RATE OF INNOVATION FOR (NON-)PERIODIC SIGNALS AND OPTIMAL LOWER STABILITY BOUND FOR FILTERING

QIYU SUN AND JUN XIAN

Abstract. One of fundamental problems in sampling theory is to reconstruct (non-)periodic signals from their filtered signals in a stable way. In this paper, we obtain a universal upper bound to the rate of innovation for signals in a closed linear space, which can be stably reconstructed, via the optimal lower stability bound for filtering on that linear space.

1. Introduction

Let $L^p(\mathbb{T}), 1 \le p \le \infty$, be the space of all p-integrable functions (signals) on the unit circle $\mathbb{T}$ with its norm denoted by $\|\cdot\|_{L^p(\mathbb{T})}$. We associate $h \in L^1(\mathbb{T})$ with a convolution operator $h*$ on $L^2(\mathbb{T})$,

$$h * f(x) := \int_{\mathbb{T}} h(x-y) f(y) dy, \quad f \in L^2(\mathbb{T}).$$

The convolution operator $h*$ is a bounded operator on $L^2(\mathbb{T})$ with operator norm $\|h\|_{L^1(\mathbb{T})}$ and it does not have bounded inverse on $L^2(\mathbb{T})$, since $\|h * e_n\|_{L^2(\mathbb{T})}/\|e_n\|_{L^2(\mathbb{T})} = |\hat{h}(n)| \to 0$ as $n \to \infty$ by Riemann-Lebesgue lemma, where $e_n(x) = \exp(2\pi i n x)$ and $\hat{h}(n) = \int_{\mathbb{T}} h(x)\overline{e_n(x)}dx, n \in \mathbb{Z}$. Thus the convolution operator associated with an integrable function does not have stability on $L^2(\mathbb{T})$. Here we say that a linear operator T on a Banach space V is *stable* if

$$A\|f\| \le \|Tf\| \le B\|f\| \quad \text{for all } f \in V \tag{1.1}$$

[1, 21, 23]. The numbers A, B in (1.1) are lower and upper stability bounds of the operator T on V. The *optimal lower (upper) stability bound* are the supremum (infimum) over all lower (upper) bounds.

The instability for the convolution operator $h*$ leads to the observation that not all periodic functions $f \in L^2(\mathbb{T})$ can be reconstructed

---

Date: July 8, 2013.

2010 *Mathematics Subject Classification.* 94A12, 42A70, 45Q05, 44A60, 94A20.

Key words and phrases. Stability, finite rate of innovation, space of homogeneous type, integral operator, filtering.

from their convolutions $h * f$ in a stable way [19, 20], even though the convolution $h*$ could be one-to-one on $L^2(\mathbb{T})$ and its range space could be dense in $L^2(\mathbb{T})$. In signal processing, stable reconstruction is always required, while on the other hand signals have a priori information, such as piecewise trigonometric polynomials. This motivates us to finding closed linear subspaces V of $L^2(\mathbb{T})$ such that signals f living in those spaces can be reconstructed from their convolutions $h * f$ in a stable way.

Let us take a look at a toy linear subspace V of $L^2(\mathbb{T})$: the band-limited subspace $\mathbb{P}_N$ of $L^2(\mathbb{T})$ that contains all trigonometric polynomials of degree at most $N \ge 1$. One may verify that a convolution operator $h*$ has stability on the closed subspace $\mathbb{P}_N \subset L^2(\mathbb{T})$ if $\min_{|n|\le N} |\hat{h}(n)| > 0$. Moreover,

$$\big(\min_{|n|\le N} |\hat{h}(n)|\big)\|f\|_{L^2(\mathbb{T})} \le \|h * f\|_{L^2(\mathbb{T})} \le \big(\max_{|n|\le N} |\hat{h}(n)|\big)\|f\|_{L^2(\mathbb{T})} \tag{1.2}$$

for all $f \in \mathbb{P}_N \subset L^2(\mathbb{T})$, and the optimal lower and upper stability bounds of the convolution operator $h*$ are given by $\min_{|n|\le N} |\hat{h}(n)|$ and $\max_{|n|\le N} |\hat{h}(n)| \le \|h\|_{L^1(\mathbb{T})}$ respectively. Observe that if the function h is Hölder continuous, i.e.,

$$\|h(\cdot + y) - h(\cdot)\|_{L^\infty(\mathbb{T})} \le C_0|y|^\alpha \text{ for all } y \in \mathbb{T} \tag{1.3}$$

where $C_0 \in (0, \infty)$ and $\alpha \in (0, 1]$, then

$$|\hat{h}(n)| = \frac{1}{2}\Big|\int_{\mathbb{T}} \Big(h(x) - h\big(x + \frac{1}{2n}\big)\Big)\overline{e_n(x)}dx\Big| \le C_0 2^{-1-\alpha}|n|^{-\alpha} \tag{1.4}$$

for all $0 \ne n \in \mathbb{Z}$. Combining (1.2) and (1.4) leads to the existence of an absolute constant C, depending on the function h only, such that

$$\dim(\mathbb{P}_N) \le C\Big(\inf_{0\ne f\in\mathbb{P}_N} \frac{\|h * f\|_{L^2(\mathbb{T})}}{\|f\|_{L^2(\mathbb{T})}}\Big)^{-1/\alpha} \quad \text{for all } N \ge 1. \tag{1.5}$$

Here $\dim V$ denotes the dimension of a linear space V.

A time signal is said to have *finite rate of innovation* (FRI) if it has finitely many degrees of freedoms per unit of time [33]. Prototypical examples of FRI signals include slow varying signals with shot noise, very narrow pulses in ultrawide band communication, electrocardiogram signals, bump algebra model of spectra, and sparse signals. They also include familiar band-limited signals in Shannon sampling theorem, time signals in wavelet spaces, time-frequency signals in Gabor theory, and signals in reproducing kernel spaces used in the investigation of heart diseases, mass spectrum used to discover disease-related proteomic patterns etc [8, 14, 16, 17, 25, 27, 28, 33].

We observe that signals in $\mathbb{P}_N$ have finite rate of innovation and their rate of innovation $R(\mathbb{P}_N) := \frac{\dim \mathbb{P}_N}{|\mathbb{T}|}$ are $(2N+1)$, as any signal $f(x) = \sum_{n=-N}^{N} c(n)e_n(x) \in \mathbb{P}_N$ is completely determined by $(2N+1)$ parameters $c(n), -N \le n \le N$. We then conclude from (1.5) that the rate of innovation for signals living in a bandlimited subspace of $L^2(\mathbb{T})$ could be estimated by the optimal lower stability bound of the convolution operator $h*$ on that bandlimited space. In this paper, we extend the above exciting estimate to the rate of innovation for bandlimited signals to periodic signals living in arbitrary closed linear subspaces V of $L^p(\mathbb{T}), 1 \le p \le \infty$.

