *then for any* $0 < \delta < 1/2$ *there exists a positive constant* C_1 *independent of* x *such that*

$$w(x) \le C_1(1+|x|)^{-\ln r_1/\ln\alpha+\delta} \quad \text{for all } x \in \mathbf{R}. \tag{3.6}$$

(ii) If w satisfies (3.4), then for any $0 < \delta < 1/2$, there exist positive constants C_2 and C_3 independent of x and n such that

$$w(\alpha^n x) \geq C_2 r_2^{-n}(1+\delta)^{-n} w(x), \quad |x| \geq 1, \tag{3.7}$$
$$w(\alpha^n x) \geq C_3 \min(1, r_2^{-n}(1+\delta)^{-n}) w(x), \quad |x| \leq 1, \tag{3.8}$$

for all $n \geq 1$.

Lemma 3.5 *Let $w(x)$ be a weight function that satisfies (1.9). Then*

$$(\min_{x\in[a,b]} w(x))\|f\|_p \leq \|f\|_{p,w} \leq (\max_{x\in[a,b]} w(x))\|f\|_p \tag{3.9}$$

for any function $f \in L^p_w([a,b])$, $1 \leq p \leq \infty$.

Proof of Theorem 3.1. To prove (3.2), we note that $L^p_w([-K,K]) \subset L^p_w(\mathbf{R})$, for any $1 \leq p \leq \infty$ and $K > 0$. Then every eigenvalue of the operator $W_{c,\alpha}$ restricted to $L^p_w([-K,K])$ is an eigenvalue of $W_{c,\alpha}$ restricted to $L^p_w(\mathbf{R})$. This together with Theorem 1.3 gives

$$\Sigma_0 = E(W_{c,\alpha}, L^p_w([-K_0,K_0])) \subset E(W_{c,\alpha}, L^p_w(\mathbf{R})). \tag{3.10}$$

Let λ be any complex number that satisfies $0 < |\lambda| < r_1$ and $\lambda \notin \Sigma_0$. Then by (3.10), the proof of (3.2) reduces to proving that $\lambda \in E(W_{c,\alpha}, L^p_w(\mathbf{R}))$. By (1.3) and (1.12), $\lambda \in P(W_{c,\alpha}, L^p_w([-K_0-1, K_0+1]))$. Thus, there exists a compactly supported function ψ_λ such that

$$(W_{c,\alpha} - \lambda I)\psi_\lambda = \phi(\,\cdot\, - 1) \tag{3.11}$$

and

$$\psi_\lambda \in L^p_w([-K_0-1, K_0+1]), \tag{3.12}$$

where ϕ is the solution of (1.2) normalized so that $\int_{\mathbf{R}} \phi(x)dx = 1$.

Set

$$\phi_\lambda = -\psi_\lambda + \sum_{n=1}^{\infty} \lambda^{n-1}\phi(\,\cdot\, - \alpha^n). \tag{3.13}$$

Then

$$\phi_\lambda \not\equiv 0, \tag{3.14}$$

because $\phi(\,\cdot\, - \alpha^n)$ are supported in the sets $[-K_0, K_0] + \alpha^n$, $n = 1, 2, \ldots$, which are mutually disjoint for sufficiently large n. Let $\delta_0 > 0$ be chosen that $\alpha^{2\delta_0}|\lambda| = r_1$. Using (3.6), (3.12) and (3.13) with δ_0 chosen as δ leads to

$$\begin{aligned}\|\phi_\lambda\|_{p,w} &\le \|\psi_\lambda\|_{p,w} + \sum_{n=1}^{\infty} |\lambda|^{n-1}\|\phi(\cdot - \alpha^n)\|_{p,w} \\ &\le C + C\sum_{n=1}^{\infty} |\lambda|^n r_1^{-n}\alpha^{\delta_0 n}\|\phi\|_p \\ &= C\left(1 + \frac{\|\phi\|_p}{\alpha^{\delta_0} - 1}\right) < \infty,\end{aligned}$$

where C is a positive constant independent of n. This shows that

$$\phi_\lambda \in L^p_w(\mathbf{R}). \tag{3.15}$$

Applying $W_{c,\alpha} - \lambda I$ to (3.13), and using (1.2), (2.10) and (3.11) lead to

$$(W_{c,\alpha} - \lambda I)\phi_\lambda = -(W_{c,\alpha} - \lambda I)\psi_\lambda + \phi(\,\cdot\, - 1) = 0. \tag{3.16}$$

It follows from (3.14), (3.15) and (3.16) that λ is an eigenvalue of $W_{c,\alpha}$ restricted to $L^p_w(\mathbf{R})$. This completes the proof of (3.2).

