Differential subalgebras and norm-controlled inversion

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In memory of Ronald G. Douglas

Abstract. In this paper, we consider the norm-controlled inversion for differential $*$-subalgebras of a symmetric $*$-algebra with common identity and involution.

1. Introduction

In [49, Lemma IIe], it states that "*If $f(x)$ is a function with an absolutely convergent Fourier series, which nowhere vanishes for real arguments, $1/f(x)$ has an absolutely convergent Fourier series.*" The above statement is now known as the classical Wiener's lemma.

We say that a Banach space $\mathcal{A}$ with norm $\|\cdot\|_{\mathcal{A}}$ is a *Banach algebra* if it has operation of multiplications possessing the usual algebraic properties, and

$$\|AB\|_{\mathcal{A}} \le K\|A\|_{\mathcal{A}}\|B\|_{\mathcal{A}} \ \text{ for all } A, B \in \mathcal{A}, \tag{1.1}$$

where K is a positive constant. Given two Banach algebras $\mathcal{A}$ and $\mathcal{B}$ such that $\mathcal{A}$ is a Banach subalgebra of $\mathcal{B}$, we say that $\mathcal{A}$ is *inverse-closed* in $\mathcal{B}$ if $A \in \mathcal{A}$ and $A^{-1} \in \mathcal{B}$ implies $A^{-1} \in \mathcal{A}$. Inverse-closedness is also known as spectral invariance, Wiener pair, local subalgebra, etc [13, 16, 30, 46]. Let $\mathcal{C}$ be the algebra of all periodic continuous functions under multiplication, and $\mathcal{W}$ be its Banach subalgebra of all periodic functions with absolutely convergent Fourier series,

$$\mathcal{W} = \Big\{ f(x) = \sum_{n\in\mathbb{Z}} \hat f(n) e^{inx}, \ \ \|f\|_{\mathcal{W}} := \sum_{n\in\mathbb{Z}} |\hat f(n)| < \infty \Big\}. \tag{1.2}$$

Then the classical Wiener's lemma can be reformulated as that $\mathcal{W}$ is an inverse-closed subalgebra of $\mathcal{C}$. Due to the above interpretation, we also call the inverse-closed property for a Banach subalgebra $\mathcal{A}$ as Wiener's lemma for that subalgebra. Wiener's lemma for Banach algebras of infinite matrices and integral operators with certain off-diagonal decay can be informally interpreted as localization preservation under inversion. Such a localization preservation is of great importance in applied harmonic analysis, numerical analysis, optimization and many mathematical and engineering fields [2, 10, 11, 23, 28, 44]. The readers may refer to the survey papers [18, 27, 37], the recent papers [14, 34, 36] and references therein for historical remarks and recent advances.

Given an element A in a Banach algebra $\mathcal{A}$ with the identity I, we define its *spectral set* $\sigma_{\mathcal{A}}(A)$ and *spectral radius* $\rho_{\mathcal{A}}(A)$ by

$$\sigma_{\mathcal{A}}(A) := \big\{\lambda \in \mathbb{C} : \lambda I - A \text{ is not invertible in } \mathcal{A}\big\}$$

and

$$\rho_{\mathcal{A}}(A) := \max\big\{|\lambda| : \lambda \in \sigma_{\mathcal{A}}(A)\big\}$$

The project is partially supported by the Basic Science Research Program through the National Research Foundation of Korea (NRF) funded by the Ministry of Education, Science and Technology (NRF-2016R1D1A1B03930571) and the National Science Foundation (DMS-1816313) .

respectively. Let $\mathcal{A}$ and $\mathcal{B}$ be Banach algebras with common identity I and $\mathcal{A}$ be a Banach subalgebra of $\mathcal{B}$. Then an equivalent condition for the inverse-closedness of $\mathcal{A}$ in $\mathcal{B}$ is that the spectral set of any $A \in \mathcal{A}$ in Banach algebras $\mathcal{A}$ and $\mathcal{B}$ are the same, i.e.,

$$\sigma_{\mathcal{A}}(A) = \sigma_{\mathcal{B}}(A).$$

By the above equivalence, a necessary condition for the inverse-closedness of $\mathcal{A}$ in $\mathcal{B}$ is that the spectral radius of any $A \in \mathcal{A}$ in the Banach algebras $\mathcal{A}$ and $\mathcal{B}$ are the same, i.e.,

$$\rho_{\mathcal{A}}(A) = \rho_{\mathcal{B}}(A). \tag{1.3}$$

The above necessary condition is shown by Hulanicki [24] to be sufficient if we further assume that $\mathcal{A}$ and $\mathcal{B}$ are $*$-algebras with common identity and involution, and that $\mathcal{B}$ is symmetric. Here we say that a Banach algebra $\mathcal{B}$ is a $*$-algebra if there is a continuous linear *involution* $*$ on $\mathcal{B}$ with the properties that

$$(AB)^* = B^*A^* \quad \text{and} \quad A^{**} = A \quad \text{for all } A, B \in \mathcal{B},$$

and that a $*$-algebra $\mathcal{B}$ is *symmetric* if

$$\sigma_{\mathcal{A}}(A^*A) \subset [0, \infty) \quad \text{for all } A \in \mathcal{B}.$$

The spectral radii approach (1.3), known as the Hulanicki's spectral method, has been used to establish the inverse-closedness of symmetric $*$-algebras [9, 20, 21, 41, 43, 45], however the above approach does not provide a norm estimate for the inversion, which is crucial for many mathematical and engineering applications.

To consider norm estimate for the inversion, we recall the concept of norm-controlled inversion of a Banach subalgebra $\mathcal{A}$ of a symmetric $*$-algebra $\mathcal{B}$, which was initiated by Nikolski [31] and coined by Gröchenig and Klotz [20]. Here we say that a Banach subalgebra $\mathcal{A}$ of $\mathcal{B}$ admits *norm-controlled inversion* in $\mathcal{B}$ if there exists a continuous function h from $[0, \infty) \times [0, \infty)$ to $[0, \infty)$ such that

$$\|A^{-1}\|_{\mathcal{A}} \le h\big(\|A\|_{\mathcal{A}}, \|A^{-1}\|_{\mathcal{B}}\big) \tag{1.4}$$

for all $A \in \mathcal{A}$ being invertible in $\mathcal{B}$ [19, 20, 34, 36].

The norm-controlled inversion is a strong version of Wiener's lemma. The classical Banach algebra $\mathcal{W}$ in (1.2) is inverse-closed in the algebra $\mathcal{C}$ of all periodic continuous functions [49], however it does not have norm-controlled inversion in $\mathcal{C}$ [5, 31]. To establish Wiener's lemma, there are several methods, including the Wiener's localization [49], the Gelfand's technique [16], the Brandenburg's trick [9], the Hulanicki's spectral method [24], the Jaffard's boot-strap argument [25], the derivation technique [21], and the Sjöstrand's commutator estimates [36, 39]. In this paper, we will use the Brandenburg's trick to establish norm-controlled inversion of a differential $*$-subalgebra $\mathcal{A}$ of a symmetric $*$-algebra $\mathcal{B}$.

This introduction article is organized as follows. In Section 2, we recall the concept of differential subalgebras and present some differential subalgebras of infinite matrices with polynomial off-diagonal decay. In Section 3, we introduce the concept of generalized differential subalgebras and present some generalized differential subalgebras of integral operators with kernels being Hölder continuous and having polynomial off-diagonal decay. In Section 4, we use the Brandenburg's trick to establish norm-controlled inversion of a differential $*$-subalgebra of a symmetric $*$-algebra, and we conclude the section with two remarks on the norm-controlled inversion with the norm control function bounded by a polynomial and the norm-controlled inversion of nonsymmetric Banach algebras.

