An Algorithm for the Construction of Symmetric and Anti-Symmetric M-Band Wavelets

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ABSTRACT

In this paper, we give an algorithm to construct semi-orthogonal symmetric and anti-symmetric M-band wavelets. As an application, some semi-orthogonal symmetric and anti-symmetric M-band spline wavelets are constructed explicitly. Also we show that if we want to construct symmetric or anti-symmetric M-band wavelets from a multiresolution, then that multiresolution has a symmetric scaling function.

Keywords: Symmetric and Antisymmetric Wavelets, M Band Wavelets, Spline Wavelets

1. INTRODUCTION

Fix an integer M larger than 2. A *multiresolution*[4] is a family of nested subspaces $\{V_j\}_{j\in\mathbf{Z}}$ of L^2 such that

- $\cap_{j\in\mathbf{Z}}V_j = \{0\}$ and $\cup_{j\in\mathbf{Z}}V_j$ is dense in L^2.
- $f \in V_j \Longleftrightarrow f(M\cdot) \in V_{j+1}$ for all $j \in \mathbf{Z}$.
- $V_j \subset V_{j+1}$ for all $j \in \mathbf{Z}$.
- There exists a compactly supported function $\phi \in V_0$ such that $\{\phi(\cdot - k) : k \in \mathbf{Z}\}$ is a Riesz basis of V_0.

We remark that the function ϕ in the multiresolution of this paper is assumed to be compactly supported instead of $\phi \in L^2$ in most of literature. From the definition of a multiresolution, we have

$$V_j = \Big\{ \sum_{k\in\mathbf{Z}} d(k)\phi(2^j\cdot - k) : \sum_{k\in\mathbf{Z}} |d(k)|^2 < \infty \Big\}, \quad j \in \mathbf{Z}. \tag{1.1}$$

The function ϕ in the multiresolution above is called a *scaling function*. We remark that there are many scaling functions for a multiresolution. Actually, $\phi - 2\phi(\cdot - 1)$ is also a scaling function if ϕ is. For a multiresolution $\{V_j\}_{j\in\mathbf{Z}}$, let $W_j, j \in \mathbf{Z}$, be the orthogonal complement of V_j in V_{j+1}. It is well known[5] that there exist an orthogonal decomposition of W_j into $W_{js}, 1 \le s \le M-1$, i.e.,

$$W_j = W_{j1} \oplus \cdots \oplus W_{j(M-1)}, \tag{1.2}$$

and some compactly supported functions $\psi_s \in W_{0s}, 1 \le s \le M-1$, such that

$$W_{js} = \Big\{ \sum_{k\in\mathbf{Z}} d(k)\psi_s(M^j\cdot - k) : \sum_{k\in\mathbf{Z}} |d(k)|^2 < \infty \Big\} \tag{1.3}$$

for all $j \in \mathbf{Z}$ and $1 \le s \le M-1$, and such that $\{\psi_s(\cdot - k) : k \in \mathbf{Z}\}$ is a Riesz basis of $W_{0s}, 1 \le s \le M-1$. The functions $\psi_s, 1 \le s \le M-1$, are called the *M-band wavelets* constructed from the multiresolution $\{V_j\}_{j\in\mathbf{Z}}$, or wavelets for short.

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In some applications, we hope that M band scaling functions and wavelets have some special properties, such as orthonormality, cardinality, symmetry and anti-symmetry.[1,3,6–9,14–17,19] Here we say that a function f has *orthonormal shifts*, or is *orthonormal* for short, if $\int_{\mathbf{R}} f(x)f(x-k)dx = \delta_k$ for all $k \in \mathbf{Z}^d$, where δ_k denotes the usual Kronecker symbol. A continuous function f is said to be *cardinal* if $f(k) = \delta_k$ for all $k \in \mathbf{Z}^d$. We say that a function f is *symmetric* if $f(x_0 + \cdot) = f(x_0 - \cdot)$ for some $x_0 \in \mathbf{R}$, and *anti-symmetric* if $f(x_0 + \cdot) = -f(x_0 - \cdot)$ for some $x_0 \in \mathbf{R}$. The point x_0 above is said to be the symmetric (anti-symmetric) point of f.

For $M = 2$, it is well known that Haar function is the only both symmetric and orthonormal scaling function,[4] and that there does not exist both orthonormal and cardinal scaling functions.[13] But for $M \geq 3$, the situation is completely different. Heller studied the construction of M band Daubechies' orthonormal scaling functions,[7] and later it was shown that the asymptotic rate of regularity of the M band Daubechies' orthonormal scaling functions for odd M is much lower than that for the classical case $M = 2$.[1,15,17] For dilation 3, Chui and Lian constructed some examples of both orthonormal and (anti)symmetric scaling functions and wavelets,[3] and for dilation 4, Han constructed some C^1 orthonormal scaling functions and wavelets which is symmetric or anti-symmetric.[6] Bi, Dai and Sun constructed some both orthonormal and cardinal scaling function for any $M \geq 3$,[1] and Ji and Shen gave some examples of orthonormal scaling functions which are also cardinal and symmetric for dilation 4.[9]

For $M = 2$, there is an explicit construction of wavelets from the scaling function.[2,4] For $M \geq 3$, some algorithm to construct M band wavelets from a multiresolution has been given. For instance, Lawton, Lee and Shen worked out a constructive algorithm to construct M band orthonormal wavelets.[12] Han give an algorithm to construct symmetric and orthonormal wavelets from a multiresolution with dilation 4.[6] In this paper we shall give a general algorithm to construct semi-orthogonal symmetric or anti-symmetric M band wavelets from a multiresolution. The wavelet functions obtained from that algorithm would be symmetric or anti-symmetric, but not orthonormal in general. It is still open to find a constructive algorithm to construct both orthonormal and (anti)symmetric wavelets from a multiresolution which has both orthonormal and symmetric scaling functions.