To prove (3.3), recall that $\sigma(W_{c,\alpha}, L^p_w(\mathbf{R}))$ is closed and contains $\{0\}\cup\Sigma_0$. Then, by (1.3) and (3.2), it suffices to prove that for any $\lambda \notin \Sigma_0$ with $0 < |\lambda| < r_1\alpha^{1-1/p}$, there exists $f_n \in L^p_w(\mathbf{R})$, $n \ge 1$, such that $f_n \ne 0$ and

$$\lim_{n\to\infty} \frac{\|(W_{c,\alpha} - \lambda I)f_n\|_{p,w}}{\|f_n\|_{p,w}} = 0. \tag{3.17}$$

Let ϕ be the solution of (1.2) normalized so that $\int_{\mathbf{R}} \phi(x)dx = 1$, and let m_0 be the minimal positive integer so that $\alpha^{m_0}(1 - \alpha^{-1}) \ge 4K_0$ and

$$\sup_{|\delta|\le 2K_0\alpha^{-m_0}} \|\phi(\cdot - \delta) - \phi\|_p \le \|\phi\|_p/2. \tag{3.18}$$

The existence of such an integer m_0 follows from the fact that ϕ is a compactly supported continuous function. For any integer $n \ge 2m_0 + 1$, let

$$g_n = \sum_{0\le k<\alpha^{n-2m_0-1}} \phi(\cdot - 2kK_0 - \alpha^n).$$

Then g_n is supported in $[\alpha^n - K_0, (1 + 2K_0\alpha^{-2m_0-1})\alpha^n + K_0]$, and hence

$$\|g_n\|_{p,w} \le C_1 r_1^{-n}\alpha^{\delta_0 n}\|g_n\|_p \tag{3.19}$$

by (3.6), where C_1 is a positive constant independent of n, and δ_0 satisfies $|\lambda|\alpha^{2\delta_0} = r_1\alpha^{1-1/p}$. Note that $\phi(\cdot - 2kK_0 - \alpha^n)$, $0 \le k < \alpha^{n-2m_0-1}$, have mutually disjoint support. Therefore,

$$\|g_n\|_p \le C\alpha^{n/p}\|\phi\|_p \tag{3.20}$$

for some positive constant C independent of n. Combining (3.19) and (3.20) leads to

$$\|g_n\|_{p,w} \le Cr_1^{-n}\alpha^{\delta_0 n+n/p}. \tag{3.21}$$

Define $f_n = \phi$ for $1 \le n \le 2m_0$, and

$$f_n = -\sum_{i=0}^{n} \lambda^{-i-1}W_{c,\alpha}^i g_n + \lambda^{-n-1}(W_{c,\alpha} - \lambda I)^{-1}W_{c,\alpha}^{n+1}g_n \tag{3.22}$$

for $n \ge 2m_0 + 1$. Note that

$$W_{c,\alpha}^i g_n = \sum_{0\le k<\alpha^{n-2m_0-1}} \phi(\cdot - 2kK_0\alpha^{-i} - \alpha^{n-i}) \tag{3.23}$$

by (1.2), and

$$\mathrm{supp}(W_{c,\alpha}^i g_n) \subset [\alpha^{n-i} - K_0, (1 + 2K_0\alpha^{-2m_0-1})\alpha^{n-i} + K_0] \tag{3.24}$$

for all $0 \le i \le n + 1$. Therefore, the functions f_n, $n \ge 1$, in (3.22) are well defined because of (3.24), Theorem 1.3 and the assumption that $\lambda \notin \Sigma_0$. By (3.24), the term $\lambda^{-n+m_0-1}W_{c,\alpha}^{n-m_0}g_n$ in the sum on the right of (3.22) has support that is disjoint from the supports of all the other terms. Therefore, $W_{c,\alpha}^{n-m_0}g_n$ and $f_n + \lambda^{-n+m_0-1}W_{c,\alpha}^{n-m_0}g_n$ have disjoint supports. These facts, together with (3.18), (3.22), (1.9), (3.23) and (3.24), lead to