2. Differential Subalgebras

Let $\mathcal{A}$ and $\mathcal{B}$ be Banach algebras such that $\mathcal{A}$ is a Banach subalgebra of $\mathcal{B}$. We say that $\mathcal{A}$ is a *differential subalgebra of order* $\theta \in (0, 1]$ in $\mathcal{B}$ if there exists a positive constant $D_0 := D_0(\mathcal{A}, \mathcal{B}, \theta)$ such that

$$\|AB\|_{\mathcal{A}} \le D_0 \|A\|_{\mathcal{A}} \|B\|_{\mathcal{A}} \Big(\Big(\frac{\|A\|_{\mathcal{B}}}{\|A\|_{\mathcal{A}}}\Big)^{\theta} + \Big(\frac{\|B\|_{\mathcal{B}}}{\|B\|_{\mathcal{A}}}\Big)^{\theta}\Big) \quad \text{for all } A, B \in \mathcal{A}. \tag{2.1}$$

The concept of differential subalgebras of order θ was introduced in [7, 26, 32] for $\theta = 1$ and [12, 20, 36] for $\theta \in (0, 1)$. We also refer the reader to [3, 15, 19, 20, 21, 25, 33, 34, 41, 42, 43, 45] for various differential subalgebras of infinite matrices, convolution operators, and integral operators with certain off-diagonal decay.

For $\theta = 1$, the requirement (2.1) can be reformulated as

$$\|AB\|_{\mathcal{A}} \le D_0 \|A\|_{\mathcal{A}} \|B\|_{\mathcal{B}} + D_0 \|A\|_{\mathcal{B}} \|B\|_{\mathcal{A}} \quad \text{for all } A, B \in \mathcal{A}. \tag{2.2}$$

So the norm $\|\cdot\|_{\mathcal{A}}$ satisfying (2.1) is also referred as a Leibniz norm on $\mathcal{A}$.

Let $C[a, b]$ be the space of all continuous functions on the interval $[a, b]$ with its norm defined by

$$\|f\|_{C[a,b]} = \sup_{t \in [a,b]} |f(t)|, \quad f \in C[a, b],$$

and $C^k[a, b], k \ge 1$, be the space of all continuously differentiable functions on the interval $[a, b]$ up to order k with its norm defined by

$$\|h\|_{C^k[a,b]} = \sum_{j=0}^{k} \|h^{(j)}\|_{C[a,b]} \text{ for } h \in C^k[a, b].$$

Clearly, $C[a, b]$ and $C^k[a, b]$ are Banach algebras under function multiplication. Moreover

$$\begin{aligned} \|h_1 h_2\|_{C^1[a,b]} &= \|(h_1 h_2)'\|_{C[a,b]} + \|h_1 h_2\|_{C[a,b]} \\ &\le \|h_1'\|_{C[a,b]} \|h_2\|_{C[a,b]} + \|h_1\|_{C[a,b]} \|h_2'\|_{C[a,b]} + \|h_1\|_{C[a,b]} \|h_2\|_{C[a,b]} \\ &\le \|h_1\|_{C^1[a,b]} \|h_2\|_{C[a,b]} + \|h_1\|_{C[a,b]} \|h_2\|_{C^1[a,b]} \text{ for all } h_1, h_2 \in C^1[a, b], \end{aligned} \tag{2.3}$$

where the second inequality follows from the Leibniz rule. Therefore we have

Theorem 2.1. *$C^1[a, b]$ is a differential subalgebra of order one in $C[a, b]$.*

Due to the above illustrative example of differential subalgebras of order one, the norm $\|\cdot\|_{\mathcal{A}}$ satisfying (2.1) is also used to describe smoothness in abstract Banach algebra [7].

Let $\mathcal{W}^1$ be the Banach algebra of all periodic functions such that both f and its derivative f' belong to the Wiener algebra $\mathcal{W}$, and define the norm on $\mathcal{W}^1$ by

$$\|f\|_{\mathcal{W}^1} = \|f\|_{\mathcal{W}} + \|f'\|_{\mathcal{W}} = \sum_{n \in \mathbb{Z}} (|n| + 1) |\hat{f}(n)| \tag{2.4}$$

for $f(x) = \sum_{n \in \mathbb{Z}} \hat{f}(n) e^{inx} \in \mathcal{W}^1$. Following the argument used in the proof of Theorem 2.1, we have

Theorem 2.2. *$\mathcal{W}^1$ is a differential subalgebra of order one in $\mathcal{W}$.*

Recall from the classical Wiener's lemma that $\mathcal{W}$ is an inverse-closed subalgebra of $\mathcal{C}$, the algebra of all periodic continuous functions under multiplication. This leads to the following natural question:

Question 2.3. *Is $\mathcal{W}^1$ a differential subalgebra of $\mathcal{C}$?*

Let $\ell^p, 1 \le p \le \infty$, be the space of all p-summable sequences on $\mathbb{Z}$ with norm denoted by $\|\cdot\|_p$. To answer the above question, we consider Banach algebras $\mathcal{C}$, $\mathcal{W}$ and $\mathcal{W}^1$ in the "frequency domain". Let $\mathcal{B}(\ell^p)$ be the algebra of all bounded linear operators on $\ell^p, 1 \le p \le \infty$, and let

$$\tilde{\mathcal{W}} = \Big\{ A := (a(i-j))_{i,j \in \mathbb{Z}}, \ \|A\|_{\tilde{\mathcal{W}}} = \sum_{k \in \mathbb{Z}} |a(k)| < \infty \Big\} \tag{2.5}$$

and

$$\tilde{\mathcal{W}}^1 = \Big\{ A := (a(i-j))_{i,j \in \mathbb{Z}}, \ \|A\|_{\tilde{\mathcal{W}}^1} = \sum_{k \in \mathbb{Z}} |k| |a(k)| < \infty \Big\} \tag{2.6}$$

be Banach algebras of Laurent matrices with symbols in $\mathcal{W}$ and $\mathcal{W}^1$ respectively. Then the classical Wiener's lemma can be reformulated as that $\tilde{\mathcal{W}}$ is an inverse-closed subalgebra of $\mathcal{B}(\ell^2)$, and an equivalent statement of Theorem 2.2 is that $\tilde{\mathcal{W}}^1$ is a differential subalgebra of order one in $\tilde{\mathcal{W}}$. Due to the above equivalence, Question 2.3 in the "frequency domain" becomes whether $\mathcal{W}^1$ is a differential subalgebra of order $\theta \in (0,1]$ in $\mathcal{C}$. In [45], the first example of differential subalgebra of infinite matrices with order $\theta \in (0,1)$ was discovered.

Theorem 2.4. *$\mathcal{W}^1$ is a differential subalgebra of $\mathcal{C}$ with order* $2/3$.

To consider differential subalgebras of infinite matrices in the noncommutative setting, we introduce three noncommutative Banach algebras of infinite matrices with certain off-diagonal decay. Given $1 \le p \le \infty$ and $\alpha \ge 0$, we define the Gröchenig-Schur family of infinite matrices by

$$\mathcal{A}_{p,\alpha} = \Big\{ A = (a(i,j))_{i,j\in\mathbb{Z}},\ \|A\|_{\mathcal{A}_{p,\alpha}} < \infty \Big\} \tag{2.7}$$

[22, 25, 29, 35, 43, 45], the Baskakov-Gohberg-Sjöstrand family of infinite matrices by

$$\mathcal{C}_{p,\alpha} = \Big\{ A = (a(i,j))_{i,j\in\mathbb{Z}},\ \|A\|_{\mathcal{C}_{p,\alpha}} < \infty \Big\} \tag{2.8}$$

[4, 17, 22, 39, 43], and the Beurling family of infinite matrices

$$\mathcal{B}_{p,\alpha} = \Big\{ A = (a(i,j))_{i,j\in\mathbb{Z}},\ \|B\|_{\mathcal{A}_{p,\alpha}} < \infty \Big\} \tag{2.9}$$