The paper is organized as follows. In Section 3, we show that if we want to construct symmetric or anti-symmetric wavelets from a multiresolution, then it is necessary to assume that multiresolution has a symmetric scaling function (Theorem 2.1). Also we reduce the construction of (anti)symmetric wavelets to that of some Laurent polynomials (Theorem 2.2). The algorithm for the construction of (anti)symmetric wavelets is given in Section 3. The last section is devoted to some examples of (anti)symmetric spline wavelets obtained by the algorithm in Section 3.

2. PRELIMINARIES

We say that a compactly supported distribution f has *linearly independent shifts* if the map

$$\{d(k)\}_{k\in\mathbf{Z}} \longmapsto \sum_{k\in\mathbf{Z}} d(k)f(\cdot - k)$$

is one-to-one. It is obvious that if f has orthonormal shifts and is compactly supported, then f has linearly independent shifts, since in this case

$$d(k) = \int_{\mathbf{R}} F(x)f(x-k)dx$$

for any sequence $\{d(k)\}_{k\in\mathbf{Z}}$, where $F(x) = \sum_{k\in\mathbf{Z}} d(k)f(x-k)$. It is well known that every compactly supported function f can be written as finite linear combination of the shifts of a function having compact support and linearly independent shifts,[10,18] i.e., there exist a sequence $\{d(k)\}_{k\in\mathbf{Z}}$ having finite support and a function g having linearly independent shifts such that $f = \sum_{k\in\mathbf{Z}} d(k)g(\cdot - k)$. Furthermore g is a scaling function too if f is. So in this paper, a scaling function is **always** assumed to be the one having linearly independent shifts. For a scaling function ϕ of a multiresolution $\{V_j\}_{j\in\mathbf{Z}}$ having linearly independent shifts and compact support, any compactly supported function $f \in V_0$ is finitely linear combination of the shifts of ϕ. Thus ϕ has minimal support. Conversely the shifts of a scaling function having minimal support are linearly independent.

Let ϕ be a scaling function having linearly independent shifts. By $V_0 \subset V_1$ in the definition of a multiresolution, the function ϕ satisfies a *refinement equation*

$$\phi = \sum_{k\in\mathbf{Z}} c_0(k)\phi(M\cdot -k). \qquad (2.1)$$

By the linearly independent shifts of ϕ, the real-valued sequence $\{c_0(k)\}_{k\in\mathbf{Z}}$ in (2.1) is unique and has finite support, i.e., $c_0(k) = 0$ for all but finitely many $k \in \mathbf{Z}$. The sequence $\{c_0(k)\}_{k\in\mathbf{Z}}$ in (2.1) and the Laurent polynomial $H(z)$ defined by

$$H(z) = \sum_{k\in\mathbf{Z}} c_0(k) z^k$$

are known as the *mask* and the *symbol* of the scaling function ϕ respectively.

Define the Fourier transform of an integrable function f by $\hat{f}(\xi) = \int_{\mathbf{R}} f(x)e^{-ix\xi}dx$. Taking Fourier transform at both sides of (2.1) leads to

$$\widehat{\phi}(M\xi) = \frac{1}{M} H(e^{-i\xi})\widehat{\phi}(\xi). \tag{2.2}$$

Obviously every scaling function ϕ is integrable since $\phi \in L^2$ and ϕ has compact support. Thus $\lim_{\xi\to\infty} \hat{\phi}(\xi) = 0$ by Riemann Lemma and hence $\hat{\phi}(2k\pi) = 0$ for all $k \in \mathbf{Z}\backslash\{0\}$ by (2.2). This together with the linearly independent shifts of ϕ leads to the following properties for the symbol H of the scaling function ϕ having linear independent shifts,

$$H(1) = M \quad \text{and} \quad H(\omega_M^j) = 0 \quad \forall\ 1 \le j \le M-1, \tag{2.3}$$

where $\omega_M = e^{-2\pi i/M}$.

Let $N_1 \le N_2$ be integers such that $c_0(k) = 0$ for all $k < N_1$ or $k > N_2$ and $c_0(N_1)c_0(N_2) \ne 0$. Then $[N_1/(M-1), N_2/(M-1)]$ is the minimal interval which ϕ is supported in. Therefore the symmetric point of the scaling function ϕ in (2.1) is $(N_2 + N_1)/(2M-2)$ if ϕ is symmetric. By (2.2), the symbol H of the scaling function ϕ satisfies

$$H(z) = z^\alpha H(z^{-1}) \tag{2.4}$$

if ϕ is symmetric and has symmetric point $\alpha/(2M-2)$, where α is an integer. Note that $\int_{\mathbf{R}} f(x)dx = 0$ for any anti-symmetric function f. Thus a scaling function ϕ is not anti-symmetric since $\int_{\mathbf{R}} \phi(x)dx = 1$.