$$\begin{aligned}
\|f_n\|_{p,w} &\ge C_1|\lambda|^{-n+m_0-1}\|W_{c,\alpha}^{n-m_0}g_n\|_{p,w} \ge C_2|\lambda|^{-n+m_0-1}\|W_{c,\alpha}^{n-m_0}g_n\|_p \\
&\ge C_3|\lambda|^{-n}\Big(\Big\| \sum_{0\le k<\alpha^{n-2m_0-1}} \phi(\cdot - \alpha^{m_0})\Big\|_p \\
&\quad - \sum_{0\le k<\alpha^{n-2m_0-1}} \|\phi(\cdot - 2K_0k\alpha^{-n+m_0} - \alpha^{m_0}) - \phi(\cdot - \alpha^{m_0})\|_p\Big) \\
&\ge C_4|\lambda|^{-n}\alpha^n\|\phi\|_p \quad \text{for all } \ n \ge 2m_0 + 1,
\end{aligned} \tag{3.25}$$

where C_i, $1 \le i \le 4$, are positive constants independent of n.

Applying $W_{c,\alpha} - \lambda I$ to both sides of (3.22) gives

$$(W_{c,\alpha} - \lambda I)f_n = g_n \quad \text{for all } \ n \ge 2m_0 + 1. \tag{3.26}$$

Combining (3.21), (3.25) and (3.26), and using the assumption that $|\lambda|\alpha^{2\delta_0} = r_1\alpha^{1-1/p}$, we obtain

$$\limsup_{n\to\infty} \frac{\|(W_{c,\alpha}-\lambda I)f_n\|_{p,w}}{\|f_n\|_{p,w}} \le C \limsup_{n\to\infty} (|\lambda| r_1^{-1}\alpha^{-1+1/p+\delta_0})^n = 0.$$

This proves (3.17) and hence Theorem 3.1. ♠

Our proof of Theorem 3.6 requires two lemmas, one on the spectral radius of the operator $W_{c,\alpha}$ on $L^p_w(\mathbf{R})$ (Lemma 3.6) and the other on an estimate of the set of eigenvalues, $E(W_{c,\alpha}, L^p_w(\mathbf{R}))$ (Lemma 3.7).

Lemma 3.6 *Let p, α, w, c and $W_{c,\alpha}$ be as in Theorem 3.2. For any $\delta, \sigma \in (0,1)$, there exists a positive constant C independent of f and n for which*

$$\|W^n_{c,\alpha} f\|_{p,w} \le C\alpha^{n-n/p} \max(1, r_2^n(1+\delta)^n)\|f\|_{p,w} \tag{3.27}$$

for all $n \ge 1$ and $f \in L^p_w(\mathbf{R})$, and

$$\|W^n_{c,\alpha} f\|_{p,w} \le C(1+\delta)^n \alpha^{n-n/p} r_2^n \|f\|_{p,w} \tag{3.28}$$

for all $n \ge 1$ and $f \in L^p_w(\mathbf{R})$ with support in $\mathbf{R}\backslash[-\sigma\alpha^n, \sigma\alpha^n]$.

Proof. For any $0 < \delta,\ \sigma < 1$, by (3.7) and (3.8), there exists a positive constant C independent of n such that

$$\|f\|_{p,w(\alpha^{-n}\cdot)} \le Cr_2^n(1+\delta)^n\|f\|_{p,w} \quad \text{for all } f \in L^p_w(\mathbf{R}\setminus[-\sigma\alpha^n, \sigma\alpha^n]), \tag{3.29}$$

and

$$\|f\|_{p,w(\alpha^{-n}\cdot)} \le C\max(1, r_2^n(1+\delta)^n)\|f\|_{p,w} \quad \text{for all } f \in L^p_w(\mathbf{R}). \tag{3.30}$$

Thus (3.27) and (3.28) follow from (3.29), (3.30) and Theorem 2.1. ♠

Lemma 3.7 *Let p, α, w, c and $W_{c,\alpha}$ be as in Theorem 3.2. Then*

$$E(W_{c,\alpha}, L^p_w(\mathbf{R})) \subset \{\lambda \in \mathbf{C} : |\lambda| \le r_2\alpha^{1-1/p}\} \cup \Sigma_0. \tag{3.31}$$