[6, 36, 41], where $u_\alpha(i,j) = (1+|i-j|)^\alpha, \alpha \ge 0$, are polynomial weights on $\mathbb{Z}^2$,

$$\|A\|_{\mathcal{A}_{p,\alpha}} = \max\Big\{ \sup_{i\in\mathbb{Z}} \big\|\big(a(i,j)u_\alpha(i,j)\big)_{j\in\mathbb{Z}}\big\|_p,\ \ \sup_{j\in\mathbb{Z}} \big\|\big(a(i,j)u_\alpha(i,j)\big)_{i\in\mathbb{Z}}\big\|_p \Big\}, \tag{2.10}$$

$$\|A\|_{\mathcal{C}_{p,\alpha}} = \Big\|\Big(\sup_{i-j=k} |a(i,j)|u_\alpha(i,j)\Big)_{k\in\mathbb{Z}}\Big\|_p, \tag{2.11}$$

and

$$\|A\|_{\mathcal{B}_{p,\alpha}} = \Big\|\Big(\sup_{|i-j|\ge|k|} |a(i,j)|u_\alpha(i,j)\Big)_{k\in\mathbb{Z}}\Big\|_p. \tag{2.12}$$

Clearly, we have

$$\mathcal{B}_{p,\alpha} \subset \mathcal{C}_{p,\alpha} \subset \mathcal{A}_{p,\alpha} \ \text{ for all } 1 \le p \le \infty \text{ and } \alpha \ge 0. \tag{2.13}$$

The above inclusion is proper for $1 \le p < \infty$, while the above three families of infinite matrices coincide for $p = \infty$,

$$\mathcal{B}_{\infty,\alpha} = \mathcal{C}_{\infty,\alpha} = \mathcal{A}_{\infty,\alpha} \ \text{ for all } \alpha \ge 0, \tag{2.14}$$

which is also known as the Jaffard family of infinite matrices [25],

$$\mathcal{J}_\alpha = \Big\{ A = (a(i,j))_{i,j\in\mathbb{Z}},\ \|A\|_{\mathcal{J}_\alpha} = \sup_{i,j\in\mathbb{Z}} |a(i,j)|u_\alpha(i-j) < \infty \Big\}. \tag{2.15}$$

Observe that $\|A\|_{\mathcal{A}_{p,\alpha}} = \|A\|_{\mathcal{C}_{p,\alpha}}$ for a Laurent matrix $A = (a(i-j))_{i,j\in\mathbb{Z}}$. Then Banach algebras $\tilde{\mathcal{W}}$ and $\tilde{\mathcal{W}}^1$ in (2.5) and (2.6) are the commutative subalgebra of the Gröchenig-Schur algebra $\mathcal{A}_{1,\alpha}$ and the Baskakov-Gohberg-Sjöstrand algebra $\mathcal{C}_{1,\alpha}$ for $\alpha = 0, 1$ respectively,

$$\tilde{\mathcal{W}} = \mathcal{A}_{1,0} \cap \mathcal{L} = \mathcal{C}_{1,0} \cap \mathcal{L} \tag{2.16}$$

and

$$\tilde{\mathcal{W}}^1 = \mathcal{A}_{1,1} \cap \mathcal{L} = \mathcal{C}_{1,1} \cap \mathcal{L}, \tag{2.17}$$

where $\mathcal{L}$ is the set of all Laurent matrices $A = (a(i-j))_{i,j\in\mathbb{Z}}$. The sets $\mathcal{A}_{p,\alpha}, \mathcal{C}_{p,\alpha}, \mathcal{B}_{p,\alpha}$ with $p = 1$ and $\alpha = 0$ are noncommutative Banach algebras under matrix multiplication, the Baskakov-Gohberg-Sjöstrand algebra $\mathcal{C}_{1,0}$ and the Beurling algebra $\mathcal{B}_{1,0}$ are inverse-closed subalgebras of $\mathcal{B}(\ell^2)$ [4, 8, 17, 39, 41], however the Schur algebra $\mathcal{A}_{1,0}$ is not inverse-closed in $\mathcal{B}(\ell^2)$ [47]. We remark that the inverse-closedness of the Baskakov-Gohberg-Sjöstrand algebra $\mathcal{C}_{1,0}$ in $\mathcal{B}(\ell^2)$ can be understood as a noncommutative extension of the classical Wiener's lemma for the commutative subalgebra $\tilde{\mathcal{W}}$ of Laurent matrices in $\mathcal{B}(\ell^2)$.

For $1 \le p \le \infty$ and $\alpha > 1 - 1/p$, one may verify that the Gröchenig-Schur family $\mathcal{A}_{p,\alpha}$, the Baskakov-Gohberg-Sjöstrand family $\mathcal{C}_{p,\alpha}$ and the Beurling family $\mathcal{B}_{p,\alpha}$ of infinite matrices form Banach algebras under matrix multiplication and they are inverse-closed subalgebras of $\mathcal{B}(\ell^2)$ [22, 25, 41, 43, 45]. In [41, 43, 45], their differentiability in $\mathcal{B}(\ell^2)$ is established.

Theorem 2.5. *Let $1 \le p \le \infty$ and $\alpha > 1 - 1/p$. Then $\mathcal{A}_{p,\alpha}$, $\mathcal{C}_{p,\alpha}$ and $\mathcal{B}_{p,\alpha}$ are differential subalgebras of order $\theta_0 = (\alpha + 1/p - 1)/(\alpha + 1/p - 1/2) \in (0, 1)$ in $\mathcal{B}(\ell^2)$.*

Proof. The following argument about differential subalgebra property for the Gröchenig-Schur algebra $\mathcal{A}_{p,\alpha}, 1 < p < \infty$, is adapted from [45]. The reader may refer to [41, 43, 45] for the detailed proof to the differential subalgebra property for the Baskakov-Gohberg-Sjöstrand algebra $\mathcal{C}_{p,\alpha}$ and the Beurling algebra $\mathcal{B}_{p,\alpha}$. Take $A = (a(i,j))_{i,j\in\mathbb{Z}}$ and $B = (b(i,j))_{i,j\in\mathbb{Z}} \in \mathcal{A}_{p,\alpha}$, and write $C = AB = (c(i,j))_{i,j\in\mathbb{Z}}$. Then

$$\begin{aligned}
\|C\|_{\mathcal{A}_{p,\alpha}} &= \max\Big\{\sup_{i\in\mathbb{Z}}\big\|\big(c(i,j)u_\alpha(i,j)\big)_{j\in\mathbb{Z}}\big\|_p,\ \sup_{j\in\mathbb{Z}}\big\|\big(c(i,j)u_\alpha(i,j)\big)_{i\in\mathbb{Z}}\big\|_p\Big\}\\
&\le 2^\alpha \max\Big\{\sup_{i\in\mathbb{Z}}\Big\|\Big(\sum_{k\in\mathbb{Z}}|a(i,k)||b(k,j)|\big(u_\alpha(i,k)+u_\alpha(k,j)\big)\Big)_{j\in\mathbb{Z}}\Big\|_p,\\
&\qquad \sup_{j\in\mathbb{Z}}\Big\|\Big(\sum_{k\in\mathbb{Z}}|a(i,k)||b(k,j)|\big(u_\alpha(i,k)+u_\alpha(k,j)\big)\Big)_{i\in\mathbb{Z}}\Big\|_p\Big\}\\
&\le 2^\alpha\|A\|_{\mathcal{A}_{p,\alpha}}\|B\|_{\mathcal{A}_{1,0}} + 2^\alpha\|A\|_{\mathcal{A}_{1,0}}\|B\|_{\mathcal{A}_{p,\alpha}},
\end{aligned}\tag{2.18}$$

where the first inequality follows from the inequality

$$u_\alpha(i,j) \le 2^\alpha\big(u_\alpha(i,k) + u_\alpha(k,j)\big),\ i,j,k \in \mathbb{Z}.$$

