Localized Stability Analysis and Design of Symmetric Spatially Distributed Systems over Sparse Proximity Graphs*

Nader Motee† and Qiyu Sun‡

Abstract— **The focus of this paper is on the finite or infinite dimensional class of spatially distributed linear systems with Hermitian and sparse state matrices. We show that exponential stability of this class of systems can be inferred in a decentralized and spatially localized manner, which is practically relevant to many real-world applications (e.g., systems with spatially discredited PDE models). Then, we obtain several sufficient conditions that allow us to adjust strength of existing couplings in a network in order to sparsify or grow a network, while ensuring global stability. Our proposed necessary and sufficient stability certificates are independent of the dimension of the entire system. Moreover, they only require localized knowledge about the state matrix of the system, which makes these verifiable conditions desirable for design of robust spatially distributed linear systems against subsystem failure and replacement.**

I. Introduction

The interest in stability verification of infinite-dimensional linear systems and distributed parameter systems dates back to couple of decades ago [4], [5], [9], where the focus has been on characterization of stability conditions in a centralized setting. Some recent studies show that for a class of spatially distributed systems, so called spatially decaying systems, stability verification can be potentially localized using spatial truncation techniques [15]–[17]. The goal of this paper is to propose spatially localized and decentralized conditions for exponential stability of finite- or infinite-dimensional spatially distributed systems. Our main focus is on the class of systems with band (sparse) and Hermitian state matrices, i.e., matrices whose rows and columns contain only a few nonzero entries and they are Hermitian.

Our Contributions: First, we provide centralized quantitative characterizations of exponential stability property of spatially distributed linear systems with finite bandwidth (i.e., sparse) and Hermitian state matrices. We show that some of these characterizations are more amenable to localized and decentralized verification schemes. Next, we derive a necessary condition for exponential stability that can be implemented locally at subsystem level using only local information. Our main contribution is introduction of some locally verifiable sufficient conditions for exponential stability. It is proven that these sufficient conditions are also necessary and almost optimal. The significant feature of our localized verifiable conditions is that they depend only on the spatially localized portions of the state matrix of the system and they are independent of the dimension of the entire system. Our results are novel with respect to our earlier work [18] as in this manuscript our focus is solely on symmetric systems and propose new design rules (in the form of several sufficient conditions) for network synthesis through sparsification (eliminating a coupling by reducing its coupling strength to zero) or growing (by strengthening the existing couplings).

Notations: The set $\mathbb{C}^+$ contains all complex numbers $z \in \mathbb{C}$ with nonnegative real-part, i.e., $\Re(z) \geq 0$. The Hermitian of a matrix $\mathbf{B}$ is shown by $\mathbf{B}^*$. The index function on a set F is denoted by χ_F and its cardinality by $\#F$. Let $\ell^2(\mathcal{V})$ (or simply ℓ_2) contain all vectors $\mathbf{c} = [c_i]_{i\in\mathcal{V}}$ with bounded norms

$$\|\mathbf{c}\|_2 := \Big(\sum_{i\in\mathcal{V}} |c_i|^2\Big)^{1/2}.$$

The set $\mathcal{B}^2$ contains all matrices $\mathbf{B}$ on ℓ^2 with bounded induced norm

$$\|\mathbf{B}\|_{\mathcal{B}^2} := \sup_{\|\mathbf{c}\|_2=1} \|\mathbf{Bc}\|_2.$$

II. Preliminary Notions on Spatial Graphs

In this section, we recall some preliminary results on geodesic metric ρ on connected simple graphs, and we present several equivalent conditions to characterize matrices whose spectra are contained in the open left-half complex plane.

The r-neighborhood of agent $i \in \mathcal{V}$ over graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ is defined by $B(i,r) = \{j \in \mathcal{V} \mid \rho(i,j) \leq r\}$. To study localization features of spatially distributed systems, we presume that:

Assumption 2.1: The counting measure $\mu_\mathcal{G}$ on the graph $\mathcal{G}$ is a doubling measure, i.e., there exists a positive number $D_0(\mathcal{G})$ such that

$$\mu_\mathcal{G}(B(i,2r)) \leq D_0(\mathcal{G})\mu_\mathcal{G}(B(i,r)) \qquad (1)$$

hold for all $i \in \mathcal{V}$ and $r \geq 0$.