THEOREM 2.1. *Let $\{V_j\}_{j\in\mathbf{Z}}$ be a multiresolution, and let ϕ be its scaling function having linearly independent shifts and compact support. Assume that ϕ is not a symmetric function. Then there does not exist any compactly supported symmetric or anti-symmetric function in V_j for any $j \in \mathbf{Z}$.*

From Theorem 2.1, we see that if we want to construct symmetric or anti-symmetric wavelets constructed from a multiresolution, then that multiresolution **must** have a symmetric scaling function with minimal support. The proof of Theorem 2.1 is postponed to the end of this section.

Recall that $W_0 \subset V_1$. Then by (1.1) and the linearly independent shifts of ϕ, there are sequences $\{c_s(k)\}_{k\in\mathbf{Z}}, 1 \le s \le M-1$, having finite support such that

$$\psi_s = \sum_{k\in\mathbf{Z}} c_s(k)\phi(M\cdot -k), \quad 1 \le s \le M-1. \tag{2.5}$$

Define

$$H_s(z) = \sum_{k\in\mathbf{Z}} c_s(k)z^k, \quad 1 \le s \le M-1.$$

Then $\widehat{\psi}_s(M\xi) = \frac{1}{M}H_s(e^{-i\xi})\widehat{\phi}(\xi)$ by taking Fourier transform at both sides of (2.5). So the construction of M-band wavelets is closely related to the construction of Laurent polynomial $H_s, 1 \le s \le M-1$, with certain properties.

Let Φ be the Laurent polynomial such that $\Phi(e^{-i\xi}) = \sum_{k\in\mathbf{Z}} |\hat{\phi}(\xi + 2k\pi)|^2$. Recall that for any scaling function ϕ, the shifts of ϕ is a Riesz basis of V_0. Then for any scaling function ϕ, we have

$$\Phi(e^{-i\xi}) > 0 \quad \forall\ \xi \in \mathbf{R}. \tag{2.6}$$

From (2.2) and the definition of Φ it follows that

$$\sum_{s=0}^{M-1} H(z\omega_M^s)H(z^{-1}\omega_M^{-s})\Phi(z\omega_M^s) = M^2\Phi(z^M). \tag{2.7}$$

Combining (2.6) and (2.7) leads to

$$C^{-1} \leq \sum_{s=0}^{M-1} H(e^{-i\xi}\omega_M^s)H(e^{i\xi}\omega_M^{-s}) \leq C \quad \forall\ \xi \in [-\pi, \pi],$$

where C is a positive constant independent of ξ.

Given any Laurent polynomial G such that $G(z) = G(z^{-1})$ and $G(z) > 0$ for all $|z| = 1$, define

$$[P, Q]_G(z^M) = \sum_{j=0}^{M-1} P(z\omega_M^j)Q(z^{-1}\omega_M^{-j})G(z\omega_M^j)$$

for Laurent polynomials P and Q. From the definition, $[P, Q]_G(z)$ is still a Laurent polynomial, and $[P, P]_G(z)$ is nonzero Laurent polynomial if P is. We say that a nonzero complex number z_0 is an *M-symmetric root* of a Laurent polynomial R if $R(z_0\omega_M^j) = 0$ for all $0 \leq j \leq M-1$. It is easy to check that if P has no M-symmetric roots, then $[P, P]_G(z) > 0$ for all complex number z with $|z| = 1$. For any Laurent polynomials $Q_1(z)$ and $Q_2(z)$ satisfying $Q_s(z) = \epsilon_s z^\alpha Q_s(z^{-1})$,

$$[Q_1, Q_2]_G(z^{-1}) = \epsilon_1\epsilon_2[Q_1, Q_2]_G(z) \tag{2.8}$$

by direct computation, where $\epsilon_s = \pm 1, s = 1, 2$ and $\alpha \in \mathbf{Z}$.

THEOREM 2.2. *Let $\{V_j\}_{j\in\mathbf{Z}}$ be a multiresolution, ϕ be its scaling function with minimal support, and H_0 be the symbol of the function ϕ. Assume that*

$$\phi\Big(\frac{\alpha}{2M-2} + x\Big) = \phi\Big(\frac{\alpha}{2M-2} - x\Big) \tag{2.9}$$

for some integer α. Let $H_s(z), 1 \leq s \leq M-1$, be Laurent polynomials such that

$$H_s(z) = \epsilon_s z^{\alpha_s} H_s(z^{-1}) \quad \forall\ 1 \leq s \leq M-1, \tag{2.10}$$

where $\epsilon_s = \pm 1$ and $\alpha_s \in \mathbf{Z}$,

$$[H_s, H_t]_\Phi(z) \equiv 0 \quad \forall\ 0 \leq s \neq t \leq M-1, \tag{2.11}$$

and

$$[H_s, H_s]_\Phi(z) > 0 \quad \forall\ |z| = 1 \quad \text{and} \quad 1 \leq s \leq M-1. \tag{2.12}$$