Theorem 1.1. *Let $1 \le p \le \infty$, and let h be a periodic function that satisfies the Hölder condition* (1.3) *for some exponent $\alpha \in (0, 1]$. Then there exists a positive constant C, depending on h and p only, such that*

$$R(V) := \frac{\dim V}{|\mathbb{T}|} \le C\Big(\inf_{0 \ne f \in V} \frac{\|h * f\|_{L^p(\mathbb{T})}}{\|f\|_{L^p(\mathbb{T})}}\Big)^{-1/\alpha}$$

for all closed linear subspaces V of $L^p(\mathbb{T})$.

By Theorem 1.1, the convolution operator $h*$ associated with a Hölder continuous periodic function h is not stable on any infinite-dimensional linear subspace of $L^p(\mathbb{T}), 1 \le p \le \infty$. The conclusion in Theorem 1.1 also provides a necessary condition for periodic signals f with their energy above δ to be observable from their filtered signals $h * f$ at the noisy level ϵ. The condition is that the rate of innovation for those periodic signals should not exceed $C(\delta/\epsilon)^{1/\alpha}$ for some positive constant C.

Next let us consider estimating rates of innovation for non-periodic signals in a closed linear subspace of $L^p(\mathbb{R}), 1 \le p \le \infty$. Here $L^p(\mathbb{R})$ is the space of all p-integrable functions on the real line $\mathbb{R}$ with its norm denoted by $\|\cdot\|_{L^p(\mathbb{R})}$. We start from considering another toy example:

$$\mathcal{S}_T := \Big\{ \sum_{k \in \mathbb{Z}} c(k)\chi_{[0,1)}(T \cdot -k) \Big| \sum_{k} |c(k)|^2 < \infty \Big\} \subset L^2(\mathbb{R})$$

where $T \ge 1$ and χ_E denotes the characteristic function on a set E. The rate of innovation $R_{a,b}(\mathcal{S}_T)$ for signals in $\mathcal{S}_T$ during the time range $[a, b)$ is $\lceil bT \rceil - \lfloor aT \rfloor$, as the restriction of such a signal f on that time range is completely determined by their evaluations $f(k/T), Ta - 1 < k < Tb$. Here $\lfloor t \rfloor$ is the integral part of a real number t and $\lceil t \rceil = -\lfloor -t \rfloor$. Thus the rate of innovation $R(\mathcal{S}_T)$ for signals living in $\mathcal{S}_T$ is

$$R(\mathcal{S}_T) := \limsup_{b-a \to \infty} \frac{R_{a,b}(\mathcal{S}_T)}{b-a} = T. \tag{1.6}$$

For a signal $f \in \mathcal{S}_T$ with $T \geq 1$ and a Hölder continuous function h with compact support,

$$\sup_{|y|\le 1} |y|^{-\alpha} \|h(\cdot + y) - h(\cdot)\|_{L^\infty(\mathbb{R})} < \infty \text{ for some } \alpha \in (0,1], \tag{1.7}$$

one may verify that

$$\inf_{0\ne f\in \mathcal{S}_T} \frac{\|h*f\|_2}{\|f\|_2} = \inf_{\xi\in\mathbb{R}} \Big(\sum_{k\in\mathbb{Z}} \Big|\hat{h}(T(\xi+k))\frac{e^{2\pi i\xi}-1}{2\pi i(\xi+k)}\Big|^2\Big)^{1/2} \le CT^{-\alpha} \tag{1.8}$$

for some positive constant C independent on $T \geq 1$. Here the Fourier transform $\hat{h}$ of an integrable function h on the real line is defined by $\hat{h}(\xi) = \int_{\mathbb{R}} e^{-2\pi i x\xi} h(x)dx$. Therefore for a compactly supported function h satisfying the Hölder condition (1.7), we obtain from (1.6) and (1.8) that the rate of innovation for non-periodic signals in $\mathcal{S}_T$ could be also estimated by the lower stability bound of the convolution operator $h*$ on that space,

$$R(\mathcal{S}_T) \le C\Big(\inf_{0\ne f\in\mathcal{S}_T} \frac{\|h*f\|_{L^2(\mathbb{R})}}{\|f\|_{L^2(\mathbb{R})}}\Big)^{-1/\alpha} \quad \text{for all } T \ge 1, \tag{1.9}$$

where C is an absolute constant independent of $T \geq 1$.

Let $1 \le p \le \infty$ and V be a locally-finite-dimensional linear subspace of $L^p(\mathbb{R})$ [4], and for a set E, denote by $V|_E$ the set of all restrictions of functions in V onto E. We define the rate of innovation $R(V)$ for signals living in V by

$$R(V) = \limsup_{b-a\to\infty} \frac{\dim V|_{[a,b)}}{b-a}.$$

There are several definitions for the rate of innovation. The above definition $R(V)$ is known in [33] as average rate of innovation for signals in V. We say that V has *limited edge effects* if for any time range $[a, b)$ and any function $f \in V$ there exists a function $g \in V$ supported in $[a - e_0, b + e_0)$ such that their restrictions on $[a, b)$ are identical, where the size e_0 of edge effective regions is independent of $f \in V$ and time range $[a, b)$. We remark that $\mathcal{S}_T, T \geq 1$, have limited edge effects, while the Paley-Wiener space of band-limited functions does not, see Remark 2.1. In this paper, we extend the estimate (1.9) to signals in locally-finite-dimensional linear subspaces V of $L^p(\mathbb{R})$ with limited edge effects, and then provide a necessary condition on non-periodic signals f in a closed space V that could be stably recovered from their filtered signals $h * f$.

Theorem 1.2. *Let $1 \le p \le \infty$ and let h be a compactly supported function satisfying the Hölder condition* (1.7) *for some $\alpha \in (0, 1]$. Then there exists a positive constant C, depending on h and p only, such that*

$$R(V) \le C\Big(\inf_{0\ne f\in V} \frac{\|h * f\|_{L^p(\mathbb{R})}}{\|f\|_{L^p(\mathbb{R})}}\Big)^{-1/\alpha}$$

for all closed locally-finite-dimensional linear subspaces V of $L^p(\mathbb{R})$ with limited edge effects.

The paper is organized as follows. In Section 2, we first recall the concept of spaces of homogenous type. An advantage of working with them is that they contain lots of our familiar spaces, such as the unit circle $\mathbb{T}$ (or general compact Riemannian manifolds with respect to geodesic metric), the real line (or high-dimensional Euclidean spaces with isotropic or anisotropic metrics), and the Cantor set (or self-similar fractals with Hausdorff measure) [11, 12, 15, 27, 30]. Then in that section we state our main theorems (Theorems 2.2 and 2.4) of this paper, which are generalizations of Theorems 1.1 and 1.2. In our main theorem, we provide universal estimates to the rate of innovation for signals living in closed linear spaces via the lower stability bounds of some time-varying filters (integral operators) on those spaces. In Section 3, we include the proofs of Theorems 2.2 and 2.4.

