# Wiener's lemma for infinite matrices with polynomial off-diagonal decay

# Le lemme de Wiener pour matrices infinies a decroissance polynomiale des termes non-diagonaux

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**Abstract**

In this note, we give a simple elementary proof to Wiener's lemma for infinite matrices with polynomial off-diagonal decay.

**Résumé**

Dans cette note, nous donnons une preuve elementaire du lemme de Wiener pour les matrices infinies a decroissance polynomiale des termes non-digonaux.

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## 1. Introduction

The classical Wiener's lemma states that *if a periodic function $f$ has an absolutely convergent Fourier series and never vanishes, then $1/f$ has an absolutely convergent Fourier series.*

Let $\ell^p, 1 \le p \le \infty$, be the space of all $p$-summable sequences on $\mathbf{Z}^d$ equipped with usual norm $\|\cdot\|_{\ell^p}$, denote by $\mathcal{B}^2$ the space of all bounded operators on $\ell^2$ equipped with usual operator norm $\|\cdot\|_{\mathcal{B}^2}$, and define $\mathcal{W} := \{(a(i-j))_{i,j\in\mathbf{Z}^d} : \sum_{j\in\mathbf{Z}^d} |a(j)| < \infty\}$ with a norm $\|A\|_{\mathcal{W}} := \sum_{j\in\mathbf{Z}^d} |a(j)|$ for every matrix $A = (a(i-j))_{i,j\in\mathbf{Z}^d} \in \mathcal{W}$. An equivalent formulation of the classical Wiener's lemma involving matrix algebra can be stated as follows: $A \in \mathcal{W}$ *and* $A^{-1} \in \mathcal{B}^2$ *imply* $A^{-1} \in \mathcal{W}$.

The classical Wiener's lemma and its various generalizations (see, for instance, [3], [8], [9], [12], [13], [14]) are important and have numerous applications, for instance, in numerical analysis ([4], [17], [18]),

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wavelets and affine frames ([5], [14]), time-frequency analysis ([2], [10], [11], [12], [13], [19]), shift-invariant spaces and polynomial spline spaces ([1], [8], [15], [19]), and non-uniform sampling ([6], [19]). Unlike the matrix algebra $\mathcal{W}$ associated with the classical Wiener's lemma, which is *commutative*, the matrix algebras in the study of spline approximation and projection ([7], [8]), affine and Gabor frame ([2], [5], [12], [13]), and non-uniform sampling ([6], [19]) are *extremely non-commutative*. But for various purposes, we still expect that those matrix algebras have the above property that the matrix algebra $\mathcal{W}$ has.

For $p \in [1, \infty]$ and $\alpha \in \mathbf{R}$, let

$$Q_{p,\alpha} := \{A := \big(A(i,j)\big)_{i,j\in\mathbf{Z}^d} : \quad \|A\|_{p,\alpha} < \infty\}, \tag{1.1}$$

where

$$\|A\|_{p,\alpha} := \sup_{i\in\mathbf{Z}^d} \big\|\big(A(i,j)(1+|i-j|)^\alpha\big)_{j\in\mathbf{Z}^d}\big\|_{\ell^p} + \sup_{j\in\mathbf{Z}^d} \big\|\big(A(i,j)(1+|i-j|)^\alpha\big)_{i\in\mathbf{Z}^d}\big\|_{\ell^p}. \tag{1.2}$$

For $p = \infty$, we see that $A = (A(i,j))_{i,j\in\mathbf{Z}^d} \in Q_{\infty,\alpha}$ if and only if $|A(i,j)| \le \|A\|_{\infty,\alpha}(1+|i-j|)^{-\alpha}$ for all $i, j \in \mathbf{Z}^d$. Because of the above interpretation of matrices in $Q_{p,\alpha}$ for $p = \infty$, we call matrices in $Q_{p,\alpha}$ to have *polynomial off-diagonal decay.*

For the matrix algebra $Q_{p,\alpha}$ with $p = \infty$ and $\alpha > d$, Jaffard use a rather delicate bootstrap argument to prove that $A \in Q_{\infty,\alpha}$ *and* $A^{-1} \in \mathcal{B}^2$ *imply* $A^{-1} \in Q_{\infty,\alpha}$ ([14]). For the matrix algebra $Q_{p,\alpha}$ with $p = 1$ and $\alpha > 0$, Barnes use the Banach algebra technique to show that $A \in Q_{1,\alpha}$ *and* $A^{-1} \in \mathcal{B}^2$ *imply* $A^{-1} \in Q_{1,\alpha}$ (see [3] for $\alpha \in (0, 1]$ and [13] for any $\alpha > 0$). In this note, we study the matrix algebra $Q_{p,\alpha}$ with $1 \le p \le \infty$ and $\alpha > d(1-1/p)$ and give a simple elementary proof to the following Wiener's lemma.