Proof. Suppose on the contrary that there exists a complex number $\lambda \in E(W_{c,\alpha}, L^p_w(\mathbf{R}))$ with $|\lambda| > r_2\alpha^{1-1/p}$, $\lambda \notin \Sigma_0$. Then

$$W_{c,\alpha} f = \lambda f \tag{3.32}$$

for some nonzero $f \in L^p_w(\mathbf{R})$. Since $\lambda \notin \Sigma_0$, f does not have compact support by Theorem 1.3. Hence there exists an integer n_0 such that $|n_0| \ge 1/(\alpha - 1) + 2K_0/(\alpha-1)^2$ and $f \not\equiv 0$ on $[n_0, n_0+1]$. Define $\Omega_0 = [n_0, n_0+1]$ and

$\Omega_k = [\alpha^k n_0 - K_0 \sum_{j=0}^{k-1} \alpha^j, \alpha^k(n_0+1) + K_0 \sum_{j=0}^{k-1} \alpha^j]$ for $k \ge 1$, and set $f_k = f\chi_{\Omega_k}$. Then Ω_k, $k \ge 1$, are mutually disjoint. By Lemma 3.3, $W_{c,\alpha} g$ is supported in $\mathbf{R}\backslash\Omega_{k-1}$ for any function g with support in $\mathbf{R}\backslash\Omega_k$, $k \ge 1$. Therefore from (3.32),

$$\begin{aligned} W_{c,\alpha}^k f_k &= \lambda^k f - W_{c,\alpha}^k (f - f_k) \\ &= \lambda^k f_0 + (f - f_0) - W_{c,\alpha}^k (f - f_k) = \lambda^k f_0 + \tilde{f}_k, \end{aligned} \tag{3.33}$$

where $\tilde{f}_k$, $k \ge 1$, are supported in $\mathbf{R}\backslash\Omega_0$. This implies that

$$\|W_{c,\alpha}^k f_k\|_{p,w} \ge |\lambda|^k \|f_0\|_{p,w}. \tag{3.34}$$

Since f_k is supported in Ω_k for any $k \ge 1$, by Lemma 3.6, there exists a positive constant C independent of $k \ge 1$ such that

$$\|W_{c,\alpha}^k f_k\|_{p,w} \le C(1+\delta_0)^k r_2^k \alpha^{k(1-1/p)} \|f_k\|_{p,w}, \tag{3.35}$$

where δ_0 is a positive constant so chosen that $|\lambda| = (1+\delta_0)^2 r_2 \alpha^{1-1/p}$. Combining (3.34) and (3.35), we obtain

$$\|f_k\|_{p,w} \ge C(1+\delta_0)^k \|f_0\|_{p,w}, \tag{3.36}$$

for sufficiently large k, where C is a positive constant independent of k and f. Since $\|f\|_{p,w} \ge \|f_k\|_{p,w}$ for all $k \ge 1$, (3.36) implies that $\|f\|_{p,w} = \infty$, which is a contradiction. ♠

Proof of Theorem 3.2. Let λ be a complex number that satisfies $|\lambda| > r_2 \alpha^{1-1/p}$ and $\lambda \notin \Sigma_0$. By Lemma 3.7, $\lambda \notin E(W_{c,\alpha}, L_w^p(\mathbf{R}))$. Therefore, $(W_{c,\alpha} - \lambda I)$ is injective on $L_w^p(\mathbf{R})$. Then it remains to show that for any $f \in L_w^p(\mathbf{R})$, we can find $g \in L_w^p(\mathbf{R})$ such that

$$\|g\|_{p,w} \le C\|f\|_{p,w} \tag{3.37}$$

and

$$(W_{c,\alpha} - \lambda I) g = f, \tag{3.38}$$

where C is a positive constant independent of f.

Write

$$f = f\chi_{[-\alpha,\alpha]} + \sum_{j=1}^{\infty} \Big(f\chi_{(\alpha^j, \alpha^{j+1}]} + f\chi_{[-\alpha^{j+1}, -\alpha^j)} \Big) = \sum_{j \in \mathbf{Z}} f_j,$$

where $f_0 = f\chi_{[-\alpha,\alpha]}$, $f_j = f\chi_{(\alpha^j,\alpha^{j+1}]}$ and $f_{-j} = f\chi_{[-\alpha^{j+1},-\alpha^j)}$ for $j \ge 1$. Then the support of f_j, $j \in \mathbf{Z}$, are mutually disjoint and

$$\|f\|_{p,w} = \|(\|f_j\|_{p,w})_{j\in\mathbf{Z}}\|_{\ell^p} \tag{3.39}$$

by the definition of the norm in $L^p_w(\mathbf{R})$. Here and hereafter, for any countable index set Λ and $1 \le p \le \infty$, we let

$$\ell^p(\Lambda) := \{D = (d_j)_{j\in\Lambda} : d_j \in \mathbf{C}\},$$

and define the norm on $\ell^p(\Lambda)$ by

$$\|D\|_{\ell^p} := \begin{cases} (\sum_{j\in\Lambda} |d_j|^p)^{1/p} & \text{if } 1 \le p < \infty \\ \sup_{j\in\Lambda} |d_j| & \text{if } p = \infty. \end{cases}$$

