STABLE PHASELESS SAMPLING AND RECONSTRUCTION OF REAL-VALUED SIGNALS WITH FINITE RATE OF INNOVATION

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Abstract. A spatial signal is defined by its evaluations on the whole domain. In this paper, we consider stable reconstruction of real-valued signals with finite rate of innovation (FRI), up to a sign, from their magnitude measurements on the whole domain or their phaseless samples on a discrete subset. FRI signals appear in many engineering applications such as magnetic resonance spectrum, ultra wide-band communication and electrocardiogram. For an FRI signal, we introduce an undirected graph to describe its topological structure, establish the equivalence between its graph connectivity and its phase retrievability by point evaluation measurements on the whole domain, apply the graph connected component decomposition to find its unique landscape decomposition, and to find all FRI signals that have the same magnitude measurements. We construct discrete sets with finite density so that magnitude measurements of an FRI signal on the whole domain are determined by its phaseless samples taken on those discrete subsets, and we show that the corresponding phaseless sampling procedure has bi-Lipschitz property with respect to a new induced metric on the signal space and the standard ℓ^p-metric on the sampling data set. In this paper, we also propose an algorithm with linear complexity to reconstruct an FRI signal from its (un)corrupted phaseless samples on the above sampling set without restriction on the noise level and apriori information whether the original FRI signal is phase retrieval. The algorithm is theoretically guaranteed to be stable, and numerically demonstrated to approximate the original FRI signal in magnitude measurements.

1. Introduction

A spatial signal f on a domain D is defined by its evaluations $f(x), x \in D$. One of fundamental problems in real/complex phase retrieval is how to determine all signals g that have the same magnitude information as f has on the domain D (i.e., $|g(x)| = |f(x)|, x \in D$), or on a discrete sampling set $\Gamma \subset D$ (i.e., $|g(\gamma)| = |f(\gamma)|, \gamma \in \Gamma$). The above problem is a highly nonlinear ill-posed problem which can be solved only if we have some extra information about the signal f, and it has been discussed for bandlimited signals [56] and wavelet signals residing in a shift-invariant space [19, 20, 55]. In this paper, we consider the phaseless sampling and reconstruction (i.e., phase retrieval by point evaluation measurements on the whole domain or on a discrete set) of *real-valued* signals residing in the linear space

$$V(\Phi) := \Big\{ \sum_{\lambda \in \Lambda} c_\lambda \phi_\lambda : \ c_\lambda \in \mathbb{R} \text{ for all } \lambda \in \Lambda \Big\}, \tag{1.1}$$

where $\Lambda \subset D$ is a discrete set with finite density, and the generator $\Phi = (\phi_\lambda)_{\lambda \in \Lambda}$ is a vector of nonzero basis signals $\phi_\lambda, \lambda \in \Lambda$, essentially supported in a neighborhood of the innovative

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position $\lambda \in \Lambda$ [23, 52, 59], i.e., any signal f in the space $V(\Phi)$ has a parametric representation

$$f(x) = \sum_{\lambda \in \Lambda} c_\lambda \phi_\lambda(x), \ x \in D, \tag{1.2}$$

where $c = (c_\lambda)_{\lambda \in \Lambda}$ is an unknown real-valued parameter vector. Signals with the above parametric representation appear in many engineering applications such as magnetic resonance spectrum, ultra wide-band communication and electrocardiogram [23, 24, 53, 59]. The linear space $V(\Phi)$ was introduced in [53, 54] to model signals with finite rate of innovation (FRI), which was introduced by Vetterli, Marziliano and Blu in [59]. Sampling and reconstruction of various FRI signals have been well studied [23, 24, 25, 40, 53, 54, 59], while there are limited literatures on phase retrievability of FRI signals [5].

Given a signal $f \in V(\Phi)$, let

$$\mathcal{M}_f := \{g \in V(\Phi) : \ |g(x)| = |f(x)|, x \in D\} \tag{1.3}$$

contain all signals $g \in V(\Phi)$ with the same magnitude measurements as f on D. As $-f$ and f have the same magnitude measurements on the whole domain, we have that

$$\mathcal{M}_f \supset \{\pm f\}.$$

A natural question is whether the above inclusion is an equality.