**Theorem 1.1** *Let* $1 \le p \le \infty$ *and* $\alpha > d(1-1/p)$. *Then* $A \in Q_{p,\alpha}$ *and* $A^{-1} \in \mathcal{B}^2$ *imply* $A^{-1} \in Q_{p,\alpha}$.

More general formulation of the above Wiener's lemma and its applications to frames and sampling will be discussed in the subsequent paper [19].

## 2. Proof of Theorem 1.1

To prove Theorem 1.1, we need the following lemma.

**Lemma 2.1** *Let* $1 \le p \le \infty$ *and* $\alpha > d(1-1/p)$. *Then there exist positive constants* $C_1$ *and* $C_2$ *such that*

$$\|A^n\|_{p,\alpha} \le C_1 \left(C_2 \frac{\|A\|_{p,\alpha}}{\|A\|_{\mathcal{B}^2}}\right)^{\frac{2-\theta}{1-\theta} n^{\log_2(2-\theta)}} (\|A\|_{\mathcal{B}^2})^n \tag{2.1}$$

*holds for all* $A \in Q_{p,\alpha}$ *and* $n \ge 1$, *where* $\theta = 1 - \frac{d}{2\alpha - 2d(1/2-1/p)}$.

*Proof*: By Hölder inequality,

$$\|A\|_{1,0} \le C\|A\|_{p,\alpha} \quad \text{for all } A \in Q_{p,\alpha}. \tag{2.2}$$

Here and hereafter, $C$ denotes an absolute constant which could be different at different occurrence.

By the definition of the operator norm $\|\cdot\|_{\mathcal{B}^2}$,

$$\|A\|_{2,0} \le \|A\|_{\mathcal{B}^2} \le \|A\|_{1,0}. \tag{2.3}$$

For any $A = (A(i,j))_{i,j\in\mathbf{Z}^d}$ and $B = (B(i,j))_{i,j\in\mathbf{Z}^d}$ in $Q_{p,\alpha}$,

$$\|AB\|_{p,\alpha} \le 2^\alpha\|A\|_{p,\alpha}\|B\|_{1,0} + 2^\alpha\|A\|_{1,0}\|B\|_{p,\alpha}, \tag{2.4}$$

by Hölder inequality and the following estimate:

$$|(AB)(i,j)|(1+|i-j|)^\alpha \le 2^\alpha \sum_{k\in\mathbf{Z}^d} |A(i,k)|(1+|i-k|)^\alpha |B(k,j)| + 2^\alpha \sum_{k\in\mathbf{Z}^d} |A(i,k)||B(k,j)|(1+|k-j|)^\alpha.$$

Let $\theta_1 = (\alpha - d(1/2-1/p))^{-1}$ and $\tau = (\|A\|_{p,\alpha})^{\theta_1}(\|A\|_{\mathcal{B}^2})^{-\theta_1}$. Then

$$\begin{aligned}\sum_{k\in\mathbf{Z}^d}|A(i,k)| &\le \sum_{|i-k|\le\tau}|A(i,k)| + \sum_{|i-k|\ge\tau}|A(i,k)| \le C\tau^{d/2}\|A\|_{2,0} + C\tau^{-\alpha+d(1-1/p)}\|A\|_{p,\alpha}\\ &\le C\tau^{d/2}\|A\|_{\mathcal{B}^2} + C\tau^{-\alpha+d(1-1/p)}\|A\|_{p,\alpha} = 2C(\|A\|_{\mathcal{B}^2})^{1-d\theta_1/2}(\|A\|_{p,\alpha})^{d\theta_1/2}\end{aligned}$$

by (2.2) and (2.3), which yields

$$\|A\|_{1,0} \le C(\|A\|_{\mathcal{B}^2})^{1-d\theta_1/2}(\|A\|_{p,\alpha})^{d\theta_1/2} \quad \text{for all } A\in Q_{p,\alpha}. \tag{2.5}$$