The doubling property of the counting measure $\mu_\mathcal{G}$ implies that numbers of agents in r-neighborhood and $(2r)$-neighborhood of any agent are comparable.

The maximal number of direct communication links for every agent in a spatially distributed system with underlying graph $\mathcal{G}$ can be measured by the maximal vertex degree, which is represented by $\deg(\mathcal{G})$. We observe that the doubling constant of the counting measure $\mu_\mathcal{G}$ on graph $\mathcal{G}$, i.e., the minimal constant $D_0(\mathcal{G}) \geq 1$ for which inequality (1)

† The author is with the Department of Mechanical Engineering and Mechanics, Packard Laboratory, Lehigh University, Bethlehem, PA. Email address: motee@lehigh.edu.
‡ The author is with the Department of Mathematics, University of Central Florida, Orlando FL. Email address: qiyu.sun@ucf.edu.
∗ This research is supported by NSF CAREER ECCS-1454022, ONR YIP N00014-16-1-2645, and NSF DMS-1412413

holds, dominates the maximal vertex degree of the graph $\mathcal{G}$, i.e., $\deg(\mathcal{G}) \leq D_0(\mathcal{G})$.

Definition 2.2: A counting measure $\mu_\mathcal{G}$ on graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ is said to have polynomial growth if there exist positive constants $D_1(\mathcal{G})$ and d such that

$$\mu_\mathcal{G}(B(i,r)) \leq D_1(\mathcal{G})(1+r)^d \tag{2}$$

for all $i \in \mathcal{V}$ and $r \geq 0$.

For a graph $\mathcal{G}$ associated with a spatially distributed system, the minimal constants d and $D_1(\mathcal{G})$ for which inequality (2) hold are so called *Beurling dimension* and *density* of that spatially distributed system, respectively. It is shown in [6] that the doubling measure $\mu_\mathcal{G}$ has polynomial degree with $D_1(\mathcal{G}) = D_0(\mathcal{G})$ and $d = \log_2 D_0(\mathcal{G})$.

Remark 2.3: For the class of spatially distributed systems whose subsystems (nodes) are located on a d-dimensional manifold and direct communication link between two nodes exists only if their spatial distance is less than a certain range, one can prove that the Beurling dimension of this class of systems is d and density is related to Ricci curvature of the underlying manifold.

III. Problem Statement

The interconnection topology of a spatially distributed system can be described by an (in)finite graph $\mathcal{G} := (\mathcal{V}, \mathcal{E})$, where $\mathcal{V}$ is the index set of all nodes and $\mathcal{E}$ is the set of all interconnection links in the system. A vertex represents a subsystem (also referred to as node) and an edge between two nodes means that a direct communication link exists between them. Throughout the paper, it is assumed that $\mathcal{G}$ is *unweighted*, *undirected*, *connected*, and *simple*, i.e., there is no graph loops nor multiple edges, which can be interpreted as follows: (i) subsystems in a spatially distributed system can communicate across the entire network, but they have direct communication links only with neighboring subsystems, (ii) direct communication links between agents are bidirectional, (iii) subsystems have identical communication capabilities, (iv) the communication component in each subsystem is not used for data transmission within that subsystem, (v) there is no multiple direct communication links between two subsystems.

Since the underlying graph $\mathcal{G}$ is connected and undirected, we can define the geodesic distance ρ on $\mathcal{G}$ such that:

- $\rho_\mathcal{G}(i,i) = 0$ for all $i \in \mathcal{G}$; and
- $\rho_\mathcal{G}(i,j)$ is the number of edges in a shortest path connecting two distinct vertices $i, j \in \mathcal{G}$ [8].

In many applications, communication between two distinct subsystems happens by transmitting information through the chain of intermediate subsystems connecting the two subsystems using their shortest path. In such cases, the communication cost between those two subsystems, say $i, j \in \mathcal{V}$, is usually proportional to their geodesic distance $\rho_\mathcal{G}(i,j)$.