Let $1/p' = 1 - 1/p$, and define

$$\tau_0 = \left\lfloor \left(\left(\frac{\alpha p'+1}{\alpha p'-1}\right)^{1/p'} \frac{\|A\|_{\mathcal{A}_{p,\alpha}}}{\|A\|_{\mathcal{B}(\ell^2)}}\right)^{1/(\alpha+1/2-1/p')}\right\rfloor, \tag{2.19}$$

where $\lfloor t \rfloor$ denotes the integer part of a real number t. Then for $i \in \mathbb{Z}$, we have

$$\begin{aligned}
\sum_{j\in\mathbb{Z}}|a(i,j)| &= \Big(\sum_{|j-i|\le\tau_0} + \sum_{|j-i|>\tau_0}\Big)|a(i,j)|\\
&\le \Big(\sum_{|j-i|\le\tau_0}|a(i,j)|^2\Big)^{1/2}\Big(\sum_{|j-i|\le\tau_0}1\Big)^{1/2}\\
&\quad + \Big(\sum_{|j-i|\ge\tau_0+1}|a(i,j)|^p(u_\alpha(i,j))^p\Big)^{1/p}\Big(\sum_{|j-i|\ge\tau_0+1}(u_\alpha(i,j))^{-p'}\Big)^{1/p'}\\
&\le \|A\|_{\mathcal{B}(\ell^2)}(2\tau_0+1)^{1/2} + 2^{1/p'}(\alpha p'-1)^{-1/p'}\|A\|_{\mathcal{A}_{p,\alpha}}(\tau_0+1)^{-\alpha+1/p'}\\
&\le D\|A\|_{\mathcal{A}_{p,\alpha}}^{1-\theta_0}\|A\|_{\mathcal{B}(\ell^2)}^{\theta_0},
\end{aligned}\tag{2.20}$$

where D is an absolute constant depending on p, α only, and the last inequality follows from (2.19) and the following estimate

$$\|A\|_{\mathcal{B}(\ell^2)} \le \|A\|_{\mathcal{A}_{1,0}} \le \Big(\sum_{k\in\mathbb{Z}}(|k|+1)^{-\alpha p'}\Big)^{1/p'}\|A\|_{\mathcal{A}_{p,\alpha}} \le \Big(\frac{\alpha p'+1}{\alpha p'-1}\Big)^{1/p'}\|A\|_{\mathcal{A}_{p,\alpha}}.$$

Similarly we can prove that

$$\sup_{j\in\mathbb{Z}}\sum_{i\in\mathbb{Z}}|a(i,j)| \le D\|A\|_{\mathcal{A}_{p,\alpha}}^{1-\theta_0}\|A\|_{\mathcal{B}(\ell^2)}^{\theta_0}. \tag{2.21}$$

Combining (2.20) and (2.21) leads to

$$\|A\|_{\mathcal{A}_{1,0}} \le D\|A\|_{\mathcal{A}_{p,\alpha}}^{1-\theta_0}\|A\|_{\mathcal{B}(\ell^2)}^{\theta_0}. \tag{2.22}$$

Replacing the matrix A in (2.22) by the matrix B gives

$$\|B\|_{\mathcal{A}_{1,0}} \le D\|B\|_{\mathcal{A}_{p,\alpha}}^{1-\theta_0}\|B\|_{\mathcal{B}(\ell^2)}^{\theta_0}. \tag{2.23}$$

Therefore it follows from (2.18), (2.22) and (2.23) that

$$\|C\|_{\mathcal{A}_{p,\alpha}} \le 2^\alpha D\|A\|_{\mathcal{A}_{p,\alpha}}\|B\|_{\mathcal{A}_{p,\alpha}}^{1-\theta_0}\|B\|_{\mathcal{B}(\ell^2)}^{\theta_0} + 2^\alpha D\|B\|_{\mathcal{A}_{p,\alpha}}\|A\|_{\mathcal{A}_{p,\alpha}}^{1-\theta_0}\|A\|_{\mathcal{B}(\ell^2)}^{\theta_0}, \tag{2.24}$$

which proves the differential subalgebra property for Banach algebras $\mathcal{A}_{p,\alpha}$ with $1 < p < \infty$ and $\alpha > 1 - 1/p$. □

The argument used in the proof of Theorem 2.5 involves a triplet of Banach algebras $\mathcal{A}_{p,\alpha}$, $\mathcal{A}_{1,0}$ and $\mathcal{B}^2$ satisfying (2.18) and (2.22). In the following theorem, we extend the above observation to general Banach algebra triplets $(\mathcal{A}, \mathcal{M}, \mathcal{B})$.

Theorem 2.6. *Let $\mathcal{A}, \mathcal{M}$ and $\mathcal{B}$ be Banach algebras such that $\mathcal{A}$ is a Banach subalgebra of $\mathcal{M}$ and $\mathcal{M}$ is a Banach subalgebra of $\mathcal{B}$. If there exist positive exponents $\theta_0, \theta_1 \in (0, 1]$ and absolute constants D_0, D_1 such that*

$$\|AB\|_{\mathcal{A}} \le D_0\|A\|_{\mathcal{A}}\|B\|_{\mathcal{A}}\Big(\Big(\frac{\|A\|_{\mathcal{M}}}{\|A\|_{\mathcal{A}}}\Big)^{\theta_0} + \Big(\frac{\|B\|_{\mathcal{M}}}{\|B\|_{\mathcal{A}}}\Big)^{\theta_0}\Big) \quad \text{for all } A, B \in \mathcal{A}, \tag{2.25}$$

and

$$\|A\|_{\mathcal{M}} \le D_1\|A\|_{\mathcal{A}}^{1-\theta_1}\|A\|_{\mathcal{B}}^{\theta_1} \quad \text{for all} \quad A \in \mathcal{A}, \tag{2.26}$$

then $\mathcal{A}$ is a differential subalgebra of order $\theta_0\theta_1$ in $\mathcal{B}$.

Proof. For any $A, B \in \mathcal{A}$, we obtain from (2.25) and (2.26) that

$$\begin{aligned} \|AB\|_{\mathcal{A}} &\le D_0\|A\|_{\mathcal{A}}\|B\|_{\mathcal{A}}\left(\Big(\frac{D_1\|A\|_{\mathcal{A}}^{1-\theta_1}\|A\|_{\mathcal{B}}^{\theta_1}}{\|A\|_{\mathcal{A}}}\Big)^{\theta_0} + \Big(\frac{D_1\|B\|_{\mathcal{A}}^{1-\theta_1}\|B\|_{\mathcal{B}}^{\theta_1}}{\|B\|_{\mathcal{A}}}\Big)^{\theta_0}\right) \\ &\le D_0 D_1^{\theta_0}\|A\|_{\mathcal{A}}\|B\|_{\mathcal{A}}\Big(\Big(\frac{\|A\|_{\mathcal{B}}}{\|A\|_{\mathcal{A}}}\Big)^{\theta_0\theta_1} + \Big(\frac{\|B\|_{\mathcal{B}}}{\|B\|_{\mathcal{A}}}\Big)^{\theta_0\theta_1}\Big), \end{aligned}$$

which completes the proof. □

Following the argument used in (2.3), we can show that $C^2[a, b]$ is a differential subalgebra of $C^1[a, b]$. For any distinct $x, y \in [a, b]$ and $f \in C^2[a, b]$, observe that

$$|f'(x)| = \frac{|f(y) - f(x) - f''(\xi)(y - x)^2/2|}{|y - x|} \le 2\|f\|_{C[a,b]}|y - x|^{-1} + \frac{1}{2}\|f''\|_{C[a,b]}|y - x|$$

for some $\xi \in [a, b]$, which implies that

$$\|f'\|_{C[a,b]} \le \max\big(4\|f\|_{C[a,b]}^{1/2}\|f''\|_{C[a,b]}^{1/2}, 8(b - a)^{-1}\|f\|_{C[a,b]}\big). \tag{2.27}$$

Therefore there exists a positive constant D such that

$$\|f\|_{C^1[a,b]} \le D\|f\|_{C^2[a,b]}^{1/2}\|f\|_{C[a,b]}^{1/2} \quad \text{for all} \quad f \in C^2[a, b]. \tag{2.28}$$

As an application of Theorem 2.6, we conclude that $C^2[a, b]$ is a differential subalgebra of order 1/2 in $C[a, b]$.