In this paper, $\#E$ denotes the cardinality of a set E; $\ell^p := \ell^p(\Lambda), 1 \le p \le \infty$, is the space of all p-summable vectors $(c(\lambda))_{\lambda\in\Lambda}$ with norm denoted by $\|\cdot\|_{\ell^p(\Lambda)}$ or $\|\cdot\|_p$ for brevity; and the capital letter C is an absolute constant which could be different at different occurrences.

2. Preliminaries and Main Results

Given a set X, a *quasi-metric* ρ on X is a function $\rho : X \times X \longmapsto [0, \infty)$ with the properties that

- (i) $\rho(x, y) = 0$ if and only if $x = y$;
- (ii) $\rho(x, y) = \rho(y, x)$ for all $x, y \in X$; and
- (iii) $\rho(x, y) \le L(\rho(x, z) + \rho(z, y))$ for all $x, y, z \in X$, where $L \ge 1$ is a positive constant.

The *quasi-metric space* (X, ρ) associated with a set X and a quasi-metric ρ on X is a topological space in which the family of balls

$$B(x, r) := \{y \in X \mid \rho(x, y) < r\}, \quad x \in X,\ r > 0,$$

forms a basis. Now we recall the definition of spaces of homogeneous type [11, 12].

Definition 2.1. A *space of homogenous type*, to be denoted by (X, ρ, μ), is a quasi-metric space (X, ρ) with a non-negative Borel measure μ that satisfies the following two conditions:

(i) the *doubling condition*

$$\mu(B(x, 2r)) \le D_1 \mu(B(x, r)) \text{ for all } x \in X \text{ and } r > 0; \tag{2.1}$$

and

(ii) the *uniform boundedness property*

$$D_2 \le \mu(B(x, 1)) \le D_3 \quad \text{for all } x \in X, \tag{2.2}$$

where D_1, D_2, D_3 are positive constants.

By (2.1) and (2.2), there exist positive constants $D_2(r)$ and $D_3(r), r > 0$, such that

$$D_2(r) \le \inf_{x\in X} \mu(B(x, r)) \le \sup_{x\in X} \mu(B(x, r)) \le D_3(r). \tag{2.3}$$

A measure μ on a quasi-metric space (X, ρ) is said to be *Ahlfors d-regular* for some $d > 0$ if there exist two positive constants A and B such that

$$Ar^d \le \mu(B(x, r)) \le Br^d \quad \text{for all } x \in X \text{ and } 0 < r \le \operatorname{diam}(X),$$

where $\operatorname{diam}(X)$ denotes the diameter of the space (X, ρ, μ) of homogeneous type. The reader may refer to [15, 30] and references therein for Ahlfors regular measure and its applications to Poincaré inequality and quasi-conformal geometry.

Let $1 \le p \le \infty$ and (X, ρ, μ) be a space of homogeneous type. Denote by $L^p(X, \rho, \mu)$, or L^p for brevity, the space of all p-integrable functions on (X, ρ, μ) with standard norm $\|\cdot\|_p$. Given a measurable kernel function K on $X \times X$, define

$$\|K\|_{\mathcal{S}} := \max\Big(\sup_{x\in X} \|K(x, \cdot)\|_1,\ \sup_{y\in X} \|K(\cdot, y)\|_1\Big),$$

and its *modulus of continuity* $\omega_\delta(K)$ by

$$\omega_\delta(K)(x, y) = \sup_{\rho(x',x)\le\delta,\ \rho(y',y)\le\delta} |K(x', y') - K(x, y)|.$$

We associate a kernel function K on $X \times X$ with an integral operator T on $L^p, 1 \le p \le \infty$, which is defined by

$$Tf(x) = \int_X K(x, y)f(y)d\mu(y), \quad f \in L^p.$$

For an integral operator T with its kernel K satisfying $\|K\|_{\mathcal{S}} < \infty$, applying Hölder inequality gives

$$\|Tf\|_p \le \|K\|_{\mathcal{S}} \|f\|_p \quad \text{for all } f \in L^p. \tag{2.4}$$

Given a closed linear subspace V of L^p and bounded operators $T_1, \ldots, T_M$ on L^p, $1 \le p \le \infty$, we say that $T_1, \ldots, T_M$ are *stable* on V if

$$0 < \inf_{g\in V, \|g\|_p=1} \sum_{m=1}^{M} \|T_m g\|_p \le \sup_{g\in V, \|g\|_p=1} \sum_{m=1}^{M} \|T_m g\|_p < \infty.$$

Theorem 2.2. *Let $1 \le p \le \infty$, (X, ρ, μ) be a space of homogeneous type with $\operatorname{diam} X < \infty$ and μ being Ahlfors d-regular for some $d > 0$, and let $T_1, \ldots, T_M$ be integral operators with kernels $K_1, \ldots, K_M$ satisfying*

$$\|K_m\|_{\mathcal{S}} \le C_0 \tag{2.5}$$

and

$$\|\omega_\delta(K_m)\|_{\mathcal{S}} \le C_0 \delta^\alpha \tag{2.6}$$

for all $0 < \delta \le \operatorname{diam} X$ and $1 \le m \le M$, where $C_0 \in (0, \infty)$ and $\alpha \in (0, d]$ are positive constants. If $T_1, \ldots, T_M$ are stable on a closed linear subspace $V \subset L^p(X, \rho, \mu)$, then

$$R(V) := \frac{\dim V}{\mu(X)} \le C\Big(\inf_{g\in V, \|g\|_p=1} \sum_{m=1}^{M} \|T_m g\|_p\Big)^{-d/\alpha}, \tag{2.7}$$

where C is an absolute constant independent on the space V and the volume $\mu(X)$.

Applying Theorem 2.2 to the unit circle $\mathbb{T}$ with Euclidean distance and Lebesgue measure, we have the following corollary about estimating the rate of innovation for periodic signals via the lower stability bound of a time-varying filtering procedure.

Corollary 2.3. *Let $1 \le p \le \infty$, and let T be an integral operator with kernel K satisfying*

$$|K(x, y)| \le C_0 \quad \text{and} \quad |K(x', y') - K(x, y)| \le C_0(|x - x'|^\alpha + |y - y'|^\alpha)$$

for all $x, x', y, y' \in \mathbb{T}$, where $C_0 \in (0, \infty)$ and $\alpha \in (0, 1]$ are positive constants. Then there exists an absolute constant C, depending on T and p only, such that

$$R(V) := \frac{\dim V}{|\mathbb{T}|} \le C\Big(\inf_{g\in V, \|g\|_{L^p(\mathbb{T})}=1} \|Tg\|_{L^p(\mathbb{T})}\Big)^{-1/\alpha}$$

hold for all stable closed linear subspaces $V \subset L^p(\mathbb{T})$.