Definition 3.1: Matrix $\mathbf{B} = [b(i,j)]_{i,j\in\mathcal{V}}$ is called σ-band for some $\sigma \geq 0$ if

$$b(i,j) = 0 \quad \text{if} \quad \rho(i,j) > \sigma. \tag{3}$$

We consider the class of linear time-invariant systems with finite or infinite dimensions whose dynamics are governed by the following differential equation

$$\frac{d}{dt}\psi(t) = \mathbf{B}\psi(t), \tag{4}$$

where state matrix $\mathbf{B}$ is a σ-band matrix and initial condition $\psi(0) = \psi_0 \in \ell_2$ are known. The exponential stability of this class of linear systems is one of fundamental problems in distributed control systems (see [3] and references in there). The exponential stability of (4) can be guaranteed by requiring the spectrum of matrix $\mathbf{B}$ to lie strictly in the left-half complex plane, i.e.,

$$\sigma(\mathbf{B}) \subset \big\{ z \in \mathbb{C} \mid \Re(z) \leq -\delta \big\} \tag{5}$$

for some $\delta > 0$.

The localized and decentralized stability verification problem is practically relevant to several real-world applications. For instance, it is often the case in a spatially distributed system that the state matrix $\mathbf{B}$ with finite bandwidth is stored by subsystems in a distributed manner. Since there are local storage limitations, each subsystem in the network should only store its own corresponding row (and perhaps its neighboring rows) in the state matrix. However, each subsystem have only partial access to information of the entire state matrix, perhaps due to privacy or security issues in the network. These limitations encourages us to utilize localized and decentralized verification tools to check the spectral set property (5). One of the important features of our results is that our localized conditions are independent of the system size.

In this paper, we consider exponential stability of a linear system (4) on a spatially distributed network with a Hermitian state matrix $\mathbf{B}$. First, we present several equivalent statements on the exponential stability of the linear system (4) on a spatially distributed network.

Theorem 3.2: Suppose that $\mathbf{B}$ is a Hermitian matrix in $\mathcal{B}^2$. Then the following statements are equivalent to each other:

(i) $\mathbf{B}$ is strictly negative definite.

(ii) There exists a positive constant A_0 such that

$$\|(z\mathbf{I} - \mathbf{B})\mathbf{c}\|_2 \geq A_0 \|\mathbf{c}\|_2 \tag{6}$$

for all $z \in \mathbb{C}^+$ and $\mathbf{c} \in \ell^2$.

(iii) There exists a positive constant A_0 such that

$$\mathbf{c}^*\mathbf{B}\mathbf{c} \leq 0 \tag{7}$$

and

$$\|\mathbf{B}\mathbf{c}\|_2 \geq A_0 \|\mathbf{c}\|_2 \tag{8}$$

hold for all $\mathbf{c} \in \ell^2$.

Condition (8) implies that constant A_0 is equal to the absolute value of the maximal eigenvalue of of negative definite matrix $\mathbf{B}$. For a spatially invariant linear system whose Toeplitz state matrix $\mathbf{A}_0 = \big[p(i-j)\big]_{i,j\in\mathbb{Z}}$ is Hermitian, $\hat{p}(\xi)$ is real-valued and takes negative values. Hence it follows that its stability threshold is equal to the lower bound of $\hat{p}(\xi)$,

$$A_0 = \inf_{\xi\in\mathbb{R}} |\hat{p}(\xi)|.$$

Derived from Theorem 3.2, we have the following necessary conditions, which can be verified by evaluating maximal or minimal eigenvalues of some localized matrices.

Theorem 3.3: Let $\mathcal{G} := (\mathcal{V}, \mathcal{E})$ be the communication graph of a spatially distributed network. Suppose that the linear system (4) is exponentially stable with stability threshold A_0 and that the state matrix $\mathbf{B}$ is Hermitian and belongs to $\mathcal{B}_\tau \cap \mathcal{B}^2$ for some $\tau \geq 0$. Then the following localized inequalities

$$\mathbf{c}^*\chi_i^N \mathbf{B}\chi_i^N \mathbf{c} \leq 0 \tag{9}$$

and

$$\mathbf{c}^*\chi_i^N \mathbf{B}^2\chi_i^N \mathbf{c} \geq A_0^2 \|\chi_i^N \mathbf{c}\|_2^2 \tag{10}$$

hold for all $N \geq \tau$, $i \in \mathcal{V}$ and $\mathbf{c} \in \ell^2$.