Then the functions $\psi_s, 1 \leq s \leq M-1$, defined by

$$\widehat{\psi}_s(\xi) = H_s(e^{-i\xi/M})\hat{\phi}(\xi/M)$$

are symmetric or anti-symmetric M-band wavelets constructed from the multiresolution $\{V_j\}_{j\in\mathbf{Z}}$. Furthermore,

$$\psi_s\Big(\frac{\alpha_s(M-1)+\alpha}{2M^2-2M} + x\Big) = \epsilon_s\psi_s\Big(\frac{\alpha_s(M-1)+\alpha}{2M^2-2M} - x\Big) \quad \forall\ 1 \leq s \leq M-1.$$

From Theorem 2.2, the construction of symmetric and anti-symmetric wavelets reduces to the construction of Laurent polynomials $H_s(z), 1 \leq s \leq M-1$, satisfying (2.10), (2.11) and (2.12). The proof of Theorem 2.2 is routine, we omit the detail here.

2.1. Proof of Theorem 2.1

To prove Theorem 2.1, we need the following lemma about the symbol of a scaling function with minimal support. The result was proved for $M = 2$ by Jia and Wang,[11] and the generalization of their result to $M \geq 3$ is straightforward.

LEMMA 2.3. *Let ϕ be a scaling function with linearly independent shifts and let $H(z)$ be its symbol. Then there do not exist Laurent polynomials $R_1(z)$ and $R_2(z)$ such that $R_1(z) \neq C_1 z^\alpha (1-z)^\beta$ for some $C_1 \in \mathbf{R}\backslash\{0\}$, $\alpha \in \mathbf{Z}$ and $\beta \in \mathbf{Z}_+$, and such that*

$$H(z) = R_2(z) R_1(z^M)/R_1(z).$$

Proof of Theorem 2.1. By the definition of a multiresolution, it suffices to prove that there does not exist any nonzero symmetric or (an) anti-symmetric compactly supported function in V_0. On the contrary, there exists a compactly supported function $\psi \in V_0$ such that $\psi \not\equiv 0$ and ψ is symmetric or anti-symmetric. By (1.1), there exists a unique sequence $\{d(k)\}_{k\in\mathbf{Z}}$ such that $\sum_{k\in\mathbf{Z}} |d(k)|^2 < \infty$ and

$$\psi = \sum_{k\in\mathbf{Z}} d(k)\phi(\cdot - k). \tag{2.13}$$

By the linearly independent shifts of ϕ, and by the assumption that ψ has compact support, the sequence $\{d(k)\}_{k\in\mathbf{Z}}$ in (2.13) has finite support, and hence $D(z) = \sum_{k\in\mathbf{Z}} d(k) z^k$ is a Laurent polynomial. Taking Fourier transform at both sides of (2.13) leads to

$$\widehat{\psi}(\xi) = D(e^{-i\xi})\widehat{\phi}(\xi). \tag{2.14}$$

By the symmetry or antisymmetry of ψ, there exist $x_0 \in \mathbf{R}$ and $\epsilon = \pm 1$ such that

$$\widehat{\psi}(\xi) = \epsilon e^{-ix_0\xi}\widehat{\psi}(-\xi). \tag{2.15}$$

Combining (2.2), (2.14) and (2.15) leads to

$$\frac{H(e^{-i\xi})}{H(e^{i\xi})} = \frac{\hat{\phi}(M\xi)\hat{\phi}(-\xi)}{\hat{\phi}(-M\xi)\hat{\phi}(\xi)} = e^{-i(M-1)x_0\xi}\frac{D(e^{iM\xi})D(e^{-i\xi})}{D(e^{-iM\xi})D(e^{i\xi})}$$

on a neighborhood of the origin. Therefore $(M-1)x_0$ is an integer, and

$$H(z)D(z^M)D(z^{-1}) = z^{(M-1)x_0} H(z^{-1})D(z^{-M})D(z) \tag{2.16}$$

for all nonzero complex number z. Let $D_1(z)$ be the maximal common polynomial factor between Laurent polynomials $D(z)$ and $D(z^{-1})$. Then

$$D_1(z) = C z^\beta D_1(z^{-1}) \tag{2.17}$$

for some $C \in \mathbf{R}\backslash\{0\}$ and $\beta \in \mathbf{Z}$. Set $\tilde{D}(z) = D(z)/D_1(z)$. Then there do not exist nonzero common roots between $\tilde{D}(z)$ and $\tilde{D}(z^{-1})$. By (2.16) and (2.17), we obtain

$$H(z)\tilde{D}(z^M)\tilde{D}(z^{-1}) = z^\alpha H(z^{-1})\tilde{D}(z^{-M})\tilde{D}(z) \quad \forall\ z \in \mathbf{C}\backslash\{0\}, \tag{2.18}$$

where α is an integer. Comparing M symmetric roots of both sides of (2.18) leads to