Combining (2.4) and (2.5) leads to the following compensated compactness estimate:

$$\|A^2\|_{p,\alpha} \le C\|A\|_{p,\alpha}^{2-\theta}\|A\|_{\mathcal{B}^2}^{\theta} \quad \text{for all } A\in Q_{p,\alpha}. \tag{2.6}$$

Applying (2.2), (2.4) and (2.6), and using $\|A^n\|_{\mathcal{B}^2} \le \|A\|_{\mathcal{B}^2}^n$ for $n\ge 1$, we obtain the following for any $n\ge1$:

$$\|A^{2n}\|_{p,\alpha} \le D(\|A^n\|_{p,\alpha})^{2-\theta}(\|A\|_{\mathcal{B}^2})^{n\theta},$$

and

$$\|A^{2n+1}\|_{p,\alpha} \le D\|A\|_{p,\alpha}(\|A^n\|_{p,\alpha})^{2-\theta}(\|A\|_{\mathcal{B}^2})^{n\theta},$$

where $D\ge1$ is a positive constant independent of $A\in Q_{p,\alpha}$ and $n\ge1$. Thus the sequence $\{b_n\}$, to be defined by $b_n = D^{-1/(1-\theta)}\|A^n\|_{p,\alpha}(\|A\|_{\mathcal{B}^2})^{-n}, n\ge1$, satisfies

$$b_{2n}\le b_n^{2-\theta} \quad \text{and} \quad b_{2n+1}\le b_1 b_n^{2-\theta} \quad \text{for all } n\ge1.$$

By induction, we have the following upper bound estimate to the sequence $\{b_n\}$:

$$b_n \le b_1^{\sum_{i=0}^{l}\epsilon_i(2-\theta)^i} \le b_1^{\frac{2-\theta}{1-\theta}n^{\log_2(2-\theta)}}$$

for $n=\sum_{i=0}^{l}\epsilon_i 2^i$, where $\epsilon_i\in\{0,1\}, 0\le i\le l$. Therefore (2.1) follows. □

**Remark 2.2** For the special case that $p=1,\alpha=0$, and $A=(q(j-j'))_{j,j'\in\mathbf{Z}}$ with $\sum_{j\in\mathbf{Z}} q(j)e^{-ij\xi}$ being reciprocal of a trigonometric polynomial $Q$, Newman proved the following better estimate than the one in (2.1) for the $Q_{1,0}$ norm of $A^n$: $\|A^n\|_{1,0}\le Cn^2\|A\|_{\mathcal{B}^2}^n$ for all $n\ge1$, where $C$ is a positive constant depending on the degree of the polynomial $Q$. That estimate is crucial for Newman's elementary proof of the classical Wiener's lemma ([16]).

Now we start to prove Theorem 1.1.

*Proof of Theorem 1.1*: For any $A=(A(i,j))_{i,j\in\mathbf{Z}^d}\in Q_{p,\alpha}$, we define its transpose $A^*$ by $A^* := \left(\overline{A(j,i)}\right)_{i,j\in\mathbf{Z}^d}$. Then $A^*A\in Q_{p,\alpha}$ by (2.2), (2.4), and the fact that $\|A^*\|_{p,\alpha}=\|A\|_{p,\alpha}$. This, together with the fact that $A^*A$ is a positive operator on $\ell^2$ by the assumption on the matrix $A$, implies that

$$A^*A = \|A^*A\|_{\mathcal{B}^2}(I-B) \tag{2.7}$$

for some $B\in\mathcal{B}^2$ with

$$\|B\|_{\mathcal{B}^2}<1 \quad \text{and} \quad \|B\|_{p,\alpha}<\infty, \tag{2.8}$$

where $I$ is the identity operator on $\ell^2$. By (2.8) and Lemma 2.1, we obtain

$$\|(I-B)^{-1}\|_{p,\alpha} \le \sum_{n=0}^{\infty}\|B^n\|_{p,\alpha} \le \sum_{n=0}^{\infty} C_1\left(C_2\frac{\|B\|_{p,\alpha}}{\|B\|_{\mathcal{B}^2}}\right)^{\frac{2-\theta}{1-\theta}n^{\log_2(2-\theta)}}(\|B\|_{\mathcal{B}^2})^n<\infty. \tag{2.9}$$

The conclusion $A^{-1} \in Q_{p,\alpha}$ then follows from (2.2), (2.4), (2.7), (2.9), and the fact that $A^{-1} = (A^*A)^{-1}A^*$. □

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