For the linear system (4) with a Hermitian state matrix, sufficient conditions for stability take simple forms.

Theorem 3.4: Let the communication graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ of a spatially distributed network be undirected and unweighted and its counting measure $\mu_\mathcal{G} : 2^\mathcal{V} \to \mathbb{Z}_+$ have the polynomial growth (2). Suppose that state matrix $\mathbf{B} \in \mathcal{B}_\tau \cap \mathcal{B}^2$ is Hermitian for some $\tau \geq 0$. Then the linear system (4) with the state matrix $\mathbf{B}$ is exponentially stable if there exist a positive integer N_0 and a positive number B_{N_0} satisfying

$$B_0 \;\geq\; 4\sqrt{\frac{C_1}{C_0}}\; D_1(\mathcal{G})\,(\sigma+1)^{d+1}\,\|\mathbf{B}\|_\infty N_0^{-1}, \tag{11}$$

where

$$\|\mathbf{B}\|_\infty = \sup_{i,j\in\mathcal{V}} |a(i,j)| < \infty,$$

such that

$$\mathbf{c}^*\chi_{i_m}^{N_0} \mathbf{B}\chi_{i_m}^{N_0} \mathbf{c} \leq 0 \tag{12}$$

and

$$\mathbf{c}^*\chi_{i_m}^{N_0} \mathbf{B}^2\chi_{i_m}^{N_0} \mathbf{c} \geq B_{N_0}^2 \|\chi_{i_m}^{N_0} \mathbf{c}\|_2^2 \tag{13}$$

hold for all $i_m \in \mathcal{V}_{N_0}$ and $\mathbf{c} \in \ell^2$.

For a given Hermitian matrix $\mathbf{B} = [b(i,j)]_{i,j\in\mathcal{V}}$ in $\mathcal{B}_\tau \cap \mathcal{B}^2$, the sufficent conditions (12) and (13) in Theorem 3.4 are spatially localized in neighborhood of each leading subsystem $i_m \in \mathcal{V}_{N_0}$, where each subsystems only need to have access to portion of state matrices $\mathbf{B}$ that is determined by truncation operator $\chi_{i_m}^{N_0}$. In particular, the requirement (12) holds if the largest eigenvalue of the spatially localized principal submatrix $[b(j,j')]_{j,j'\in B(i_m,N)}$ is nonpositive for every $i_m \in \mathcal{V}_{N_0}$.

For a Hermitian matrix $\mathbf{B}$, define

$$\tilde{B}_N(i) = \inf_{\|\chi_i^N \mathbf{c}\|_2=1} \|\mathbf{B}\chi_i^N \mathbf{c}\|_2, \tag{14}$$

where $N \geq 1$ and $i \in \mathcal{V}$. The quantity $B_N(i)$ is the same as square root of the smallest eigenvalue of the spatially localized matrix

$$\chi_i^N \mathbf{B}^2 \chi_i^N = \left[\sum_{k\in B(j,\tau)\cap B(j',\tau)} b(j,k)b(k,j')\right]_{j,j'\in B(i,N)}, \tag{15}$$

and it can be evaluated in a distributed manner. Then the constant $\tilde{B}_{N_0}$ in (13) can be thought of as the uniform stability threshold for small-scale systems with state matrices $\chi_{i_m}^{N_0}\mathbf{B}^2\chi_{i_m}^{N_0}, i_m \in \mathcal{V}_{N_0}$.

We can prove that $\{\tilde{B}_N(i)\}_{N=\tau}^\infty$ is a decreasing sequence that converges to A_0 for every $i \in \mathcal{V}$,

$$\lim_{N\to\infty} \tilde{B}_N(i) = A_0. \tag{16}$$

Therefore one can conclude that a symmetric large-scale linear system on a spatially distributed graph can achieve higher levels of performance whenever the stability threshold of its properly-localized small-scale systems are improved.