$$H(z)\tilde{D}(z^{-1}) = \tilde{D}(z^{-M})R(z)$$

for some Laurent polynomial $R(z)$. Hence

$$\tilde{D}(z) = C z^\alpha (1-z)^\beta \tag{2.19}$$

for some $C \in \mathbf{R}\backslash\{0\}, \alpha \in \mathbf{Z}$ and $\beta \in \mathbf{Z}_+$ by Lemma 2.3. Using (2.18) and (2.19), we have

$$H(z) = z^\gamma H(-z) \quad \text{for some} \quad \gamma \in \mathbf{Z}.$$

Therefore, ϕ is symmetric since

$$\hat{\phi}(\xi) = \prod_{n=0}^{\infty} \frac{H(e^{-iM^{-n}\xi})}{M},$$

which is a contradiction. □

3. ALGORITHM

In this section, we shall give an algorithm to construct symmetric and anti-symmetric wavelets. Let ϕ be a symmetric scaling function with symmetric point $\alpha/(2M-2)$, and let H be its symbol, where α is an integer. Write $\alpha = -2\gamma + \delta$ where $\gamma \in \mathbf{Z}$ and $\delta = 0$ or 1. Then $z^{\gamma} H(z) = z^{\delta-\gamma} H(z^{-1})$.

Initial Step *Define $H_{0,0} = H$, and $H_{0,s}, 1 \leq s \leq M-1$, by*

$$H_{0,s}(z) = z^{-\gamma} \begin{cases} z^{s} + z^{-s} & 1 \leq s \leq \left[\frac{M-1}{2}\right] \\ z^{M-s} - z^{s-M} & \left[\frac{M+1}{2}\right] \leq s \leq M-1 \end{cases}$$

if $\delta = 0$,

$$H_{0,s}(z) = z^{-\gamma} \begin{cases} z^{s+1} + z^{-s} & 1 \leq s \leq M/2-1 \\ z^{M-s} - z^{1+s-M} & M/2 \leq s \leq M-1 \end{cases}$$

if $\delta = 1$ and M is even, and

$$H_{0,s}(z) = z^{-\gamma} \begin{cases} z^{s} + z^{1-s} & 1 \leq s \leq (M-1)/2 \\ z^{M-s} - z^{1+s-M} & (M+1)/2 \leq s \leq M-1 \end{cases}$$

if $\delta = 1$ and M is odd, where $[x]$ is the greatest integer smaller than or equals to x.

For the functions $H_{0,s}, 0 \leq s \leq M-1$, constructed above, we have the following properties which proof is postponed to the end of this section.

Proposition 3.1. *Let $H_{0,s}(z), 0 \leq s \leq M-1$, be Laurent polynomials defined in the initial step. Then*

$$z^{\gamma} H_{0,s}(z) = \epsilon_s z^{\delta-\gamma} H_{0,s}(z^{-1}) \quad \forall\, 0 \leq s \leq M-1, \tag{3.1}$$

where $\epsilon_s = \pm 1$, and $\det \left(H_{0,s}(z\omega_M^j) \right)_{0 \leq s,j \leq M-1}$ *is a nonzero Laurent polynomial.*

Main Step *For $1 \leq t \leq M-1$, we inductively define $H_{t,s} = H_{t-1,s}$ for all $0 \leq s \leq t-1$, and $H_{t,s}, t \leq s \leq M-1$, by*

$$H_{t,s}(z) = [H_{t-1,t-1,}, H_{t-1,t-1}]_{\Phi}(z^{-M}) H_{t-1,s}(z) - [H_{t-1,t-1}, H_{t-1,s}]_{\Phi}(z^{-M}) H_{t-1,t-1}(z).$$

For the above functions $H_{t,s}$, we have the following properties, which proof is postponed to the end of this section.

Proposition 3.2. *Let $H_{t,s}(z), 1 \leq t \leq M-1, 0 \leq s \leq M-1$, be defined above. Then for any $1 \leq t \leq M-1$,*

$$z^{\gamma} H_{t,s}(z) = \epsilon_s z^{\delta-\gamma} H_{t,s}(z^{-1}) \quad \forall\, 0 \leq s \leq M-1, \tag{3.2}$$

where $\epsilon_s = \pm 1$ independent of t,

$$[H_{t,s}, H_{t,s'}]_{\Phi}(z) = 0 \quad \forall\, 0 \leq s \leq t-1, s < s' \leq M-1, \tag{3.3}$$

and $\det(H_{ts}(z\omega^j))_{0 \leq s,j \leq M-1}$ *is a nonzero Laurent polynomial.*

Last Step *Write*

$$H_{M-1,s}(z) = H_s(z) \tilde{H}_{M,s}(z^M), \; 1 \leq s \leq M-1 \tag{3.4}$$

such that $H_s, 1 \leq s \leq M-1$, have no M-symmetric roots.