Question 1.1. *Can we characterize all signals $f \in V(\Phi)$ so that $\mathcal{M}_f = \{\pm f\}$?*

An equivalent statement to the above question is whether a signal f is determined, up to a sign, from the magnitude information $|f(x)|, x \in D$. The above question is an *infinite-dimensional* phase retrieval problem with point evaluation measurements, which has been discussed for bandlimited signals [56], wavelet signals in a shift-invariant space [19, 20, 55]. The reader may refer to [1, 2, 12, 32, 42, 45, 50] for historical remarks and additional references on phase retrieval in an infinite-dimensional linear space. In Section 3, we introduce an undirected graph $\mathcal{G}_f$ for an FRI signal $f \in V(\Phi)$, and we provide an answer to Question 1.1 by showing that $\mathcal{M}_f = \{\pm f\}$ if and only if $\mathcal{G}_f$ is connected, see Theorem 3.2.

For a signal $f \in V(\Phi)$, the corresponding graph $\mathcal{G}_f$ is not always connected. This leads to our next question.

Question 1.2. *Can we find the set $\mathcal{M}_f$ for any signal $f \in V(\Phi)$?*

For a signal $f \in V(\Phi)$, we can decompose its graph $\mathcal{G}_f$ uniquely as a union of connected components $\mathcal{G}_i, i \in I$,

$$\mathcal{G}_f = \cup_{i \in I} \mathcal{G}_i. \tag{1.4}$$

Then we can construct signals $f_i \in V(\Phi), i \in I$, with $\mathcal{G}_{f_i} = \mathcal{G}_i, i \in I$, such that

$$f_i f_{i'} = 0 \text{ for all distinct } i, i' \in I, \tag{1.5}$$

$$\mathcal{M}_{f_i} = \{\pm f_i\}, \ i \in I, \tag{1.6}$$

and

$$f = \sum_{i \in I} f_i, \tag{1.7}$$

see Theorem 4.4. Due to the mutually disjoint support property (1.5) for signals $f_i, i \in I$, and the connectivity of the graphs $\mathcal{G}_{f_i}, i \in I$, we can interpret the above adaptive decomposition visually that the landscape of original signal f is composed by islands of signals $f_i, i \in I$, see the top left plot in Figure 1 and the left plot in Figure 2. Therefore the conclusion in Theorem 4.4 shows that landscapes of signals $g \in \mathcal{M}_f$ are combinations of islands of the original signal f and their reflections. We remark that landscape decomposition for signals in a linear space has been used in Gabor phase retrieval [1, 32]. By (1.5) and (1.7), we have

$$\mathcal{M}_f \supset \Big\{ \sum_{i\in I} \delta_i f_i : \ \delta_i \in \{-1, 1\}, i \in I \Big\}.$$

In Section 4, we provide an answer to Question 1.2 by showing in Theorem 4.1 that the above inclusion is in fact an equality for any signal $f \in V(\Phi)$, and hence there are $2^{\#I}$ elements in the set $\mathcal{M}_f$.

Now we consider phaseless sampling and reconstruction on a discrete set $\Gamma \subset D$. For a signal $f \in V(\Phi)$, let $\mathcal{M}_{f,\Gamma}$ contain all signals $g \in V(\Phi)$ such that

$$|g(\gamma)| = |f(\gamma)|, \ \gamma \in \Gamma, \tag{1.8}$$

and $\mathcal{N}_\Gamma$ contain all signals $h \in V(\Phi)$ such that

$$h(\gamma) = 0, \ \gamma \in \Gamma. \tag{1.9}$$

By (1.3), (1.8) and (1.9), we have

$$\mathcal{M}_f = \mathcal{M}_{f,D}, \ \mathcal{N}_D = \{0\}, \tag{1.10}$$

and

$$\mathcal{M}_f + \mathcal{N}_\Gamma \subset \mathcal{M}_{f,\Gamma} \text{ for all } \Gamma \subset D. \tag{1.11}$$

This leads to the third question.

Question 1.3. *Can we find all discrete sets Γ such that $\mathcal{M}_{f,\Gamma} = \mathcal{M}_f$ for all signals $f \in V(\Phi)$?*

An equivalent statement to the equality $\mathcal{M}_{f,\Gamma} = \mathcal{M}_f$ is that magnitude measurements $|f(x)|, x \in D$, on the whole domain D are determined by their samples $|f(\gamma)|, \gamma \in \Gamma$, taken on a discrete set Γ. By (1.11), a necessary condition such that the equality $\mathcal{M}_{f,\Gamma} = \mathcal{M}_f$ holds for some signal $f \in V(\Phi)$ is that $\mathcal{N}_\Gamma = \{0\}$, which means that all signals in the linear space $V(\Phi)$ are determined from their samples taken on Γ. The reader may refer to [24, 51, 54, 59] and references therein for sampling and reconstruction of FRI signals. In Section 5, we show the existence of a discrete set Γ with finite density such that $\mathcal{M}_{f,\Gamma} = \mathcal{M}_f$ for all signals $f \in V(\Phi)$. In Theorem 5.3, we construct such a discrete set Γ explicitly under the assumption that the linear space $V(\Phi)$ has local complement property on a family of open sets. The local complement property, see Definition 3.1, is introduced in [20] and it is closely related to the complement property for ideal sampling functionals in [19] and the complement property for frames in Hilbert/Banach spaces [2, 7, 10, 12]. The local complement property on a bounded open set can be characterized by phase retrievable frames associated with the generator Φ and the sampling set Γ on a finite-dimensional space, see Proposition 5.2. The reader may refer to [6, 7, 13, 16, 17, 27, 34, 36, 48, 61] and references therein for historical remarks and recent advances on finite-dimensional phase retrievable frames.