IV. Design of Spatially Distributed Networks

Given an exponentially stable linear dynamical system (4), we consider a specific class of state feedback control laws that alters its dynamics in the following manner

$$\frac{d}{dt}\psi(t) = \mathbf{B}\psi(t) + u(t), \tag{17}$$

with

$$u(t) = w\mathbf{E}_{kl}\psi(t), \tag{18}$$

where w is a scalar feedback gain and $\mathbf{E}_{kl}, k, l \in \mathcal{V}$, are matrices whose have zero entries except (k,l)-th and (l,k)-th entries taking value one. Our design goal is to apply Theorem 3.4 to compute an interval for the scalar feedback gain such that the resulting closed-loop network

$$\frac{d}{dt}\psi(t) = (\mathbf{B} + w\mathbf{E}_{kl})\psi(t) \tag{19}$$

remains exponentially stable with guaranteed stability thresholds and the stability margin being at the same or higher level as the system (4). This application can be interpreted as weight adjustment problem in general symmetric linear dynamical networks in order to enhance stability threshold via local adjustments and it can be particularly utilized to strengthen existing couplings (e.g., when $b(k,l), w > 0$) or sparsify (e.g., when $b(k,l) > 0$ and $w < 0$) coupling structure of a linear dynamical network. For $\tau, M \geq 0$, let $\mathcal{B}_\tau(M)$ denote the set of all band matrices $\mathbf{B} \in \mathcal{B}_\tau$ with bounded entries $\|\mathbf{B}\|_\infty < M$.

Theorem 4.1: Let the communication graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ of the linear system (17) be undirected and unweighted and its counting measure $\mu_\mathcal{G} : 2^\mathcal{V} \to \mathbb{Z}_+$ have the polynomial growth property (2). Assume that the linear system (17) is exponentiable stable and its state matrix $\mathbf{B} = [b(i,j)]_{i,j\in\mathcal{V}}$ is a Hermitian matrix in $\mathcal{B}_\tau(M) \cap \mathcal{B}^2$ for some $\tau \geq 0$ and $M > 0$. Take an integer N_0 such that

$$\tilde{B}_{N_0} := \inf_{i_m\in\mathcal{V}_{N_0}} \tilde{B}_{N_0}(i_m) \geq 4M\sqrt{\frac{\alpha_2}{\alpha_1}}\, D_1(\mathcal{G})\tau(\tau+1)^d N_0^{-1}. \tag{20}$$

For $k,l\in\mathcal{V}$ with $\rho(k,l)\le\tau$ define

$$\eta_{kl}=\inf_{\substack{\rho(k,i_m)\le N_0\\ \rho(l,i_m)\le N_0}}\ \inf_{\|\mathbf{P}_{i_m}^{N_0}\mathbf{c}\|_2=1}\Re\big((\mathbf{P}_{i_m}^{N_0}\mathbf{c})^*\mathbf{E}_{kl}\mathbf{B}\mathbf{P}_{i_m}^{N_0}\mathbf{c}\big)\quad(21)$$

and

$$\beta_{kl}=\sup_{\substack{\rho(k,i_m)\le N_0\\ \rho(l,i_m)\le N_0}}\ \sup_{\|\mathbf{P}_{i_m}^{N_0}\mathbf{c}\|_2=1}\Re\big((\mathbf{P}_{i_m}^{N_0}\mathbf{c})^*\mathbf{E}_{kl}\mathbf{B}\mathbf{P}_{i_m}^{N_0}\mathbf{c}\big),\quad(22)$$

where $\tilde{B}_N(i)$ is given by (14), $\mathbf{P}_{i_m}^{N_0}$ is the projection matrix onto the eigenspace of the localized matrix $\chi_{i_m}^{N_0}\mathbf{B}^2\chi_{i_m}^{N_0}$ corresponding to its smallest eigenvalue $(\tilde{B}_{N_0}(i_m))^2$. Then the following statements hold:

(i) If $\eta_{kl}>0$, then there exists $\epsilon_0>0$ such that for every weight $w\in(0,\epsilon_0)$ such that the modified network (19) is exponentially stable and the state matrix $\mathbf{B}+w\mathbf{E}_{kl}$ still belongs to $\mathcal{B}_\tau(M)$.

(ii) If $\beta_{kl}<0$, then there exists $\epsilon_1>0$ such that for every weight $w\in(-\epsilon_1,0)$ such that the modified network (19) is exponentially stable and $\mathbf{B}+w\mathbf{E}_{kl}$ still belongs to $\mathcal{B}_\tau(M)$.