For the functions $H_s, 1 \leq s \leq M-1$, in the last step, we have

Proposition 3.3. *Let $H_s(z), 1 \leq s \leq M-1$, be as in (3.4). Then $H_s, 1 \leq s \leq M-1$, satisfies (2.10), (2.11) and (2.12).*

3.1. Proofs

Proof of Proposition 3.1 Obviously (3.1) follows from the definition of $H_{0,s}, 1 \le s \le M-1$. The assertion $\det(H_{0,s}(z\omega_M^j))_{0\le s,j\le M-1} \not\equiv 0$ for the case $\delta = 1$, and for the case $\delta = 0$ and odd integer M, can be proved in the same way as the proof for the case $\delta = 0$ and even integer M. So we only give the proof for the case $\delta = 0$ and even integer M in detail. In this case,

$$\begin{cases} H_{0,M/2}(z) = (1 - z^M) z^{-M/2-\gamma} \\ H_{0,s}(z) + H_{0,M-s}(z) = 2z^{s-\gamma} \quad \forall\, 1 \le s \le M/2 - 1 \\ H_{0,s}(z) - H_{0,M-s}(z) = 2z^{-s-\gamma} \quad \forall\, 1 \le s \le M/2 - 1. \end{cases}$$

Write $z^\gamma H(z) = \sum_{j=-M/2}^{M/2-1} z^j P_j(z^M)$. Then $P_j(1) = 1$ for all $-M/2 \le j \le M/2-1$ by (2.3) and the assumption on $H(z)$. Therefore,

$$\begin{aligned} & \det\left(H_{0,s}(z\omega_M^j)\right)_{0\le s,j\le M-1} \\ = \;& C_1 z^{-(M-1)\gamma}(1-z^M) \times \\ & \det\left(H(z\omega_M^j), (z\omega_M^j)^{-M/2}, \ldots, (z\omega_M^j)^{-1}, (z\omega_M^j), \ldots, (z\omega_M^j)^{M/2-1}\right)_{0\le j\le M-1} \\ = \;& C_2 z^{-M\gamma}(1-z^M) P_0(z^M) \det((z\omega_M^j)^s)_{-M/2\le s\le M/2-1, 0\le j\le M-1}, \end{aligned}$$

where C_1 and C_2 are nonzero constants. Hence $\det(H_{0,s}(z\omega_M^j))_{0\le s,j\le M-1} \not\equiv 0$ since $P_0(z) \not\equiv 0$ by $P_0(1) = 1/M$ and $\det((z\omega_M^j)^s)_{-M/2\le s\le M/2-1, 0\le j\le M-1}$ is a Vandermonde determinant. □

Proof of Proposition 3.2 We prove the assertion by induction on t. For $t = 1$, (3.3) follows directly from the definition. By Proposition 3.1 and $H_{0,0} \not\equiv 0$, we obtain

$$\begin{aligned} & \det(H_{1,s}(z\omega_M^j))_{0\le s,j\le M-1} \\ = \;& \left([H_{0,0}, H_{0,0}]_\Phi(z^{-1})\right)^{M-1} \det(H_{0,s}(z\omega_M^j))_{0\le s,j\le M-1} \not\equiv 0. \end{aligned}$$

Then it remains to prove (3.2). For $s = 0$, $H_{1,0}(z)$ satisfies (3.2) since $H_{0,0}$ does. For $1 \le s \le M-1$,

$$\begin{aligned} z^\alpha H_{1,s}(z^{-1}) &= z^\alpha [H_{0,0}, H_{0,0}]_\Phi(z^M) H_{0,s}(z^{-1}) - z^\alpha [H_{0,0}, H_{0,s}]_\Phi(z^M) H_{0,0}(z^{-1}) \\ &= \epsilon_s [H_{0,0}, H_{0,0}]_\Phi(z^{-M}) H_{0,s}(z) - \epsilon_s [H_{0,0}, H_{0,s}]_\Phi(z^M) H_{0,0}(z) \\ &= H_{1,s}(z), \end{aligned}$$

where we have used (2.8) and (3.1). This prove the assertion for $t = 1$. The assertion for $2 \le t \le M-1$ can be proved inductively by using the same procedure above. We omit the detail here. □

Proof of Proposition 3.3 By Proposition 3.2,

$$H_s(z)\tilde{H}_{M,s}(z^M) = \epsilon_s z^\alpha H_s(z^{-1})\tilde{H}_{M,s}(z^{-M}). \tag{3.5}$$

Comparing all nonzero M-symmetric roots at both sides of (3.5) leads to $\tilde{H}_{M,s}(z) = z^{\alpha'}\tilde{H}_{M,s}(z^{-1})$. Substituting this into (3.5) implies (2.10).

By (3.4) and Proposition 3.2,

$$[H_{M-1,s}, H_{M-1,t}]_\Phi(z) = \tilde{H}_{M,s}(z)\tilde{H}_{M,t}(z^{-1})[H_s, H_t]_\Phi(z).$$

This together with (3.3) leads to (2.11).

Note that $H_s, 1 \le s \le M-1$, has not M symmetric roots. Thus $\sum_{j=0}^{M-1} |H_s(z\omega_M^j)| > 0$ for nonzero z. Hence (2.12) follows. □

4. SPLINE WAVELETS

In this section, we use the algorithm in Section 3 to construct symmetric or anti-symmetric spline wavelets for $N = 2, 3$ and $M = 3$. Some examples of bi-orthogonal (anti)symmetric wavelets are given by Soardi,[16] where the premier scaling functions and wavelets are B-splines and spline wavelets.