We finish the section with the proof of Theorem 2.4.

Proof of Theorem 2.4. The conclusion follows from (2.17) and Theorem 2.5 with $p = 1$ and $\alpha = 1$. □

3. Generalized differential subalgebras

By (2.1), a differential subalgebra $\mathcal{A}$ satisfies the Brandenburg's requirement:

$$\|A^2\|_{\mathcal{A}} \le 2D_0 \|A\|_{\mathcal{A}}^{2-\theta} \|A\|_{\mathcal{B}}^{\theta},\ A \in \mathcal{A}. \tag{3.1}$$

To consider the norm-controlled inversion of a Banach subalgebra $\mathcal{A}$ of $\mathcal{B}$, the above requirement (3.1) could be relaxed to the existence of an integer $m \ge 2$ such that the m-th power of elements in $\mathcal{A}$ satisfies

$$\|A^m\|_{\mathcal{A}} \le D \|A\|_{\mathcal{A}}^{m-\theta} \|A\|_{\mathcal{B}}^{\theta},\quad A \in \mathcal{A}, \tag{3.2}$$

where $\theta \in (0, m-1]$ and $D = D(\mathcal{A}, \mathcal{B}, m, \theta)$ is an absolute positive constant, see Theorem 4.1 in the next section. For $h \in C^1[a,b]$ and $m \ge 2$, we have

$$\|h^m\|_{C^1[a,b]} = m\|h^{m-1}h'\|_{C[a,b]} + \|h^m\|_{C[a,b]} \le m\|h\|_{C^1[a,b]} \|h\|_{C[a,b]}^{m-1},$$

and hence the differential subalgebra $C^1[a,b]$ of $C[a,b]$ satisfies (3.2) with $\theta = m-1$. In this section, we introduce some sufficient conditions so that (3.2) holds for some integer $m \ge 2$.

Theorem 3.1. *Let $\mathcal{A}, \mathcal{M}$ and $\mathcal{B}$ be Banach algebras such that $\mathcal{A}$ is a Banach subalgebra of $\mathcal{M}$ and $\mathcal{M}$ is a Banach subalgebra of $\mathcal{B}$. If there exist an integer $k \ge 2$, positive exponents θ_0, θ_1, and absolute constants E_0, E_1 such that*

$$\|A_1 A_2 \cdots A_k\|_{\mathcal{A}} \le E_0 \Big(\prod_{i=1}^{k} \|A_i\|_{\mathcal{A}}\Big) \sum_{j=1}^{k} \Big(\frac{\|A_i\|_{\mathcal{M}}}{\|A_i\|_{\mathcal{A}}}\Big)^{\theta_0},\quad A_1, \ldots, A_k \in \mathcal{A} \tag{3.3}$$

and

$$\|A^2\|_{\mathcal{M}} \le E_1 \|A\|_{\mathcal{A}}^{2-\theta_1} \|A\|_{\mathcal{B}}^{\theta_1},\ A \in \mathcal{A}, \tag{3.4}$$

then (3.2) *holds for $m = 2k$ and $\theta = \theta_0\theta_1$.*

Proof. By (1.1), (3.3) and (3.4), we have

$$\|A^{2k}\|_{\mathcal{A}} \le kE_0 \|A^2\|_{\mathcal{A}}^{k-\theta_0} \|A^2\|_{\mathcal{M}}^{\theta_0} \le kE_0 E_1^{\theta_0} K^{k-\theta_0} \|A\|_{\mathcal{A}}^{2k-\theta_0\theta_1} \|A\|_{\mathcal{B}}^{\theta_0\theta_1},\quad A \in \mathcal{A}, \tag{3.5}$$

which completes the proof. □

For a Banach algebra triplet $(\mathcal{A}, \mathcal{M}, \mathcal{B})$ in Theorem 2.6, we obtain from (2.25) and (2.26) that

$$\begin{aligned} \|A_1 A_2 \cdots A_k\|_{\mathcal{A}} &\le D_0 \|A_1\|_{\mathcal{A}} \|A_2 \cdots A_k\|_{\mathcal{A}} \left(\Big(\frac{\|A_1\|_{\mathcal{M}}}{\|A_1\|_{\mathcal{A}}}\Big)^{\theta_0} + \Big(\frac{\|A_2 \cdots A_k\|_{\mathcal{M}}}{\|A_2 \cdots A_k\|_{\mathcal{A}}}\Big)^{\theta_0} \right) \\ &\le \tilde{D}_0 \Big(\prod_{i=1}^{k} \|A_i\|_{\mathcal{A}}\Big) \sum_{j=1}^{k} \Big(\frac{\|A_j\|_{\mathcal{M}}}{\|A_j\|_{\mathcal{A}}}\Big)^{\theta_0},\quad A_1, \ldots, A_k \in \mathcal{A}, \end{aligned} \tag{3.6}$$

and

$$\|A^2\|_{\mathcal{M}} \le \tilde{K} \|A\|_{\mathcal{M}}^2 \le D_1^2 \tilde{K} \|A\|_{\mathcal{A}}^{2-2\theta_1} \|A\|_{\mathcal{B}}^{2\theta_1},\quad A \in \mathcal{A}, \tag{3.7}$$

where $\tilde{D}_0$ is an absolute constant and $\tilde{K}$ is the constant K in (1.1) for the Banach algebra $\mathcal{M}$. Therefore the assumptions (3.3) and (3.4) in Theorem 3.1 are satisfied for the Banach algebra triplet $(\mathcal{A}, \mathcal{M}, \mathcal{B})$ in Theorem 2.6.

For a differential subalgebra $\mathcal{A}$ of order θ_0 in $\mathcal{B}$, we observe that the requirements (3.3) and (3.4) with $\mathcal{M} = \mathcal{B}$, $k = 2$ and $\theta_1 = 2$ are met, and hence (3.2) holds for $m = 4$ and $\theta = 2\theta_0$. Recall that $\mathcal{B}$ is a trivial differential subalgebra of $\mathcal{B}$. In the following corollary, we can extend the above conclusion to arbitrary differential subalgebras $\mathcal{M}$ of $\mathcal{B}$.