Theorem 1.1 follows from Corollary 2.3 with the kernel (time-varying filter) $K(x, y)$ replaced by (time-invariant filter) $h(x - y)$.

Now let us consider spaces (X, ρ, μ) of homogeneous type with infinite diameter (i.e., $\text{diam}X = \infty$), and a locally-finite-dimensional linear subspace V of $L^p(X, \rho, \mu), 1 \le p \le \infty$. We define rate of innovation $R(V)$ for signals living in V by

$$R(V) = \limsup_{r\to\infty} \sup_{x_0\in X} \frac{\dim V|_{B(x_0,r)}}{\mu(B(x_0, r))}. \tag{2.8}$$

We say that V has *limited edge effects* if for any ball $B(x_0, r)$ and any function $f \in V$ there exists a function $g \in V$ supported in $B(x_0, r + e_0)$ such that their restrictions on $B(x_0, r)$ are identical, where e_0 is independent of balls $B(x_0, r)$ and functions $f \in V$, see Remark 2.1.

Theorem 2.4. *Let $1 \le p \le \infty$, (X, ρ, μ) be a space of homogeneous type with* $\text{diam}X = \infty$ *and μ being Ahlfors d-regular for some $d > 0$, and let $T_1, \ldots, T_M$ be integral operators with kernels $K_1, \ldots, K_M$ satisfying*

$$K_m(x, y) = 0 \quad \text{for all } x, y \in X \text{ with } \rho(x, y) \ge R_0, \tag{2.9}$$

$$|K_m(x, y)| \le C_1 \quad \text{for all } x, y \in X, \tag{2.10}$$

and

$$|K_m(x, y) - K_m(x', y')| \le C_1(\rho(x, x') + \rho(y, y'))^\alpha \tag{2.11}$$

for all $x, x', y, y' \in X$ and $1 \le m \le M$, where $R_0, C_1 \in (0,\infty)$ and $\alpha \in (0, d]$ are positive constants. If $T_1, \ldots, T_M$ are stable on a closed locally-finite-dimensional linear subspace $V \subset L^p(X, \rho, \mu)$ with limited edge effects, then there exists an absolute constant C, independent on the space V, such that

$$R(V) \le C\Big(\inf_{g\in V,\|g\|_p=1} \sum_{m=1}^{M} \|T_m g\|_p\Big)^{-d/\alpha}. \tag{2.12}$$

Theorem 1.2 follows from Theorem 2.4 with the space of homogeneous type replaced by the real line with Euclidean distance and Lebesgue measure, and integral operators by the convolution operator $h*$.

Let $\Lambda := \{\lambda_i\}_{i=-\infty}^{\infty}$ satisfy

$$0 < \inf_i(\lambda_{i+1} - \lambda_i) \le \sup_i(\lambda_{i+1} - \lambda_i) < \infty, \tag{2.13}$$

and let $\mathcal{S}_{\Lambda,N} \subset L^2(\mathbb{R}), N \ge 1$, be the space of all non-uniform splines of order N having $N - 1$ continuity at each knot $\lambda_i \in \Lambda$ [3, 22, 26, 29, 32]. One may verify that the rate of innovation $R(\mathcal{S}_{\Lambda,N})$ for spline signals in $\mathcal{S}_{\Lambda,N}$ is the same as the Beurling upper density $D^+(\Lambda) := \limsup_{b-a\to\infty} \frac{\#(\Lambda\cap[a,b])}{b-a}$ of the set Λ, i.e.,

$$R(\mathcal{S}_{\Lambda,N}) = D^+(\Lambda).$$

This, together with Theorem 2.4, leads to the following corollary about the Beurling upper density.

Corollary 2.5. *Let $1 \le p \le \infty, N \ge 1$, and let h be a compactly supported function satisfying the Hölder condition* (1.7) *for some $\alpha \in (0,1]$. Then there exists a positive constant C, depending on h, p and N only, such that*

$$D^+(\Lambda) \le C\Big(\inf_{g\in\mathcal{S}_{\Lambda,N}, \|g\|_{L^p(\mathbb{R})}=1} \|h * g\|_{L^p(\mathbb{R})}\Big)^{-1/\alpha}$$

for all sets $\Lambda := \{\lambda_i\}_{i=-\infty}^{\infty}$ satisfying (2.13).

We finish this section with a remark on limited edge effects.

Remark 2.1. The Paley-Wiener spaces

$$B_\sigma = \{f \in L^2(\mathbb{R}) :\ \hat{f}(\xi) = 0 \text{ for all } \xi \notin [-\sigma,\sigma]\},\ \sigma > 0,$$

are not locally-finite-dimensional, and they do not have limited edge effects. But there are a lot of linear spaces with limited edge effects for time signals with finite rate of innovation to reside. Here are a few of them:

(1) The shift-invariant space

$$V_p(\varphi_1,\ldots,\varphi_M) = \Big\{\sum_{m=1}^{M}\sum_{k\in\mathbb{Z}} c_m(k)\varphi_m(\cdot - k) :\ (c_m(k))_{k\in\mathbb{Z}} \in \ell^p(\mathbb{Z})\Big\}$$

generated by compactly supported functions $\varphi_1,\ldots,\varphi_M \in L^p(\mathbb{R})$ [2, 5, 6, 10, 26, 31].

(2) The space

$$V(\Phi,\Lambda) = \Big\{\sum_{\lambda\in\Lambda} c(\lambda)\phi_\lambda(\cdot - \lambda) :\ c \in \ell^p(\Lambda)\Big\}$$

containing all superpositions of impulse response ϕ_λ of the signal generating device at the location $\lambda, \lambda \in \Lambda$, via coefficients in $\ell^p(\Lambda)$, where Λ is a discrete subset of $\mathbb{R}$ with $\sum_{\lambda\in\Lambda}\chi_{\lambda+[0,1)} \in L^\infty(\mathbb{R})$ and $\phi_\lambda, \lambda \in \Lambda$, are supported in a compact set K and have bounded $L^p(\mathbb{R})$ norm [7, 8, 13, 17, 24, 25, 27, 28].

(3) The range space

$$V_p = \big\{Tf :\ f \in L^p(\mathbb{R})\big\}$$

of an idempotent integral operator T,

$$Tf(x) = \int_{\mathbb{R}} K(x,y)f(y)dy, \ f \in L^p(\mathbb{R}),$$

with kernel K being bounded and supported in a fixed compact neighborhood of the diagonal, i.e., (2.9) and (2.10) hold, [9, 18].