Case 1 *Spline wavelets with order two and dilation three.*

In this case, the scaling function is the hat function

$$\phi_2(x) = \begin{cases} 0 & x < 0 \quad or \quad x > 2 \\ x & 0 \le x \le 1 \\ 2 - x & 1 < x \le 2, \end{cases}$$

which is symmetric, and whose filter is $H_3^2(z) = (1 + z + z^2)^2/3$. By the algorithm in Section 3, we may choose two wavelet filters $H_{3,1}^2$ and $H_{3,2}^2$ by

$$\begin{aligned} H_{3,1}^2(z) &= -2 + 5z - 6z^2 + 5z^3 - 2z^4 \\ H_{3,2}^2(z) &= -2z^{-6} + 5z^{-5} - 6z^{-4} - 14z^{-2} + 12z^{-1} + 64 - 51z \\ &\quad +51z^3 - 64z^4 - 12z^5 + 14z^6 + 6z^8 - 5z^9 + 2z^{10}. \end{aligned}$$

Hence the functions $\psi_{3,1}^2$ and $\psi_{3,2}^2$ defined by

$$\begin{cases} \widehat{\psi_{3,1}^2}(\xi) = H_{3,1}^2(e^{-i\xi/3})\widehat{\phi_2}(\xi/3) \\ \widehat{\psi_{3,2}^2}(\xi) = H_{3,2}^2(e^{-i\xi/3})\widehat{\phi_2}(\xi/3) \end{cases}$$

are symmetric and anti-symmetric spline wavelets respectively (see Figure 1).

Case 2 *Spline wavelets with order three and dilation three.*

In this case the scaling function is

$$\phi_3(x) = \begin{cases} 0 & x < 0 \quad or \quad x > 3 \\ 0.5x^2 & 0 \le x < 1 \\ -x^2 + 3x - 1.5 & 1 \le x < 2 \\ 0.5(x-3)^2 & 2 \le x \le 3 \end{cases}$$

and the filter is $H_3^3(z) = \frac{1}{9}(1 + z + z^2)^3$. By the algorithm in Section 3, $\psi_{3,1}^3$ and $\psi_{3,2}^3$ defined by

$$\begin{cases} \widehat{\psi_{3,1}^3}(\xi) = H_{3,1}^3(e^{-i\xi/3})\widehat{\phi_3}(\xi/3) \\ \widehat{\psi_{3,2}^3}(\xi) = H_{3,2}^3(e^{-i\xi/3})\widehat{\phi_3}(\xi/3) \end{cases}$$

are symmetric or antisymmetric wavelets respectively, where

$$\begin{aligned} H_{3,1}^3(z) &= -3.1216049 + 2.1092593(z + z^{-1}) - 0.0481481(z^2 + z^{-2}) - 1.3083333(z^3 + z^{-3}) \\ &\quad +1.1787037(z^4 + z^{-4}) - 0.312037(z^5 + z^{-5}) - 0.1305555(z^6 + z^{-6}) \\ &\quad +0.0731481(z^7 + z^{-7}) - 0.000925925(z^8 + z^{-8}) - 0.000308641(z^9 + z^{-9}) \\ H_{3,2}^3(z) &= 153.08162(z - z^{-1}) - 102.45087(z^2 - z^{-2}) + 20.508609(z^3 - z^{-3}) \\ &\quad +117.05492(z^4 - z^{-4}) - 32.004276(z^5 - z^{-5}) + 14.708013(z^6 - z^{-6}) \\ &\quad +50.554366(z^7 - z^{-7}) - 3.4053212(z^8 - z^{-8}) + 3.5721543(z^9 - z^{-9}) \\ &\quad +14.159285(z^{10} - z^{-10}) - 0.0644075(z^{11} - z^{-11}) + 0.1464843(z^{12} - z^{-12}) \\ &\quad +3.0332681(z^{13} - z^{-13}) - 0.0921772(z^{14} - z^{-14}) - 0.0353921(z^{15} - z^{-15}) \\ &\quad +0.4548712(z^{16} - z^{-16}) - 0.0080759(z^{17} - z^{-17}) - 0.0029319(z^{18} - z^{-18}) \\ &\quad +0.259919(z^{19} - z^{-19}) - 0.00017672(z^{20} - z^{-20}) + 0.000059834(z^{21} - z^{-21}) \\ &\quad +0.000474545(z^{22} - z^{-22}). \end{aligned}$$