Note that $W^j_{c,\alpha}(f_j + f_{-j})$ is supported in $[-\alpha - K_0, \alpha + K_0]$ by Lemma 3.3. Therefore, by Theorem 1.3 and the assumption that $\lambda \notin \Sigma_0 \cup \{0\}$, there exist functions $\psi_j \in L^p_w([-\alpha - K_0, \alpha + K_0])$, $j \ge 0$, such that

$$\begin{cases} (W_{c,\alpha} - \lambda I)\psi_0 = f_0, \\ \|\psi_0\|_{p,w} \le C\|f_0\|_{p,w}, \end{cases} \tag{3.40}$$

and

$$\begin{cases} (W_{c,\alpha} - \lambda I)\psi_j = W^j_{c,\alpha}(f_j + f_{-j}), \\ \|\psi_j\|_{p,w} \le C\|W^j_{c,\alpha}(f_j + f_{-j})\|_{p,w}, \end{cases} \tag{3.41}$$

where $j \ge 1$, and C is a positive generic constant, which is independent of j and f.

Let δ_1 be a positive constant chosen so that

$$|\lambda| = (1 + \delta_1)^2 r_2 \alpha^{1-1/p}. \tag{3.42}$$

Since $f_j + f_{-j}$ is supported in $\mathbf{R}\backslash[-\alpha^j, \alpha^j]$, by (3.41) and Lemma 3.6, we have

$$\|\psi_j\|_{p,w} \le C(1 + \delta_1)^j \alpha^{j-j/p} r_2^j \|f_j + f_{-j}\|_{p,w} \tag{3.43}$$

for $j \ge 1$, where C is independent of j and $f \in L^p_w(\mathbf{R})$. Therefore, it follows from (3.40), (3.42) and (3.43) that

$$\begin{aligned} \Big\| \sum_{j=0}^{\infty} \lambda^{-j}\psi_j \Big\|_{p,w} &\le \sum_{j=0}^{\infty} |\lambda|^{-j} \|\psi_j\|_{p,w} \\ &\le C\|f_0\|_{p,w} + C\sum_{j=1}^{\infty} (1 + \delta_1)^{-j} (\|f_j\|_{p,w} + \|f_{-j}\|_{p,w}). \end{aligned} \tag{3.44}$$

Combining (3.39) and (3.44) gives

$$\begin{aligned} \Big\| \sum_{j=0}^{\infty} \lambda^{-j}\psi_j \Big\|_{p,w} &\le C\|f\|_{p,w} + C\|f\|_{p,w} \sum_{j=1}^{\infty} (1 + \delta_1)^{-j} \\ &= C(1 + \delta_1^{-1})\|f\|_{p,w}. \end{aligned} \tag{3.45}$$

We now define

$$\phi_j = \sum_{i=1}^{\infty} W_{c,\alpha}^j (f_{i+j} + f_{-i-j}), \quad j \geq 0. \tag{3.46}$$

By Lemma 3.3, $W_{c,\alpha}^j(f_{i+j} + f_{-i-j})$ is supported in $[-\alpha^{i+1} - K_0, -\alpha^i + K_0] \cup [\alpha^i - K_0, \alpha^{i+1} + K_0]$, which have finite overlaps for any given j. This together with (3.39) and Lemma 3.6 leads to

$$\begin{aligned}\|\phi_j\|_{p,w} &\leq C_1 \|(\|W_{c,\alpha}^j(f_{i+j} + f_{-i-j})\|_{p,w})_{i\geq 1}\|_{\ell^p} \\ &\leq C_2 r_2^{-j} \alpha^{(1-1/p)j} (1+\delta_1)^j \|(\|f_{i+j}\|_{p,w} + \|f_{-i-j}\|_{p,w})_{i\geq 1}\|_{\ell^p} \\ &\leq C_3 r_2^{-j} \alpha^{(1-1/p)j} (1+\delta_1)^j \|f\|_{p,w} \quad \text{for all} \quad j \geq 0, \end{aligned} \tag{3.47}$$