3. Proofs

We recall finite covering and partition of the unity on a space of homogenous type in the first subsection, we present a technical lemma in the second subsection, and we provide the proofs of Theorems 2.2 and 2.4 in last two subsections.

3.1. **Finite covering and partition of the unity.** Let (X,ρ,μ) be a space of homogeneous type, and let $L \ge 1$ be the constant in the definition of the quasi-metric ρ. For any $\delta > 0$, define

$$\mathbf{H}_\delta := \{B(x,\delta/(2L)) : \ x \in X\}$$

and let

$$\mathbf{G}_\delta := \{B(x_i,\delta/(2L)) : \ x_i \in X_\delta\} \subset \mathbf{H}_\delta \tag{3.1}$$

be a maximal disjoint subcollection of $\mathbf{H}_\delta$, i.e.,

$$B(x_i,\delta/(2L)) \cap B(x_{i'},\delta/(2L)) = \emptyset \quad \text{for all } x_i \ne x_{i'}, \tag{3.2}$$

and

$$B(x,\delta/(2L)) \cap \big(\cup_{x_i \in X_\delta} B(x_i,\delta/(2L))\big) \ne \emptyset \quad \text{for all } x \in X. \tag{3.3}$$

For a discrete subset Γ of a space (X,ρ,μ) of homogeneous type, we define

$$A_\Gamma(\delta) = \inf_{x\in X} \sum_{\gamma\in\Gamma} \chi_{B(\gamma,\delta)}(x)$$

and

$$B_\Gamma(\delta) = \sup_{x\in X} \sum_{\gamma\in\Gamma} \chi_{B(\gamma,\delta)}(x).$$

For any $x \in X$, there exists $x_i \in X_\delta$ by (3.3) such that $B(x,\delta/(2L)) \cap B(x_i,\delta/(2L)) \ne \emptyset$. Therefore $\rho(x,x_i) \le L(\rho(x,z)+\rho(z,x_i)) \le \delta$ where $z \in B(x,\delta/(2L)) \cap B(x_i,\delta/(2L))$. This proves that $\{B(x_i,\delta)\}_{x_i\in X_\delta}$ is a covering of the set X, i.e.,

$$X \subset \cup_{x_i \in X_\delta} B(x_i,\delta).$$

For any $x_i, x_j \in X_\delta \cap B(x,\delta)$ and $z \in B(x_j,\delta/(2L))$,

$$\rho(z,x_i) \le L^2\rho(z,x_j) + L^2\rho(x_j,x) + L\rho(x,x_i) \le (L^2+3L/2)\delta,$$

which implies that

$$\cup_{\rho(x_j,x)<\delta, x_j\in X_\delta} B(x_j,\delta/(2L)) \subset B(x_i,(L^2+3L/2)\delta)$$

for all $x\in X$, where $x_i\in X_\delta$ satisfies $\rho(x_i,x)<\delta$. Hence

$$\begin{aligned} 1 \ &\le\ A_{X_\delta}(\delta)\le B_{X_\delta}(\delta)\le \frac{\mu\big(\cup_{\rho(x_j,x)<\delta,x_j\in X_\delta} B(x_j,\delta/(2L))\big)}{\inf_{\rho(x_j,x)<\delta,x_j\in X_\delta}\mu\big(B(x_j,\delta/(2L))\big)} \\ &\le \sup_{\rho(x_j,x)<\delta,x_j\in X_\delta} \frac{\mu\big(B(x_j,(L^2+3L/2)\delta)\big)}{\mu\big(B(x_j,\delta/(2L))\big)} \le D_1^{3\log_2 L+4}<\infty \end{aligned} \tag{3.4}$$

(and $\{B(x_i,\delta)\}_{x_i\in X_\delta}$ is a *finite covering* of the set X), where D_1 is the constant in (2.1) for the doubling measure μ.

We may associate a finite covering $\{B(\gamma,\delta)\}_{\gamma\in\Gamma}$ of the set X with a family $U:=\{h_\gamma\}_{\gamma\in\Gamma}$ of functions on X,

$$h_\gamma(x):=\frac{\chi_{B(\gamma,\delta)}(x)}{\sum_{\gamma'\in\Gamma}\chi_{B(\gamma',\delta)}(x)},\ \gamma\in\Gamma, \tag{3.5}$$

that forms a *partition of the unity*,

$$\begin{cases} 0\le h_\gamma(x)\le 1 \quad \text{for all } x\in X \text{ and } \gamma\in\Gamma,\\ h_\gamma(x)=0 \text{ if } x\notin B(\gamma,\delta), \text{ and}\\ \sum_{\gamma\in\Gamma} h_\gamma(x)=1 \quad \text{for all } x\in X. \end{cases} \tag{3.6}$$

Moreover,

$$\frac{D_2(\delta)}{B_\Gamma(\delta)}\le \|h_\gamma\|_1\le \frac{D_3(\delta)}{A_\Gamma(\delta)} \quad \text{for all } \gamma\in\Gamma \tag{3.7}$$

by (2.3).

3.2. **A technical lemma.** To prove Theorems 2.2 and 2.4, we need a technical lemma.

Lemma 3.1. *Let $1\le p\le\infty$, (X,ρ,μ) be a space of homogeneous type, and let $T_1,\ldots,T_M$ be integral operators with kernels $K_1,\ldots,K_M$ satisfying*

$$\|K_m\|_{\mathcal S}<\infty,\quad 1\le m\le M. \tag{3.8}$$

Then for any closed linear subspace V of $L^p(X,\rho,\mu)$ and any discrete set Γ with $1\le A_\Gamma(\delta_0)\le B_\Gamma(\delta_0)<\infty$ for some $\delta_0>0$, we have that

$$\Big(\frac{A_\Gamma(\delta_0)}{D_3(\delta_0)}\Big)^{1/p}\Big(\inf_{g\in V,\|g\|_p=1}\sum_{m=1}^M\|T_mg\|_p-\sum_{m=1}^M\|\omega_{\delta_0}(K_m)\|_{\mathcal S}\Big)\|f\|_p$$
$$\le\sum_{m=1}^M\big\|\big(T_mf(\gamma)\big)_{\gamma\in\Gamma}\big\|_p\le\Big(\frac{B_\Gamma(\delta_0)}{D_2(\delta_0)}\Big)^{1/p}$$
$$\times\Big(\sup_{g\in V,\|g\|_p=1}\sum_{m=1}^M\|T_mg\|_p+\sum_{m=1}^M\|\omega_{\delta_0}(K_m)\|_{\mathcal S}\Big)\|f\|_p\quad\text{for all } f\in V. \tag{3.9}$$

Proof. Let $\delta_0>0$ and Γ be a discrete subset of X with

$$1\le A_\Gamma(\delta_0)\le B_\Gamma(\delta_0)<\infty. \tag{3.10}$$

Associated with the finite covering $\{B(\gamma,\delta_0)\}_{\gamma\in\Gamma}$, the family $\{h_\gamma\}_{\gamma\in\Gamma}$ of functions in (3.5) forms a partition of the unity (3.6) and satisfies (3.7). For $1\le m\le M$,

$$\|\omega_{\delta_0}(T_mf)\|_p\le\|\omega_{\delta_0}(K_m)\|_{\mathcal S}\|f\|_p\quad\text{for all } f\in L^p, \tag{3.11}$$

by (2.4), and

$$|T_mf(x)-T_mf(\gamma)|\le\omega_{\delta_0}(T_mf)(x)\ \ \text{for all } x\in B(\gamma,\ \delta_0) \text{ and } \gamma\in\Gamma, \tag{3.12}$$

by the definition of the modulus of continuity.