In many real world applications, phaseless samples are usually corrupted by some bounded deterministic/random noises $\eta(\gamma), \gamma \in \Gamma$, and the available noisy phaseless samples are

$$z_\eta(\gamma) = |f(\gamma)| + \eta(\gamma),\ \ \gamma \in \Gamma.$$

Set $\eta = (\eta(\gamma))_{\gamma\in\Gamma}$ and $z_\eta = (z_\eta(\gamma))_{\gamma\in\Gamma}$. This leads to the fourth question to be discussed in this paper.

Question 1.4. *Can we find an algorithm Δ such that the reconstructed signal $g_\eta = \Delta(z_\eta)$ is an "approximation" to the original signal f?*

For a finite-dimensional phase retrieval problem, there are various algorithms available, such as the alternating minimization, semidefinite programming, and Wirtinger flow method [14, 15, 16, 26, 28, 29, 44, 48, 63], however applicability of the algorithms, to our knowledge, requires that the original signal f is phase retrieval, i.e., $\mathcal{M}_f = \{\pm f\}$. In [19, 20], an MAPS algorithm is proposed to reconstruct a signal f in a shift-invariant space, up to a sign, from its phaseless samples taken on a shift-invariant set, where the original signal f is phase retrieval.

Given a Borel measure μ on the domain D, let $L^p := L^p(D, \mu), 1 \le p \le \infty$, be the linear space of all p-integrable signals with standard norm $\|\cdot\|_{L^p} := \|\cdot\|_{L^p(D,\mu)}$ and $\ell^p := \ell^p(\Gamma)$ be the space of all p-summable sequences η on Γ with its standard p-norm denoted by $\|\eta\|_{l^p} := \|\eta\|_{\ell^p(\Gamma)}$. Let $1 \le p \le \infty$ and define

$$V_p(\Phi) = \Big\{ \sum_{\lambda\in\Lambda} c_\lambda \phi_\lambda :\ (c_\lambda)_{\lambda\in\Lambda} \in \ell^p \Big\} \subset V(\Phi) \cap L^p,\ 1 \le p \le \infty. \tag{1.12}$$

In Section 6, we propose a robust algorithm with linear complexity so that the reconstructed signal g_η is a good approximation to the original signal in the linear space $V_p(\Phi)$, see Theorem 6.5 and Remarks 6.1–6.4 and 6.6. This provides an affirmative answer to Question 1.5 for the original signals in $V_p(\Phi)$.

Stability of a sampling scheme is an important concept for the robustness and uniqueness for sampling and reconstruction of signals in a linear space, see [3, 43, 54, 57]. Due to the nonlinearity, stability of the phaseless sampling scheme

$$S_\Gamma : V(\Phi) \ni f \longmapsto (|f(\gamma)|)_{\gamma\in\Gamma} \tag{1.13}$$

on a sampling set Γ should be described by its bi-Lipschitz property in some metrics on the signal space and the sampling data set. This leads to the fifth question to be discussed in this paper.

Question 1.5. *Let Γ be a sampling set such that $\mathcal{M}_{f,\Gamma} = \mathcal{M}_f$ for all $f \in V(\Phi)$. Can we define appropriate metrics on the signal space and on the sampling data set such that the phaseless sampling operator S_Γ in* (1.13) *has the bi-Lipschitz property?*

For $1 \le p \le \infty$, we define the natural metric for phase retrievability on the signal space $V_p(\Phi)$ by

$$m_p(f, g) = \min(\|f + g\|_{L^p}, \|f - g\|_{L^p}) \text{ for all } f, g \in V_p(\Phi), \tag{1.14}$$

and the ℓ^p-metric on the phaseless sampling data set by

$$D_p(S_\Gamma f, S_\Gamma g) = \|S_\Gamma f - S_\Gamma g\|_{l^p} \text{ for all } f, g \in V_p(\Phi), \tag{1.15}$$

cf. [1, 9, 10, 27]. The nonlinear sampling operator S_Γ in (1.13) does not have the bi-Lipschitz property with respect to the natural metric m_p on the signal space $V_p(\Phi)$ and the ℓ^p-metric D_p in the phaseless sampling data set, i.e., there does not exist positive constants C_1 and C_2 such that

$$C_1 m_p(f,g) \le D_p(S_\Gamma f, S_\Gamma g) \le C_2 m_p(f,g) \text{ for all } f,g \in V_p(\Phi).$$