Corollary 3.2. *Let $\mathcal{A}, \mathcal{M}$ and $\mathcal{B}$ be Banach algebras such that $\mathcal{A}$ is a differential subalgebra of order θ_0 in $\mathcal{M}$ and $\mathcal{M}$ is a differential subalgebra of order θ_1 in $\mathcal{B}$. Then* (3.2) *holds for $m = 4$ and $\theta = \theta_0\theta_1$.*

Following the argument used in the proof of Theorem 3.1, we can show that (3.2) holds for $m = 4$ if the requirement (3.3) with $k = 3$ is replaced by the following strong version

$$\|ABC\|_{\mathcal{A}} \le E_0 \|A\|_{\mathcal{A}} \|C\|_{\mathcal{A}} \|B\|_{\mathcal{A}}^{1-\theta_0} \|B\|_{\mathcal{M}}^{\theta_0},\quad A, B, C \in \mathcal{A}. \tag{3.8}$$

Theorem 3.3. *Let $\mathcal{A}, \mathcal{M}$ and $\mathcal{B}$ be Banach algebras such that $\mathcal{A}$ is a Banach subalgebra of $\mathcal{M}$ and $\mathcal{M}$ is a Banach subalgebra of $\mathcal{B}$. If there exist positive exponents $\theta_0, \theta_1 \in (0,1]$ and absolute constants E_0, E_1 such that* (3.4) *and* (3.8) *hold, then* (3.2) *holds for $m = 4$ and $\theta = \theta_0\theta_1$.*

Let $L^p := L^p(\mathbb{R}), 1 \le p \le \infty$, be the space of all p-integrable functions on $\mathbb{R}$ with standard norm $\|\cdot\|_p$, and $\mathcal{B}(L^p)$ be the algebra of bounded linear operators on L^p with the norm $\|\cdot\|_{\mathcal{B}(L^p)}$. For $1 \le p \le \infty, \alpha \ge 0$ and $\gamma \in [0,1)$, we define the norm of a kernel K on $\mathbb{R} \times \mathbb{R}$ by

$$\|K\|_{\mathcal{W}^\gamma_{p,\alpha}} = \begin{cases} \max\Big(\sup_{x\in\mathbb{R}} \big\|K(x,\cdot)u_\alpha(x,\cdot)\big\|_p,\ \sup_{y\in\mathbb{R}} \big\|K(\cdot,y)u_\alpha(\cdot,y)\big\|_p\Big) & \text{if } \gamma = 0 \\ \|K\|_{\mathcal{W}^0_{p,\alpha}} + \sup_{0<\delta\le 1} \delta^{-\gamma} \|\omega_\delta(K)\|_{\mathcal{W}^0_{p,\alpha}} & \text{if } 0 < \gamma < 1, \end{cases} \tag{3.9}$$

where the modulus of continuity of the kernel K is defined by

$$\omega_\delta(K)(x,y) := \sup_{|x'|\le\delta, |y'|\le\delta} |K(x+x', y+y') - K(x,y)|,\ x, y \in \mathbb{R}, \tag{3.10}$$

and $u_\alpha(x,y) = (1+|x-y|)^\alpha, x, y \in \mathbb{R}$ are polynomial weights on $\mathbb{R}\times\mathbb{R}$. Consider the set $\mathcal{W}^\gamma_{p,\alpha}$ of integral operators

$$Tf(x) = \int_{\mathbb{R}} K_T(x,y) f(y) dy, \quad f \in L^p, \tag{3.11}$$

whose integral kernels K_T satisfy $\|K_T\|_{\mathcal{W}^\gamma_{p,\alpha}} < \infty$, and define

$$\|T\|_{\mathcal{W}^\gamma_{p,\alpha}} := \|K_T\|_{\mathcal{W}^\gamma_{p,\alpha}},\ T \in \mathcal{W}^\gamma_{p,\alpha}.$$

Integral operators in $\mathcal{W}^\gamma_{p,\alpha}$ have their kernels being Hölder continuous of order γ and having off-diagonal polynomial decay of order α. For $1 \le p \le \infty$ and $\alpha > 1 - 1/p$, one may verify that $\mathcal{W}^\gamma_{p,\alpha}, 0 \le \gamma < 1$, are Banach subalgebras of $\mathcal{B}(L^2)$ under operator composition. The Banach algebras $\mathcal{W}^\gamma_{p,\alpha}, 0 < \gamma < 1$, of integral operators may not form a differential subalgebra of $\mathcal{B}(L^2)$, however the triple $(\mathcal{W}^\gamma_{p,\alpha}, \mathcal{W}^0_{p,\alpha}, \mathcal{B}(L^2))$ is proved in [42] to satisfy the following

$$\|T_0\|_{\mathcal{B}} \le D\|T_0\|_{\mathcal{W}^0_{p,\alpha}} \le D\|T_0\|_{\mathcal{W}^\gamma_{p,\alpha}}, \tag{3.12}$$

$$\|T_0^2\|_{\mathcal{W}^0_{p,\alpha}} \le D\|T_0\|^{1+\theta}_{\mathcal{W}^\gamma_{p,\alpha}} \|T_0\|^{1-\theta}_{\mathcal{B}(L^2)} \tag{3.13}$$

and

$$\|T_1T_2T_3\|_{\mathcal{W}^\gamma_{p,\alpha}} \le D\|T_1\|_{\mathcal{W}^\gamma_{p,\alpha}} \|T_2\|_{\mathcal{W}^0_{p,\alpha}} \|T_3\|_{\mathcal{W}^\gamma_{p,\alpha}} \tag{3.14}$$

holds for all $T_i \in \mathcal{W}^\gamma_{p,\alpha}, 0 \le i \le 3$, where D is an absolute constant and

$$\theta = \frac{\alpha + \gamma + 1/p}{(1+\gamma)(\alpha + 1/p)}.$$

Then the requirements (3.4) and (3.8) in Theorem 3.3 are met for the triplet $(\mathcal{W}^\gamma_{p,\alpha}, \mathcal{W}^0_{p,\alpha}, \mathcal{B}(L^2))$, and hence the Banach space pair $(\mathcal{W}^\gamma_{p,\alpha}, \mathcal{B}(L^2))$ satisfies the Brandenburg's condition (3.2) with $m = 4$ [15, 42].

4. Brandenburg trick and norm-controlled inversion

Let $\mathcal{A}$ and $\mathcal{B}$ are $*$-algebras with common identity and involution, and let $\mathcal{B}$ be symmetric. In this section, we show that $\mathcal{A}$ has norm-controlled inversion in $\mathcal{B}$ if it meets the Brandenburg requirement (3.2).

Theorem 4.1. *Let $\mathcal{B}$ be a symmetric $*$-algebra with its norm $\|\cdot\|_{\mathcal{B}}$ being normalized in the sense that* (1.1) *holds with $K = 1$,*

$$\|\tilde{A}\tilde{B}\|_{\mathcal{B}} \le \|\tilde{A}\|_{\mathcal{B}} \|\tilde{B}\|_{\mathcal{B}},\ \tilde{A}, \tilde{B} \in \mathcal{B}, \tag{4.1}$$

and $\mathcal{A}$ be a $*$-algebra with its norm $\|\cdot\|_{\mathcal{A}}$ being normalized too,

$$\|AB\|_{\mathcal{A}} \le \|A\|_{\mathcal{A}} \|B\|_{\mathcal{A}},\ A, B \in \mathcal{A}. \tag{4.2}$$

If $\mathcal{A}$ is a $*$-subalgebra of $\mathcal{B}$ with common identity I and involution $*$, and the pair $(\mathcal{A},\mathcal{B})$ satisfies the Brandenburg requirement (3.2), then $\mathcal{A}$ has norm-controlled inversion in $\mathcal{B}$. Moreover, for any $A\in\mathcal{A}$ being invertible in $\mathcal{B}$ we have

$$\begin{aligned}\|A^{-1}\|_{\mathcal{A}} \ \le\ & \|A^*A\|_{\mathcal{B}}^{-1}\|A^*\|_{\mathcal{A}}\\ & \times\begin{cases}\big(2t_0+(1-2^{\log_m(1-\theta/m)})^{-1}(\ln a)^{-1}\big)a\exp\Big(\frac{\ln m-\ln(m-\theta)}{\ln(m-\theta)}t_0\ln a\Big) & \text{if } \theta<m-1\\ a^2(\ln a)^{-1}(Db)^{m-1}\Gamma\Big(\frac{(m-1)\ln(Db)}{\ln m\ln a}+1\Big) & \text{if } \theta=m-1,\end{cases}\end{aligned}\tag{4.3}$$

where $\Gamma(s)=\int_0^\infty t^{s-1}e^{-t}dt$ is the Gamma function, $m\ge 2$ and $0<\theta\le m-1$ are the constants in (3.2)*, $\kappa(A^*A)=\|A^*A\|_{\mathcal{B}}\|(A^*A)^{-1}\|_{\mathcal{B}}$, $a=\big(1-(\kappa(A^*A))^{-1}\big)^{-1}>1$,*

$$b=\frac{\|I\|_{\mathcal{A}}+\|A^*A\|_{\mathcal{B}}^{-1}\|A^*A\|_{\mathcal{A}}}{1-(\kappa(A^*A))^{-1}}\ge a>1,$$

and

$$t_0=\Big(\frac{(m-1)(m-\theta)\log_m(m-\theta)\ln(Db)}{(m-1-\theta)\ln a}\Big)^{\ln m/(\ln m-\ln(m-\theta))}\quad\text{for } 0<\theta<m-1.\tag{4.4}$$