The conclusion (3.9) for $p=\infty$ follows directly from (3.10), (3.11) and (3.12).

Now we consider $1\le p<\infty$. We obtain from (3.6), (3.7), (3.11) and (3.12) that

$$\sum_{m=1}^M\|(T_mf(\gamma))_{\gamma\in\Gamma}\|_p\le\Big(\frac{B_\Gamma(\delta_0)}{D_2(\delta_0)}\Big)^{1/p}\sum_{m=1}^M\Big(\sum_{\gamma\in\Gamma}|T_mf(\gamma)|^p\|h_\gamma\|_1\Big)^{1/p}$$
$$\le\Big(\frac{B_\Gamma(\delta_0)}{D_2(\delta_0)}\Big)^{1/p}\sum_{m=1}^M\big\||T_mf|+\omega_{\delta_0}(T_mf)\big\|_p$$
$$\le\Big(\frac{B_\Gamma(\delta_0)}{D_2(\delta_0)}\Big)^{1/p}\Big(\sum_{m=1}^M\|T_mf\|_p+\sum_{m=1}^M\|\omega_{\delta_0}(K_m)\|_{\mathcal S}\|f\|_p\Big)$$
$$\le\Big(\frac{B_\Gamma(\delta_0)}{D_2(\delta_0)}\Big)^{1/p}\Big(\sup_{g\in V,\|g\|_p=1}\big(\sum_{m=1}^M\|T_mg\|_p\big)+\sum_{m=1}^M\|\omega_{\delta_0}(K_m)\|_{\mathcal S}\Big)\|f\|_p$$

and

$$\begin{aligned}
&\Big(\frac{A_\Gamma(\delta_0)}{D_3(\delta_0)}\Big)^{1/p}\inf_{g\in V, \|g\|_p=1}\Big(\sum_{m=1}^M \|T_m g\|_p\Big)\|f\|_p\\
\le\ & \Big(\frac{A_\Gamma(\delta_0)}{D_3(\delta_0)}\Big)^{1/p}\Big(\sum_{m=1}^M \|T_m f\|_p\Big)\\
\le\ & \Big(\frac{A_\Gamma(\delta_0)}{D_3(\delta_0)}\Big)^{1/p}\Big(\sum_{m=1}^M \Big(\sum_{\gamma\in\Gamma}|T_m f(\gamma)|^p\|h_\gamma\|_1\Big)^{1/p}+\sum_{m=1}^M\|\omega_{\delta_0}(T_m f)\|_p\Big)\\
\le\ & \sum_{m=1}^M \big\|\big(T_m f(\gamma)\big)_{\gamma\in\Gamma}\big\|_p+\Big(\frac{A_\Gamma(\delta_0)}{D_3(\delta_0)}\Big)^{1/p}\Big(\sum_{m=1}^M\|\omega_{\delta_0}(K_m)\|_{\mathcal S}\Big)\|f\|_p
\end{aligned}$$

for all $f\in V$. This proves (3.9) for $1\le p<\infty$ and completes the proof. □

We finish this subsection with a remark on the stability of integral operators.

Remark 3.1. Bounded operators $T_1,\ldots,T_M$ on L^p are said to be *stable samplers* on $V\subset L^p$ for sampling sets having small gap if there exists a sufficiently small positive number δ_0 such that

(3.13)
$$0<\inf_{g\in V,\|g\|_p=1}\sum_{m=1}^M\big\|\big(T_m g(\gamma)\big)_{\gamma\in\Gamma}\big\|_p\le \sup_{g\in V,\|g\|_p=1}\sum_{m=1}^M\big\|\big(T_m g(\gamma)\big)_{\gamma\in\Gamma}\big\|_p<\infty$$

for any sampling set Γ with sampling gap $\delta\in(0,\delta_0)$ (i.e., $1\le A_\Gamma(\delta)\le B_\Gamma(\delta)<\infty$). By Lemma 3.1, integral operators $T_1,\ldots,T_M$ with kernels $K_m, 1\le m\le M$, satisfying

$$\|K_m\|_{\mathcal S}<\infty \quad\text{and}\quad \lim_{\delta\to 0}\|\omega_\delta(K_m)\|_{\mathcal S}=0,\quad 1\le m\le M$$

are stable samplers on V for sampling sets having small gap. Following the argument in [5], we can show that the converse is also true if kernels $K_m, 1\le m\le M$, satisfy

$$\|K_m\|_{\mathcal W}<\infty \quad\text{and}\quad \lim_{\delta\to 0}\|\omega_\delta(K_m)\|_{\mathcal W}=0,\quad 1\le m\le M$$

and the measure μ is Ahlfors d-regular for some $d>0$. Here for a kernel function K on $X\times X$, its *Wiener amalgam norm* $\|K\|_{\mathcal W}$ is defined by

$$\|K\|_{\mathcal W}:=\max\Big(\sup_{x\in X}\Big\|\sup_{z\in B(\cdot,1)}|K(x,z)|\Big\|_1,\ \sup_{y\in X}\Big\|\sup_{z\in B(\cdot,1)}|K(z,y)|\Big\|_1\Big).$$

This generalizes the equivalence between stability on V and stable samplers on V for sampling sets with sufficiently small gap in [5, 18], where the linear space V is assumed to be a finitely-generated shift-invariant space and a reproducing kernel subspace of $L^p(\mathbb{R}^d)$ respectively.