The proof of Theorem 4.1 is technical and eliminated due to space limitations. Suppose that $\{e_{i_{m_1}}^{N_0},\dots,e_{i_{m_k}}^{N_0}\}$ is an orthonormal basis of the eigenspace corresponding to the smallest eigenvalue $(\tilde{B}_{N_0}(i_m))^2$ of the matrix in (15). Then, the projection matrix in Theorem 4.1 can be represented by

$$\mathbf{P}_{i_m}^{N_0}=\sum_{s=1}^{k}e_{i_{m_s}}^{N_0}(e_{i_{m_s}}^{N_0})^T.$$

When the smallest eigenvalue is simple with normalized eigenvector $\mathbf{q}_{i_m}^{N_0}$, the project matrix is given by

$$\mathbf{P}_{i_m}^{N_0}=\mathbf{q}_{i_m}^{N_0}(\mathbf{q}_{i_m}^{N_0})^*,$$

where constraint $\|\mathbf{P}_{i_m}^{N_0}\mathbf{c}\|_2=1$ implies that

$$\mathbf{P}_{i_m}^{N_0}\mathbf{c}=\mathbf{q}_{i_m}^{N_0}.$$

This results in the following closed-form solutions

$$\inf_{\|\mathbf{P}_{i_m}^{N_0}\mathbf{c}\|_2=1}\Re\big((\mathbf{P}_{i_m}^{N_0}\mathbf{c})^*\mathbf{E}_{kl}\mathbf{B}\mathbf{P}_{i_m}^{N_0}\mathbf{c}\big)=\Re\left((\mathbf{q}_{i_m}^{N_0})^*\mathbf{E}_{kl}\mathbf{B}\mathbf{q}_{i_m}^{N_0}\right)$$

and

$$\sup_{\|\mathbf{P}_{i_m}^{N_0}\mathbf{c}\|_2=1}\Re\big((\mathbf{P}_{i_m}^{N_0}\mathbf{c})^*\mathbf{E}_{kl}\mathbf{B}\mathbf{P}_{i_m}^{N_0}\mathbf{c}\big)=\Re\left((\mathbf{q}_{i_m}^{N_0})^*\mathbf{E}_{kl}\mathbf{B}\mathbf{q}_{i_m}^{N_0}\right)$$

that can be useful to evaluate η_{kl} and β_{kl} in (21) and (22).

The quantities η_{kl} in (21) and β_{kl} in (22) can be evaluated by using entries $b(i,j)$ of the matrix $\mathbf{B}$ with

$$i,j\in B(k,2N_0+\tau)\cap B(l,2N_0+\tau).$$

Therefore, the requirements $\eta_{kl}>0$ and $\beta_{kl}<0$ in Theorem 4.1 can be verified by applying local information of the matrix $\mathbf{B}$ in a localized neighborhood of nodes $k,l\in\mathcal{V}$. Given $k,l\in\mathcal{V}$, we should select positive weight w if $\eta_{kl}>0$ and negative weight if $\beta_{kl}<0$.

The results of Theorem 4.1 will remain true if matrix $\mathbf{E}_{kl}$ is replaced by $\mathbf{R}_{kl}(\theta)$ whose (k,l)'th and (l,k)'th entries are $\sin\theta$, (k,k)'th entry is $\cos\theta$, and (l,l)'th entry is $-\cos\theta$ for some $0\le\theta\le\pi$. We can establish similar result when the matrix $\mathbf{E}_{kl}$ in Theorem 4.1 is replaced by $\mathbf{L}_{ij}$ in which $\mathbf{L}_{ij}=\mathbf{e}_i\mathbf{e}_i^*+\mathbf{e}_j\mathbf{e}_j^*-\mathbf{E}_{ij}$ and $\mathbf{e}_i$'s are the standard basis for ℓ_2.

References

[1] A. Aldroubi and K. Gröchenig, Nonuniform sampling and reconstruction in shift-invariant spaces, *SIAM Review*, vol. 43, pp. 585–620, 2001.