The reason is that some signals in $V_p(\Phi)$ may not be determined, up to a sign, from their phaseless samples on Γ, and hence we can find $f,g \in V_p(\Phi)$ such that

$$m_p(f,g) \neq 0 \text{ and } D_p(S_\Gamma f, S_\Gamma g) = 0.$$

In this paper, we induce a new metric on the signal space $V_p(\Phi), 1 \le p \le \infty$,

$$M_p(f,g) = \inf_{\tilde f, \tilde g \in V_p(\Phi) \text{ satisfying } \mathcal{M}_{\tilde f} = \mathcal{M}_{\tilde g}} \max\big(\|f-\tilde f\|_{L^p}, \|g-\tilde g\|_{L^p}\big). \tag{1.16}$$

Clearly we have that

$$\||f|-|g|\|_{L^p}/2 \le M_p(f,g) \le m_p(f,g) \text{ for all } f,g \in V_p(\Phi),\ 1 \le p \le \infty. \tag{1.17}$$

In Theorem 6.7, we show that the phaseless sampling operator S_Γ has the bi-Lipschitz property with respect to the above new metric M_p on the infinite-dimensional signal space $V_p(\Phi)$ and the ℓ^p-metric on the phaseless sampling data set.

1.1. **Contributions and Comparisons.** This paper is the continuation of [19, 20]. In [19, 20], we discuss the phaseless sampling and reconstruction of wavelet signals in a shift-invariant space

$$V(\phi) := \Big\{ \sum_{k \in \mathbb{Z}^d} c(k)\phi(\cdot - k) :\ c(k) \in \mathbb{R} \text{ for all } k \in \mathbb{Z}^d \Big\} \tag{1.18}$$

generated by a compactly supported function ϕ, while spatial signals considered in this paper belong to the linear space $V(\Phi)$ in (1.1). Our representative examples of the linear space $V(\Phi)$, different from the shift-invariant space $V(\phi)$, are the linear space of all graph signals to describe structured data in applications such as social networks, smart power grids, wireless sensor networks, and drone/UAV fleets [21], and the linear space of superpositions

$$f(x) = \sum_{\lambda \in \Lambda} c_\lambda \phi(x-\lambda),\ x \in \mathbb{R}^d, \tag{1.19}$$

of non-uniform translations $\phi_\lambda = \phi(\cdot - \lambda), \lambda \in \Lambda \neq \mathbb{Z}^d$ of a basis signal ϕ, which has been used in some sampling and approximation problems [4, 33, 53, 54]. Similarity between the shift-invariant space $V(\phi)$ considered in [19, 20] and the linear space $V(\Phi)$ used in this paper is that we both assume that any signal in those two linear spaces has a unique parametric representation, see (2.6), while the main difference is that the linear space $V(\Phi)$ does not have a shift-invariant structure. Our first challenge is how to define the local complementary property of the linear space $V(\Phi)$ appropriately, and our first main contribution is to characterize all phase retrieval signals f in the linear space $V(\Phi)$, i.e., $\mathcal{M}_f = \{\pm f\}$, see Theorem 3.2, which has been discussed in [19, 20] for signals in the shift-invariant spaces $V(\phi)$.

Spatial signals $f \in V(\Phi)$ are not always determined, up to a sign, from the magnitude information $|f(x)|, x \in D$ on the domain D, i.e., $\mathcal{M}_f \neq \{\pm f\}$. In such a scenario, we aim at finding all FRI signals in the set $\mathcal{M}_f$ which have the same magnitude information on their domain D as the original FRI signal f has. In [19, Lemma 6.9], it has been shown that any signal in some shift-invariant space $V(\phi)$ on the real line has a unique landscape decomposition and the set $\mathcal{M}_f$ is fully described. For signals in the shift-invariant space $V(\phi)$ on $\mathbb{R}^d, d \geq 2$, we can apply connected component decomposition to the associated graphs in [20], and to find their landscape decompositions, however the uniqueness of landscape decompositions is not established and the set $\mathcal{M}_f$ is not discussed. The second challenge is the uniqueness of such a landscape decomposition, and the second main contribution is that we provide a *full* description to the set $\mathcal{M}_f$ and we discover a *unique* landscape decomposition for any signal f in the linear space $V(\Phi)$ using connected component decomposition of the associated graph $\mathcal{G}_f$, see Theorems 4.1 and 4.4.