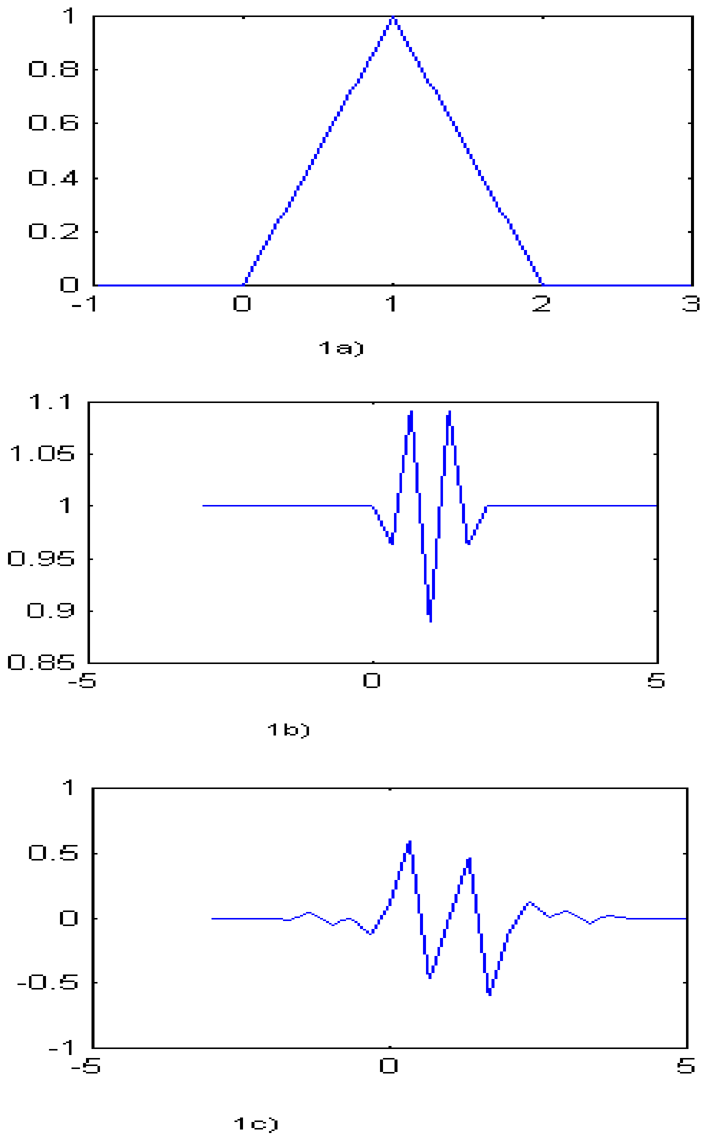

Figure 1. 1a) ϕ_2, 1b) $\psi_{3,1}^2$, 1c) $\psi_{3,1}^2$

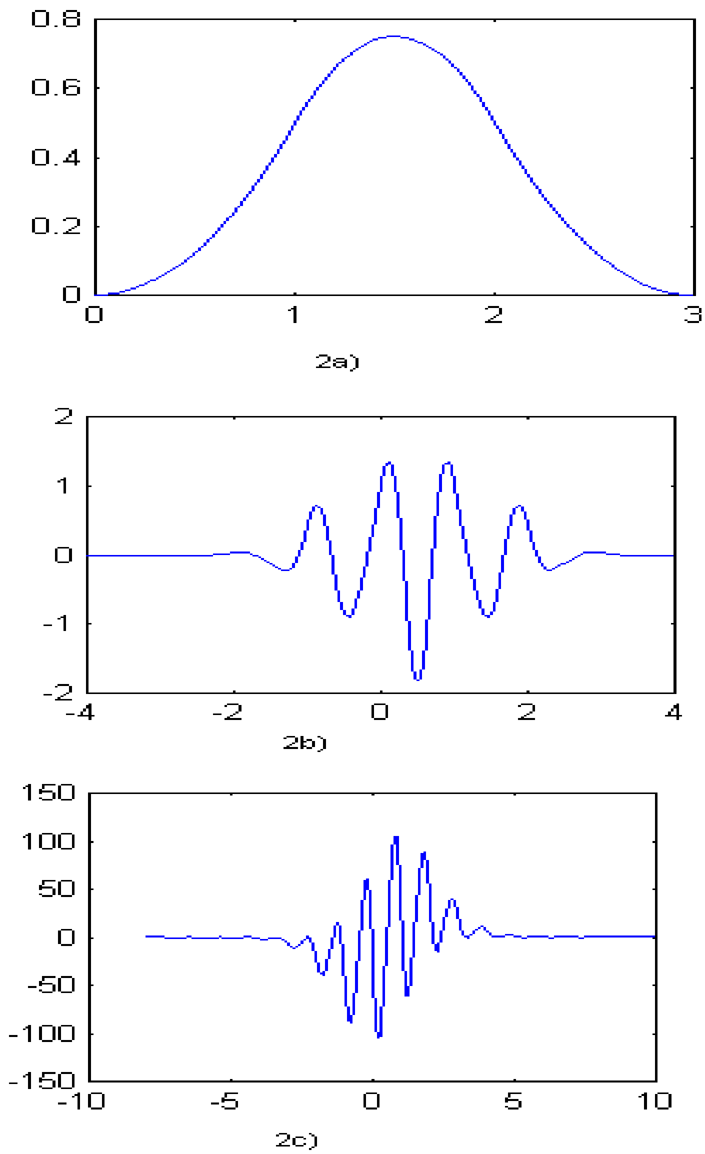

Figure 2. 2a) ϕ_3, 2b) $\psi^3_{3,1}$, 2c) $\psi^3_{3,2}$

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