where C_1, C_2 and C_3 are positive constants independent of j and f. Combining (3.42) and (3.47), we obtain

$$\begin{aligned} \Big\| \sum_{j=0}^{\infty} \lambda^{-j-1} \phi_j \Big\|_{p,w} &\leq \sum_{j=0}^{\infty} |\lambda|^{-j-1} \|\phi_j\|_{p,w} \\ \leq C_3 \sum_{j=0}^{\infty} (1+\delta_1)^{-j} \|f\|_{p,w} &= C_3(1+\delta^{-1}) \|f\|_{p,w}. \end{aligned} \tag{3.48}$$

By (3.45), (3.48) and (3.49), the function

$$g = \sum_{j=0}^{\infty} \lambda^{-j} \psi_j - \sum_{j=0}^{\infty} \lambda^{-j-1} \phi_j. \tag{3.49}$$

belongs to $L_w^p(\mathbf{R})$. Furthermore

$$\|g\|_{p,w} \leq C \|f\|_{p,w}$$

for some positive constant C independent of $f \in L_w^p(\mathbf{R})$. This proves (3.37).

Applying $W_{c,\alpha} - \lambda I$ to (3.49) and using (3.40), (3.41) and (3.46) give

$$\begin{aligned} &(W_{c,\alpha} - \lambda I) g \\ &= f_0 + \sum_{j=1}^{\infty} \lambda^{-j} W_{c,\alpha}^j (f_j + f_{-j}) - \sum_{j=0}^{\infty} \sum_{i=1}^{\infty} \lambda^{-j-1} W_{c,\alpha}^{j+1} (f_{i+j} + f_{-i-j}) \\ &\quad + \sum_{j=0}^{\infty} \sum_{i=1}^{\infty} \lambda^{-j} W_{c,\alpha}^j (f_{i+j} + f_{-i-j}) \\ &= f_0 - \sum_{j=1}^{\infty} \sum_{i=1}^{\infty} \lambda^{-j} W_{c,\alpha}^j (f_{i+j} + f_{-i-j}) + \sum_{j=0}^{\infty} \sum_{i=1}^{\infty} \lambda^{-j} W_{c,\alpha}^j (f_{i+j} + f_{-i-j}) \end{aligned}$$

$$= f_0 + \sum_{i=1}^{\infty}(f_i + f_{-i}) = f.$$

This proves (3.38), and hence completes the proof of Theorem 3.2. ♠

4 Characterization of Compactness of $W_{c,\alpha}$ on $L^p_w(\mathbf{R})$

It is known that $W_{c,\alpha}$ restricted to the invariant subspace $L^p([-K,K])$ is a compact operator for all sufficiently large K. One would expect that it is also a compact operator if restricted to a subspace of $L^p(\mathbf{R})$ comprising functions with fast decay. This result is contained in Theorem 1.6. A proof is given below.

Proof of Theorem 1.6. We first prove the compactness of the operator $W_{c,\alpha}$ on $L^p_w(\mathbf{R})$ under the assumption that w satisfies (1.15). Let f_n, $n \ge 1$, be any bounded sequence in $L^p_w(\mathbf{R})$. Since $L^p_w(\mathbf{R})$ is a Banach space, it suffices to prove the existence of a subsequence g_n, $n \ge 1$ of f_n, $n \ge 1$, such that $W_{c,\alpha} g_n$ is a Cauchy sequence in $L^p_w(\mathbf{R})$.

By the assumption on f_n, $n \ge 1$, the set $\{f_n \chi_{[-\alpha^k K_0, \alpha^k K_0]} : n \ge 1\}$ is a bounded set in $L^p_w([-\alpha^k K_0, \alpha^k K_0])$ for $k \ge 1$. Setting $f_{n,0} := f_n$ for all $n \ge 1$, an inductive argument using Theorem 1.3 shows that there exist sequences $f_{n,k}$, $n \ge 1$, for $k \ge 0$, such that $f_{n,k}$, $n \ge 1$, is a subsequence of $f_{n,k-1}$, $n \ge 1$, for any $k \ge 1$, and $W_{c,\alpha}(f_{n,k}\chi_{[-\alpha^k K_0, \alpha^k K_0]})$, $n \ge 1$, is a Cauchy sequence in $L^p_w([-\alpha^k K_0, \alpha^k K_0])$. Then $g_n := f_{n,n}$, $n \ge 1$, is a subsequence of f_n, $n \ge 1$, and also g_n, $n \ge k$, is a subsequence of the sequences $f_{n,k}$, $n \ge 1$, for any $k \ge 1$.