Proof. Obviously it suffices to prove (4.3). In this paper, we follow the argument in [36] to give a sketch proof. Let $A\in\mathcal{A}$ so that $A^{-1}\in\mathcal{B}$. As $\mathcal{B}$ is a symmetric $*$-algebra, the spectrum of A^*A in $\mathcal{B}$ lies in an interval on the positive real axis,

$$\sigma(A^*A)\subset\big[\|(A^*A)^{-1}\|_{\mathcal{B}}^{-1},\ \|A^*A\|_{\mathcal{B}}\big].\tag{4.5}$$

Therefore $B:=I-\|A^*A\|_{\mathcal{B}}^{-1}A^*A\in\mathcal{A}$ satisfies

$$\|B\|_{\mathcal{B}}\le 1-(\kappa(A^*A))^{-1}=a^{-1}<1\tag{4.6}$$

and

$$\|B\|_{\mathcal{A}}\le\|I\|_{\mathcal{A}}+\|A^*A\|_{\mathcal{B}}^{-1}\|A^*A\|_{\mathcal{A}}=ba^{-1}.\tag{4.7}$$

For a positive integer $n=\sum_{i=0}^l\varepsilon_i m^i$, define $n_0=n$ and $n_k, 1\le k\le l$, inductively by

$$n_k=\frac{n_{k-1}-\varepsilon_{k-1}}{m}=\sum_{i=k}^{l}\varepsilon_i m^{i-k}, 1\le k\le l,\tag{4.8}$$

where $\varepsilon_i\in\{0,1,\ldots,m-1\}, 1\le i\le l-1$ and $\varepsilon_l\in\{1,\ldots,m-1\}$. By (3.2) and (4.1), we have

$$\|B^{mn_k}\|_{\mathcal{A}}\le D\|B^{n_k}\|_{\mathcal{A}}^{m-\theta}\|B^{n_k}\|_{\mathcal{B}}^{\theta}\le D\|B^{n_k}\|_{\mathcal{A}}^{m-\theta}\|B\|_{\mathcal{B}}^{n_k\theta},\quad k=1,\cdots,l-1.\tag{4.9}$$

By (4.1), (4.2), (4.6), (4.7), (4.8) and (4.9), we obtain

$$\begin{aligned}\|B^n\|_{\mathcal{A}} &= \|B^{n_0}\|_{\mathcal{A}}\le\|B^{mn_1}\|_{\mathcal{A}}\|B\|_{\mathcal{A}}^{\varepsilon_0}\le D\|B^{n_1}\|_{\mathcal{A}}^{m-\theta}\|B\|_{\mathcal{A}}^{\varepsilon_0}\|B\|_{\mathcal{B}}^{n_1\theta}\\ &\le D^{1+(m-\theta)}\|B^{n_2}\|_{\mathcal{A}}^{(m-\theta)^2}\|B\|_{\mathcal{A}}^{\varepsilon_0+\varepsilon_1(m-\theta)}\|B\|_{\mathcal{B}}^{n_1\theta+n_2\theta(m-\theta)}\\ &\le \cdots\\ &\le D^{\sum_{k=0}^{l-1}(m-\theta)^k}\|B\|_{\mathcal{A}}^{\sum_{k=0}^{l}\varepsilon_k(m-\theta)^k}\|B\|_{\mathcal{B}}^{\theta\sum_{k=1}^{l}n_k(m-\theta)^{k-1}}\\ &= D^{\sum_{k=0}^{l-1}(m-\theta)^k}\|B\|_{\mathcal{A}}^{\sum_{k=0}^{l}\varepsilon_k(m-\theta)^k}\|B\|_{\mathcal{B}}^{n-\sum_{k=0}^{l}\varepsilon_k(m-\theta)^k}\\ &\le D^{\sum_{k=0}^{l-1}(m-\theta)^k}b^{\sum_{k=0}^{l}\varepsilon_k(m-\theta)^k}a^{-n}\\ &\le \begin{cases}(Db)^{\frac{(m-1)(m-\theta)}{m-1-\theta}}n^{\log_m(m-\theta)}a^{-n} & \text{if } \theta<m-1\\ (Db)^{(m-1)\log_m(mn+1)}a^{-n} & \text{if } \theta=m-1,\end{cases}\end{aligned}\tag{4.10}$$

where the last inequality holds since

$$\begin{aligned} \sum_{k=0}^{l} \varepsilon_k (m-\theta)^k &\le (m-1)\sum_{k=0}^{l}(m-\theta)^k \le (m-1)\begin{cases} \frac{(m-\theta)^{l+1}-1}{m-1-\theta} & \text{if } \theta < m-1 \\ l+1 & \text{if } \theta = m-1 \end{cases} \\ &\le (m-1)\begin{cases} \frac{m-\theta}{m-1-\theta} n^{\log_m (m-\theta)} & \text{if } \theta < m-1 \\ \log_m(mn+1) & \text{if } \theta = m-1. \end{cases} \end{aligned}$$

Observe that $A^*A = \|A^*A\|_{\mathcal{B}}(I-B)$. Hence

$$A^{-1} = (A^*A)^{-1}A^* = \|A^*A\|_{\mathcal{B}}^{-1}\Big(\sum_{n=0}^{\infty} B^n\Big)A^*.$$

This together with (4.2), (4.10) and (4.11) implies that

$$\begin{aligned} \|A^{-1}\|_{\mathcal{A}} &\le \|A^*A\|_{\mathcal{B}}^{-1}\|A^*\|_{\mathcal{A}} \sum_{n=0}^{\infty}\|B^n\|_{\mathcal{A}} \\ &\le \|A^*A\|_{\mathcal{B}}^{-1}\|A^*\|_{\mathcal{A}} \times \begin{cases} \sum_{n=0}^{\infty} (Db)^{\frac{(m-1)(m-\theta)}{m-1-\theta} n^{\log_m(m-\theta)}} a^{-n} & \text{if } \theta < m-1 \\ \sum_{n=0}^{\infty} (Db)^{(m-1)\log_m(mn+1)} a^{-n} & \text{if } \theta = m-1. \end{cases} \end{aligned} \tag{4.11}$$

By direct calculation, we have

$$\begin{aligned} \sum_{n=0}^{\infty} (Db)^{(m-1)\log_m(mn+1)} a^{-n} &\le a\sum_{n=0}^{\infty}\int_n^{n+1} (Db)^{(m-1)\log_m(mt+1)} a^{-t}\,dt \\ &\le a^2 (Db)^{m-1}\int_0^{\infty} (t+1)^{(m-1)\log_m(Db)} e^{-(t+1)\ln a}\,dt \\ &\le a^2 (Db)^{m-1}(\ln a)^{-1}\Gamma\Big(\frac{(m-1)\ln(Db)}{\ln m \ln a}+1\Big). \end{aligned} \tag{4.12}$$

This together with (4.11) proves (4.3) for $\theta = m-1$.