3.3. **Proof of Theorem 2.2.** Without loss of generality, we assume that $T_1, \ldots, T_M$ are stable on $V \subset L^p$. Let δ_0 be so chosen that

$$C_0 M \delta_0^\alpha = \frac{1}{2} \inf_{g\in V, \|g\|_p=1} \sum_{m=1}^M \big\|T_m g\|_p > 0, \tag{3.14}$$

where C_0 is the constant in (2.5) and (2.6). Then $\delta_0 \in (0,1)$ by (2.4), (2.5) and the stability assumption for integral operators $T_1, \ldots, T_M$. Let X_{δ_0} be as in (3.1) with $\delta = \delta_0$, and set $\Gamma = X_{\delta_0}$. Then $1 \le A_\Gamma(\delta_0) \le B_\Gamma(\delta_0) \le D_1^{3\log_2 L+4}$ by (3.4). This, together with the Ahlfors d-regularity of the measure μ, implies that

$$\begin{aligned} A\delta_0^d \#\Gamma &\le \sum_{\gamma\in\Gamma} \mu(B(\gamma,\delta_0)) \le \int_X \sum_{\gamma\in\Gamma} \chi_{B(\gamma,\delta_0)}(x) d\mu(x) \\ &\le B_\Gamma(\delta_0)\mu(X) \le D_1^{3\log_2 L+4}\mu(X), \end{aligned} \tag{3.15}$$

where A is a lower bound for the Ahlfors d-regular measure μ. By (2.6), (3.14) and Lemma 3.1,

$$\sum_{m=1}^M \|(T_m f(\gamma))_{\gamma\in\Gamma}\|_p \ge \frac{1}{2}\Big(\frac{A_\Gamma(\delta_0)}{D_3(\delta_0)}\Big)^{1/p}\big(\inf_{g\in V,\|g\|_p=1}\sum_{m=1}^M \big\|T_m g\|_p\big)\|f\|_p > 0$$

for all $0 \ne f \in V$. Then

$$\dim V \le M\#\Gamma. \tag{3.16}$$

Combining (3.14), (3.15) and (3.16) proves (2.7). This completes the proof.

3.4. **Proof of Theorem 2.4.** Without loss of generality, we assume that $T_1, \ldots, T_M$ are stable on V. Take $x_0 \in X$ and $r > 0$. Find a basis $f_i, 1 \le i \le N$, for the linear space $V|_{B(x_0,r)}$, which contains all restrictions of functions in V on $B(x_0, r)$. By the limited edge effect assumption, there exist $\tilde{f}_1, \ldots, \tilde{f}_N \in V$ such that the restriction of $\tilde{f}_i$ coincides with f_i for every $i \in \{1, 2, \ldots, N\}$, and

$$\tilde{f}_1, \ldots, \tilde{f}_N \in V \text{ are supported on } B(x_0, r + e_0). \tag{3.17}$$

Let $W(x_0, r)$ be the linear space spanned by $\tilde{f}_1, \ldots, \tilde{f}_N$. Due to the linear independence for the restrictions of $\tilde{f}_1, \ldots, \tilde{f}_N$ on $B(x_0, r)$,

$$\dim W(x_0, r) = \dim V|_{B(x_0,r)}. \tag{3.18}$$

By (2.9), (2.10) and (2.11), we have that

$$|\omega_\delta(K_m)(x, y)| \le C_1(2\delta)^\alpha \quad \text{for all } x, y \in X,$$

and

$$\omega_\delta(K_m)(x,y)=0 \quad \text{for all } x,y\in X \text{ with } \rho(x,y)\ge L^2R_0+(L+1)\delta.$$

This, together with the Ahlfors d-regularity of the measure μ, implies that

$$\sum_{m=1}^{M}\|\omega_\delta(K_m)\|_{\mathcal S}\le BMC_1(L^2R_0+(L+1)\delta)^d(2\delta)^\alpha. \tag{3.19}$$

Let $\delta_0\in(0,1)$ be so chosen that

$$BMC_1(L^2R_0+L+1)^d(2\delta_0)^\alpha=\frac{1}{2}\inf_{g\in V,\|g\|_p=1}\sum_{m=1}^{M}\|T_mg\|_p>0. \tag{3.20}$$

The existence of such a number $\delta_0\in(0,1)$ follows as $T_1,\dots,T_M$ are stable on V, and

$$\|T_mg\|_p\le BC_1(R_0)^d\|g\|_p \quad \text{for all } g\in V$$

by (2.4). Let X_{δ_0} be as in (3.1) with $\delta=\delta_0<1$. Applying Lemma 3.1 and using (3.19) and (3.20), we obtain

$$\sum_{m=1}^{M}\|(T_mf(\gamma))_{\gamma\in X_{\delta_0}}\|_p\ge\frac{1}{2}\Big(\frac{A_\Gamma(\delta_0)}{D_3(\delta_0)}\Big)^{1/p}\Big(\inf_{g\in V,\|g\|_p=1}\sum_{m=1}^{M}\|T_mg\|_p\Big)\|f\|_p \tag{3.21}$$

for all $f\in W(x_0,r)\subset V$. From the support assumption (2.9) for kernels $K_m, 1\le m\le M$, and the support property (3.17) for functions $f\in W(x_0,r)$, it follows that

$$T_mf(\gamma)=0 \quad \text{for all } \rho(\gamma,x_0)>L(r+R_0+e_0). \tag{3.22}$$

Combining (3.21) and (3.22) leads to

$$\sum_{m=1}^{M}\|(T_mf(\gamma))_{\gamma\in X_{\delta_0}\cap B(x_0,L(r+R_0+e_0))}\|_p>0 \quad \text{for all } 0\ne f\in W(x_0,r),$$

which, in turn, implies that

$$\dim W(x_0,r)\le M\#\Gamma_{x_0,r}, \tag{3.23}$$

where $\Gamma_{x_0,r}=X_{\delta_0}\cap B(x_0,L(r+R_0+e_0))$. Following the argument in (3.15), we obtain that

$$\begin{aligned} A\delta_0^d\#\Gamma_{x_0,r} &\le \int_{B(x_0,L^2(r+R_0+e_0)+L\delta_0)}\sum_{\gamma\in\Gamma_{x_0,r}}\chi_{B(\gamma,\delta_0)}(x)d\mu(x)\\ &\le B_\Gamma(\delta_0)\mu(B(x_0,L^2(r+R_0+e_0)+L\delta_0))\\ &\le BD_1^{3\log_2 L+4}(L^2(r+R_0+e_0)+L)^d. \qquad (3.24)\end{aligned}$$

Combining (2.8), (3.18), (3.23) and (3.24), we have that

$$\begin{aligned}
R(V) &= \limsup_{r\to\infty} \sup_{x_0\in X} \frac{\dim V|_{B(x_0,r)}}{\mu(B(x_0,r))} = \limsup_{r\to\infty} \sup_{x_0\in X} \frac{\dim W(x_0,r)}{\mu(B(x_0,r))} \\
&\le M \limsup_{r\to\infty} \sup_{x_0\in X} \frac{\#\Gamma_{x_0,r}}{\mu(B(x_0,r))} \\
&\le \frac{MBD_1^{3\log_2 L+4}}{A\delta_0^d} \limsup_{r\to\infty} \sup_{x_0\in X} \frac{(L^2(r+R_0+e_0)+L)^d}{\mu(B(x_0,r))} \\
&\le MBA^{-2}D_1^{3\log_2 L+4} L^{2d}\delta_0^{-d}.
\end{aligned}$$

This together with (3.20) proves (2.12), and hence completes the proof of Theorem 2.4.