We now prove that $W_{c,\alpha} g_n$, $n \ge 1$, is a Cauchy sequence in $L^p_w(\mathbf{R})$. Without loss of generality, we assume that

$$\|f_n\|_{p,w} \le 1 \quad \text{for all} \quad n \ge 1. \tag{4.1}$$

For any positive integers n and k, let $g_{n,k} = g_n \chi_{[-\alpha^k K_0, \alpha^k K_0]}$. Then by (4.1), we have

$$\|g_n\|_{p,w} + \|g_{n,k}\|_{p,w} \le 2 \qquad \text{for all} \quad n \ge 1 \quad k \ge 1.$$

This, together with Theorem 2.1, lead to

$$\|W_{c,\alpha} g_n\|_{p,w(\alpha\cdot)} + \|W_{c,\alpha} g_{n,k}\|_{p,w(\alpha\cdot)} \le C_0, \tag{4.2}$$

where C_0 is a positive constant independent of positive integers n and k. Since $g_n - g_{n,k}$ is supported in $\mathbf{R}\backslash[-\alpha^k K_0, \alpha^k K_0]$, $W_{c,\alpha}(g_n - g_{n,k})$ is supported in

$\mathbf{R}\backslash[-\alpha^{k-2}K_0, \alpha^{k-2}K_0]$. Hence for any $n, k \ge 1$, it follows from that (4.2) that

$$\begin{aligned} \|W_{c,\alpha}(g_n - g_{n,k})\|_{p,w} &\le \left\{ \max_{|x|>\alpha^{k-2}K_0} \frac{w(x)}{w(\alpha x)} \right\} \|W_{c,\alpha}(g_n - g_{n,k})\|_{p,w(\alpha\cdot)} \\ &\le C_0 \left\{ \max_{|x|>\alpha^{k-2}K_0} \frac{w(x)}{w(\alpha x)} \right\}, \end{aligned} \tag{4.3}$$

where C_0 is the positive constant in (4.2). By (1.15), for any $\epsilon > 0$ there exists a positive integer k_ϵ so that

$$\max_{|x|>\alpha^{k-2}K_0} \frac{w(x)}{w(\alpha x)} \le C_0^{-1}\epsilon \quad \text{for all } \ k \ge k_\epsilon. \tag{4.4}$$

Therefore, (4.3) and (4.4) give

$$\|W_{c,\alpha}(g_n - g_{n,k})\|_{p,w} \le \epsilon \quad \text{for all } \ k \ge k_\epsilon \quad \textit{and} \quad n \ge 1. \tag{4.5}$$

Recall that $W_{c,\alpha}g_{n,k_\epsilon}$, $n \ge 1$, is a Cauchy sequence in $L^p_w([-\alpha^{k_\epsilon}K_0, \alpha^{k_\epsilon}K_0])$, and $W_{c,\alpha}g_{n,k_\epsilon}$, $n \ge 1$, are supported in $[-\alpha^{k_\epsilon}K_0, \alpha^{k_\epsilon}K_0]$. Then $W_{c,\alpha}g_{n,k_\epsilon}$, $n \ge 1$, is a Cauchy sequence in $L^p_w(\mathbf{R})$. Therefore there exists an integer n_ϵ such that for all $n, m \ge n_\epsilon$,

$$\|W_{c,\alpha}g_{n,k_\epsilon} - W_{c,\alpha}g_{m,k_\epsilon}\|_{p,w} \le \epsilon. \tag{4.6}$$

Combining (4.5) and (4.6), we obtain

$$\|W_{c,\alpha}g_n - W_{c,\alpha}g_m\|_{p,w} \le 3\epsilon \ \text{ for all } \ m, n \ge n_\epsilon.$$

This proves that the sequence $W_{c,\alpha}g_n$, $n \ge 1$, is a convergent sequence in $L^p_w(\mathbf{R})$, and hence $W_{c,\alpha}$ is a compact operator in $L^p_w(\mathbf{R})$.