For $0<\theta<m-1$, set

$$s(t) = t - \frac{(m-1)(m-\theta)\ln(Db)}{(m-1-\theta)\ln a} t^{\log_m(m-\theta)}.$$

Observe that

$$s'(t) = 1 - \frac{(m-1)(m-\theta)\ln(Db)}{(m-1-\theta)\ln a}\log_m(m-\theta) t^{\log_m(1-\theta/m)}.$$

Therefore

$$\min_{t\ge 0} s(t) = s(t_0) = -\frac{\ln m - \ln(m-\theta)}{\ln(m-\theta)} t_0 < 0 \tag{4.13}$$

and

$$1 \ge s'(t) \ge s'(2t_0) = 1 - 2^{\log_m(1-\theta/m)} \qquad \text{for all } \ t \ge 2t_0, \tag{4.14}$$

where t_0 is given in (4.4). By (4.13) and (4.14), we have

$$\begin{aligned} &\sum_{n=0}^{\infty}(Db)^{\frac{(m-1)(m-\theta)}{m-1-\theta}n^{\log_m(m-\theta)}} a^{-n} \le a\sum_{n=0}^{\infty}\int_n^{n+1}(Db)^{\frac{(m-1)(m-\theta)}{m-1-\theta}t^{\log_m(m-\theta)}} a^{-t}\,dt \\ = \;& a\Big(\int_0^{2t_0} + \int_{2t_0}^{\infty}\Big)\exp(-s(t)\ln a)\,dt \\ \le \;& 2at_0\exp(-s(t_0)\ln a) + (1-2^{\log_m(1-\theta/m)})^{-1} a\int_{s(2t_0)}^{\infty}\exp(-u\ln a)\,du \\ \le \;& \Big(2t_0 + (1-2^{\log_m(1-\theta/m)})^{-1}(\ln a)^{-1}\Big) a\exp\Big(\frac{\ln m - \ln(m-\theta)}{\ln(m-\theta)} t_0 \ln a\Big). \end{aligned} \tag{4.15}$$

Combining the above estimate with (4.11) proves (4.3) for $\theta < m-1$. □

For $m = 2$, the estimate (4.3) to the inverse $A^{-1} \in \mathcal{A}$ is essentially established in [19, 20] for $\theta = 1$ and [36, 40] for $\theta \in (0, 1)$, and a similar estimate is given in [34]. The reader may refer to [15, 21, 42, 43, 45] for norm estimation of the inverse of elements in Banach algebras of infinite matrices and integral operators with certain off-diagonal decay.

Remark 4.2. A good estimate for the norm control function h in the norm-controlled inversion (1.4) is important for some mathematical and engineering applications. For an element $A \in \mathcal{A}$ with $A^{-1} \in \mathcal{B}$, we obtain the following estimate from Theorem 4.1:

$$\|(A^*A)^{-1}\|_{\mathcal{A}} \le C\|A^*A\|_{\mathcal{B}}^{-1} a(\ln a)^{-1} \times \begin{cases} t_1 \exp(Ct_1) & \text{if } \theta < m-1 \\ ab^{m-1} \exp\left(C\frac{\ln b}{\ln a} \ln\left(\frac{\ln b}{\ln a}\right)\right) & \text{if } \theta = m-1, \end{cases} \tag{4.16}$$

where C is an absolute constant independent of A and

$$t_1 = (\ln b)^{\ln m/(\ln m - \ln(m-\theta))}(\ln a)^{-\ln(m-\theta)/(\ln m - \ln(m-\theta))}.$$

We remark that the above norm estimate to the inversion is far away from the optimal estimation for our illustrative differential subalgebra $C^1[a, b]$. In fact, give any $f \in C^1[a, b]$ being invertible in $C[a, b]$, we have

$$\|1/f\|_{C^1[a,b]} \le \|f'\|_{C[a,b]}\|f^{-1}\|_{C[a,b]}^2 + \|1/f\|_{C[a,b]} \le \|1/f\|_{C[a,b]}^2 \|f\|_{C^1[a,b]}.$$

Therefore $C^1[a, b]$ has norm-controlled inversion in $C[a, b]$ with the norm control function $h(s, t)$ in (1.4) being $h(s, t) = st^2$. Gröchenig and Klotz first considered norm-controlled inversion with the norm control function h having polynomial growth, and they show in [19] that the Baskakov-Gohberg-Sjöstrand algebra $\mathcal{C}_{1,\alpha}, \alpha > 0$ and the Jaffard algebra $\mathcal{J}_\alpha, \alpha > 1$ have norm-controlled inversion in $\mathcal{B}(\ell^2)$ with the norm control function h bounded by a polynomial. In [36], we proved that the Beurling algebras $\mathcal{B}_{p,\alpha}$ with $1 \le p \le \infty$ and $\alpha > 1-1/p$ admit norm-controlled inversion in $\mathcal{B}(\ell^2)$ with the norm control function bounded by some polynomials. Following the commutator technique used in [36, 39], we can establish a similar result for the Baskakov-Gohberg-Sjöstrand algebras $\mathcal{C}_{p,\alpha}$ with $1 \le p \le \infty$ and $\alpha > 1 - 1/p$.

Theorem 4.3. *Let $1 \le p \le \infty$ and $\alpha > 1 - 1/p$. Then the Baskakov-Gohberg-Sjöstrand algebra $\mathcal{C}_{p,\alpha}$ and the Beurling algebra $\mathcal{B}_{p,\alpha}$ admit norm-controlled inversion in $\mathcal{B}(\ell^2)$ with the norm control function bounded by a polynomial.*

It is still unknown whether Gröchenig-Schur algebras $\mathcal{A}_{p,\alpha}, 1 \le p < \infty, \alpha > 1-1/p$, admit norm-controlled inversion in $\mathcal{B}(\ell^q), 1 \le q < \infty$, with the norm control function bounded by a polynomial. In [19], Gröchenig and Klotz introduce a differential operator $\mathcal{D}$ on a Banach algebra and use it to define a differential $*$-algebra $\mathcal{A}$ of a symmetric $*$-algebra $\mathcal{B}$, which admits norm-controlled inversion with the norm control function bounded by a polynomial. However, the differential algebra in [19] does not include the Gröchenig-Schur algebras $\mathcal{A}_{p,\alpha}$, the Baskakov-Gohberg-Sjöstrand algebra $\mathcal{C}_{p,\alpha}$ and the Beurling algebra $\mathcal{B}_{p,\alpha}$ with $1 \le p < \infty$ and $\alpha > 1 - 1/p$. It could be an interesting problem to extend the conclusions in Theorem 4.3 to general Banach algebras such that the norm control functions in the norm-controlled inversion have polynomial growth.

Remark 4.4. A crucial step in the proof of Theorem 4.1 is to introduce $B := I - \|A^*A\|_{\mathcal{B}}^{-1}A^*A \in \mathcal{A}$, whose spectrum is contained in an interval on the positive real axis. The above reduction depends on the requirements that $\mathcal{B}$ is symmetric and both $\mathcal{A}$ and $\mathcal{B}$ are $*$-algebras with common identity and involution. For the applications to some mathematical and engineering fields, the widely-used algebras $\mathcal{B}$ of infinite matrices and integral operators are the operator algebras $\mathcal{B}(\ell^p)$ and $\mathcal{B}(L^p), 1 \le p \le \infty$, which are symmetric only when $p = 2$. In [1, 15, 36, 38, 42, 48], inverse-closedness of localized matrices and integral operators in $\mathcal{B}(\ell^p)$ and $\mathcal{B}(L^p), 1 \le p \le \infty$, are discussed, and in [14], Beurling algebras $\mathcal{B}_{p,\alpha}$ with $1 \le p < \infty$ and $\alpha > d(1 - 1/p)$ are shown to admit polynomial norm-controlled inversion in nonsymmetric algebras $\mathcal{B}(\ell^p), 1 \le p < \infty$. It is still widely open to discuss Wiener's lemma and norm-controlled inversion when $\mathcal{B}$ and $\mathcal{A}$ are not $*$-algebras and $\mathcal{B}$ is not a symmetric algebra.

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