Acknowledgement: The authors would like to thank the reviewers for their remarks and suggestions. This research is partially supported by National Science Foundation (DMS-1109063) and Guangdong Provincial Government of China through the Computational Science Innovative Research Team program and Guangdong Province Key Laboratory of Computational Science at the Sun Yat-sen University.

References

[1] A. Aldroubi, A. Baskakov and I. Krishtal, Slanted matrices, Banach frames, and sampling, *J. Funct. Anal.*, **255**(2008), 1667–1691.

[2] A. Aldroubi and K. Gröchenig, Nonuniform sampling and reconstruction in shift-invariant spaces, *SIAM Rev.*, **43**(2001), 585–620.

[3] A. Aldroubi and K. Gröchenig, Beurling-Landau-type theorems for non-uniform sampling in shift invariant spline spaces, *J. Fourier Anal. Appl.*, **6**(2000), 91–101.

[4] A. Aldroubi and Q. Sun, Locally finite dimensional shift-invariant spaces in $\mathbb{R}^d$, *Proc. Amer. Math. Soc.*, **130**(2002), 2641–2654.

[5] A. Aldroubi, Q. Sun and W.-S. Tang, Convolution, average sampling, and a Calderon resolution of the identity for shift-invariant spaces, *J. Fourier. Anal. Appl*, **22**(2005), 215–244.

[6] A. Aldroubi, Q. Sun and W.-S. Tang, p-frames and shift invariant subspaces of L^p, *J. Fourier Anal. Appl.*, **7**(2001), 1–21.

[7] N. D. Atreas, On a class of non-uniform average sampling expansions and partial reconstruction in subspaces of $L^2(\mathbb{R})$, *Adv. Comput. Math.*, **36**(2012), 21–38.

[8] N. Bi, M. Z. Nashed and Q. Sun, Reconstructing signals with finite rate of innovation from noisy samples, *Acta Appl. Math.*, **107**(2009), 339–372.

[9] J. G. Christensen, Sampling in reproducing kernel Banach spaces on Lie groups, *J. Approx. Theory*, **164**(2012), 179–203.

[10] C. K. Chui and Q. Sun, Affine frame decompositions and shift-invariant spaces, *Appl. Computat. Harmonic Anal.*, **20**(2006), 74–107.

[11] R. Coifman and G. Weiss, *Analyses Harmoniques Noncommutative sur Certains Espaces Homogenes*, Springer, 1971.
[12] D. Deng and Y. Han, *Harmonic Analysis on Spaces of Homogeneous Type*, Springer, 2008.
[13] D. Donoho, Compressive sampling, *IEEE Trans. Inform. Theory*, **52**(2006), 1289–1306.
[14] P. L. Dragotti, M. Vetterli and T. Blu, Sampling moments and reconstructing signals of finite rate of innovation: Shannon meets Strang-Fix, *IEEE Trans. Signal Processing*, **55**(2007), 1741–1757.
[15] S. Keith and X. Zhong, The Poincaré inequality is an open ended condition, *Ann. Math.*, **167**(2008), 575–599.
[16] I. Maravic and M. Vetterli, Sampling and reconstruction of signals with finite rate of innovation in the presence of noise, *IEEE Trans. Signal Processing*, **53**(2005), 2788–2805.
[17] T. Michaeli and Y. C. Eldar, Xampling at the rate of innovation, *IEEE Trans. Signal Processing*, **60**(2011), 1121–1133.
[18] M. Z. Nashed and Q. Sun, Sampling and reconstruction of signals in a reproducing kernel subspace of $L^p(\mathbb{R}^d)$, *J. Funct. Anal.*, **258**(2010), 2422–2452.
[19] M. Z. Nashed, Q. Sun and W.-S. Tang, Average sampling in L^2, *C. Acad. Sci. Paris, Ser I*, **347**(2009), 1007–1010.
[20] A. Olevskii and A. Ulanovskii, Almost integer translates. do nice generators exist?, *J. Fourier Anal. Appl.*. **10**(2004), 93–104.
[21] K. S. Rim, C. E. Shin and Q. Sun, Stability of localized integral operators on weighted L^p spaces, *Numer. Funct. Anal. Optim.*, **33**(2012), 1166–1193.
[22] L. L. Schumaker, *Spline Functions: Basic Theory*, John Wiley & Sons, 1981.
[23] C. E. Shin and Q. Sun, Stability of localized operators, *J. Funct. Anal.*, **256**(2009), 2417–2439.
[24] P. Shukla and P. L. Dragotti, Sampling schemes for multidimensional signals with finite rate of innovation, *IEEE Trans. on Signal Processing*, **55**(2007), 3670–3686.
[25] Q. Sun, Localized nonlinear functional equations and two sampling problems in signal processing, *Adv. Comput. Math.*, accepted. DOI: 10.1007/s10444-013-9314-3
[26] Q. Sun, Local reconstruction for sampling in shift-invariant spaces, *Adv. Comput. Math.*, **32**(2010), 335–352.
[27] Q. Sun, Frames in spaces with finite rate of innovations, *Adv. Comput. Math.*, **28**(2008), 301–329.
[28] Q. Sun, Non-uniform sampling and reconstruction for signals with finite rate of innovations, *SIAM J. Math. Anal.*, **38**(2006/07), 1389–1422.
[29] W. C. Sun and X. W. Zhou, Characterization of local sampling sequences for spline subspaces, *Adv. Comput. Math.*, **30**(2009), 153–175.
[30] J. T. Tyson, Metric and geometric quasiconformality in Ahlfors regular Loewner spaces, *Conform. Geom. Dyn.*, **5**(2001), 21–73.
[31] M. Unser, Sampling – 50 years after Shannon, *Proc. IEEE*, **88**(2000), 569–587.
[32] J. Xian and W. C. Sun, Local sampling and reconstruction in shift-invariant spaces and their applications in spline subspaces, *Numer. Funct. Anal. Optim,* **3**1(2010), 366–386.

[33] M. Vetterli, P. Marziliano, and T. Blu, Sampling signals with finite rate of innovation, *IEEE Trans. Signal Processing*, **50**(2002), 1417–1428.

Q. Sun, Department of Mathematics, University of Central Florida, Orlando, FL 32816, USA
E-mail address: qiyu.sun@ucf.edu

J. Xian, Department of Mathematics and Guangdong Province Key Laboratory of Computational Science, Sun Yat-sen (Zhongshan) University, Guangzhou, 510275, China
E-mail address: xianjun@mail.sysu.edu.cn