We now prove that if $W_{c,\alpha}$ is a compact operator on $L^p_w(\mathbf{R})$, then w satisfies (1.15). If (1.15) does not hold, then by (1.9) there exists an $\epsilon_0 > 0$ independent of n such that $\lim_{n\to\infty} x_n = \infty$, $[x_n - K_0, x_n + K_0]$ are mutually disjoint, and $w(x) \ge \epsilon_0 w(\alpha x)$ for almost all $x \in [x_n - K_0, x_n + K_0]$. Define $\phi_n := \phi(\cdot - \alpha x_n)/\|\phi(\cdot - \alpha x_n)\|_{p,w}$, where ϕ is the compactly supported eigenfunction of $W_{c,\alpha}$ with eigenvalue 1. Then ϕ_n, $n \ge 1$, is a bounded sequence in $L^p_w(\mathbf{R})$, and

$$W_{c,\alpha}\phi_n = \phi(\cdot - x_n)/\|\phi(\cdot - \alpha x_n)\|_{p,w}$$

by (1.2) and (2.10). Therefore $W_{c,\alpha}\phi_n$ converges to zero pointwise since it is supported in $[x_n - K_0, x_n + K_0]$ and $\lim_{n\to\infty} x_n = \infty$.

On the other hand,

$$\|W_{c,\alpha}\phi_n\|_{p,w} = \|\phi(\cdot - x_n)\|_{p,w}/\|\phi(\cdot - \alpha x_n)\|_{p,w} \ge Cw(x_n)/w(\alpha x_n) \ge C\epsilon_0,$$

for some constant C independent of n. Therefore there is no convergent subsequence of $\{W_{c,\alpha}\phi_n\}_{n\geq 1}$ in $L^p_w(\mathbf{R})$, which contradicts the compactness of the operator $W_{c,\alpha}$ on $L^p_w(\mathbf{R})$.

Finally, we prove (1.16) and (1.17). Note that (1.17) follows from (1.16) and the compactness of the operator $W_{c,\alpha}$ on $L^p_w(\mathbf{R})$. Hence it suffices to prove (1.16). Since $L^p_w([-K,K]) \subset L^p_w(\mathbf{R})$, every eigenvalue of $W_{c,\alpha}$ with eigenfunctions in $L^p_w([-K,K])$ is an eigenvalue of $W_{c,\alpha}$ restricted to $L^p_w(\mathbf{R})$. Therefore,

$$\Sigma_0 = E(W_{c,\alpha}, L^p_w([-K,K])) \subset E(W_{c,\alpha}, L^p_w(\mathbf{R})). \tag{4.7}$$

For any nonnegative number s and weight w that satisfies (1.9) and (1.15), there exists a positive constant C_s such that $w(x) \geq C_s(1+|x|)^s$. This implies that $L^p_w(\mathbf{R}) \subset L^p_{(1+|\cdot|)^s}(\mathbf{R})$ for any $s \geq 1$. Hence any eigenvalue of the operator $W_{c,\alpha}$ restricted to $L^p_w(\mathbf{R})$ is an eigenvalue of the operator $W_{c,\alpha}$ restricted to $L^p_{(1+|\cdot|)^s}(\mathbf{R})$. Therefore by Corollary 1.5,

$$\begin{aligned} E(W_{c,\alpha}, L^p_w(\mathbf{R})) &\subset E(W_{c,\alpha}, L^p_{(1+|\cdot|)^s}(\mathbf{R})) \subset \sigma(W_{c,\alpha}, L^p_{(1+|\cdot|)^s}(\mathbf{R})) \\ &\subset \Sigma_0 \cup \{\lambda \in \mathbf{C} : |\lambda| \leq \alpha^{-s+1-1/p}\} \end{aligned}$$

for all $s \geq 1$, which implies

$$E(W_{c,\alpha}, L^p_w(\mathbf{R})) \subset \Sigma_0 \cup \{0\}. \tag{4.8}$$

Note that $0 \notin E(W_{c,\alpha}, L^p_w(\mathbf{R}))$ since $L^p_w(\mathbf{R}) \subset L^1(\mathbf{R})$. This together with (4.7) and (4.8) lead to (1.16). ♠

Acknowledgments. The research is supported by the Wavelets Strategic Research Programme, National University of Singapore, under a grant from the National Science and Technology Board and the Ministry of Education, Singapore.

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