[2] M. S. Andersen, S. K. Pakazad, A. Hansson, and A. Rantzer, "Robust stability analysis of sparsely interconnected uncertain systems," *IEEE Trans. Autom. Control*, vol. 59, no. 8, pp. 2151–2156, 2014.

[3] B. Bamieh, F. Paganini, and M. A. Dahleh. "Distributed control of spatially-invariant systems," *IEEE Trans. Autom. Control*, vol. 47, no. 7, pp. 1091–1107, July 2002.

[4] A. Bensoussan, G. Da Prato, M. C. Delfour, and S.K. Mitter, *Representation and control of infinite dimensional systems,* Birkhäuser Boston, vol. 2, 1993.

[5] J. W. Bunce, "Stabilizability of linear systems defined over C^*-algebras," *Math. Systems Theory*, vol. 18, no. 1, pp. 237–250, 1985.

[6] C. Cheng, Y. Jiang and Q. Sun, Spatially distributed sampling and reconstruction, arXiv:1511.08541

[7] O. Christensen, *An Introduction to Frames and Riesz Bases*, Applied and Numerical Harmonic Analysis, Birkhuser Boston Inc., Boston, MA, 2003.

[8] F. R. K. Chung, *Spectral Graph Theory*, American Mathematical Society, 1997.

[9] R. F. Curtain and H. J. Zwart, *An Introduction to Infinite-Dimensional Linear Systems Theory*, Springer, 1995.

[10] R. D'Andrea and G. E. Dullerud, "Distributed control design for spatially interconnected systems," *IEEE Trans. Autom. Control*, vol. 48, no. 9, pp. 1478–1495, 2003.

[11] D. Kincaid and W. Cheney, *Numerical Analysis: Mathematics of Scientific Computing,* Brooks/Cole, Third edition, 2002.

[12] C. Langbort, R. S. Chandra, and R. D'Andrea, "Distributed control design for systems interconnected over an arbitrary graph," *IEEE Trans. Autom. Control*, vol. 49, no. 9, pp. 1502–1519, 2004.

[13] N. Motee and A. Jadbabaie, "Approximation methods and spatial interpolation in distributed control systems," in *Proc. American Control Conference,* pp. 860–865, 2009.

[14] N. Motee and A. Jadbabaie, "Distributed multi-parametric quadratic programming," *IEEE Trans. Autom. Control*, vol. 54, no. 10, pp. 2279–2289, Oct. 2009.

[15] N. Motee and A. Jadbabaie, "Optimal control of spatially distributed systems," *IEEE Trans. Autom. Control*, vol. 53, no. 7, pp. 1616–1629, Aug. 2008.

[16] N. Motee and Q. Sun, "Measuring sparsity in spatially interconnected systems," in *Proc. IEEE 52nd Conference Decision and Control*, Dec. 2013.

[17] N. Motee and Q. Sun, "Sparsity measures for spatially decaying systems," in *Proc. American Control Conference,* pp. 5459–5464, June 2014.

[18] N. Motee and Q. Sun, "Localized Stability Certificates for Spatially Distributed Systems with Sparse Graph Topologies," *The 55th IEEE Conference on Decision and Control*, Las Vegas, NV, December 2016.

[19] J. Ploeg, N. Van De Wouw, and H. Nijmeijer, "L_p string stability of cascaded systems: Application to vehicle platooning," *IEEE Transactions on Control Systems Technology,* vol. 22, no. 2, pp. 786–793, 2014.

[20] C. E. Shin and Q. Sun, "Stability of localized operators," *J. Funct. Anal.*, vol. 256, pp. 2417–2439, 2009.

[21] Q. Sun, "Nonuniform average sampling and reconstruction of signals with finite rate of innovation," *SIAM J. Math. Anal.*, vol. 38, pp. 1389–1422, 2006.

[22] Q. Sun and J. Xian, Rate of innovation for (non-)periodic signals and optimal lower stability bound for filtering, *J. Fourier Anal. Appl.*, vol. 20, pp. 119–134, 2014.

[23] Q. Sun, Stability criterion for convolution-dominated infinite matrices, *Proc. Amer. Math. Soc.*, vol. 138, pp. 3933–3943, 2010.

[24] M. Unser, Sampling–50 years after Shannon, *Proc. IEEE*, vol. 88, no. 4, pp. 569–587, 2000.