In [19], an MAPS algorithm is proposed to reconstruct a signal f in the shift-invariant space on the real line, up to a sign, from its phaseless samples $|f(x_n + k)|, x_n \in X \subset [0, 1]$, when the original signal f is phase retrieval. The algorithm has linear complexity and it consists of three steps: 1) minimization to find local approximations; 2) phase adjustment for local approximations; and 3) sewing local approximation together to reconstruct the original signal, up to a sign. In [19, Theorem 4.1], it is shown that the MAPS algorithm is robust against small noises. A high-dimensional version of the MAPS algorithm is introduced in [20] to reconstruct a signal f in the shift-invariant space on the d-dimensional Euclidean space, up to a sign, from its phaseless samples $|f(x_n + k)|, x_n \in X \subset [0, 1]^d$, when the original signal f is phase retrieval. In this paper, we introduce a new strategy in the phase adjustment step and propose a new MAPS algorithm to reconstruct signals in the linear space $V_p(\Phi), 1 \leq p \leq \infty$ from their (un)corrupted phaseless samples. The third main contribution is that the reconstructed signal obtained from the proposed MAPS algorithm is an "approximation" to the original signal in the linear space $V_p(\Phi)$ *without* restriction on noise level and *apriori* information on the original signal f, see Theorem 6.5 and Remark 6.6. Moreover the proposed algorithm is robust and non-iterative, and it has linear complexity, see Remarks 6.2–6.4.

In [19, 20], we consider the *local* stability of a phaseless sampling operator S_Γ in natural metric m_∞ in the shift-invariant space $V_\infty(\phi)$, where

$$V_p(\phi) := \Big\{ \sum_{k\in\mathbb{Z}^d} c(k)\phi(\cdot - k) :\ (c(k))_{k\in\mathbb{Z}^d} \in \ell^p \Big\},\ 1 \leq p \leq \infty. \tag{1.20}$$

It is shown in [19, Theorem 4.1] that for any phase retrieval signal $f \in V_\infty(\phi)$ on the real line, there exist positive constants A and ϵ_f (depending on f) such that

$$A m_\infty(g, f) \leq D_\infty(S_\Gamma g, S_\Gamma f)$$

hold for all signals $g \in V_\infty(\phi)$ satisfying $D_\infty(S_\Gamma g, S_\Gamma f) \leq \epsilon_f$. The fourth main contribution is that we construct sampling sets Γ with finite density so that the nonlinear sampling operator S_Γ in (1.13) has bi-Lipschitz property with respect to the metric M_p in (1.16) on a linear subspace $V_p(\Phi), 1 \leq p \leq \infty$, i.e., there exists positive constants A_1 and A_2 such that

$$A_1 M_p(g, f) \leq D_p(S_\Gamma g, S_\Gamma f) \leq A_2 M_p(g, f) \text{ for all } g, f \in V_p(\Phi),$$

see Theorem 6.7. To the best of our knowledge, the above stability inequality is the first *global* estimation for certain phase retrieval signals in an infinite-dimensional linear space.

1.2. **Organization.** In Section 2, we present some preliminaries on the linear space $V(\Phi)$. In Section 3, we introduce a graph structure for any signal in $V(\Phi)$ and use its connectivity to provide an answer to Question 1.1. In Section 4, we introduce a landscape decomposition for a signal $f \in V(\Phi)$ and use it to find all signals in $\mathcal{M}_f$. In Section 5, we construct a discrete set Γ with finite density such that $\mathcal{M}_{f,\Gamma} = \mathcal{M}_f$ for all $f \in V(\Phi)$. In Section 6, we introduce a stable algorithm Δ with linear complexity to reconstruct signals in $V(\Phi)$ from their noisy phaseless samples taken on a discrete set Γ and show that the phaseless sampling operator S_Γ in (1.13) has bi-Lipschitz property with respect to the metric M_p in (1.16). In Section 7, we demonstrate the stable reconstruction of our proposed algorithm Δ by reconstructing one-dimensional non-uniform spline signals and two-dimensional piecewise affine signals on triangulations from their noisy phaseless samples. In Appendix A, we show that the density of a discrete set Γ with $\mathcal{M}_{f,\Gamma} = \mathcal{M}_f$, $f \in V(\Phi)$, must be no less than the innovative rate of signals in $V(\Phi)$.

2. Preliminaries

Spatial signals considered in the paper are defined on a domain D. Our representing models of the domain D are the d-dimensional Euclidean space $\mathbb{R}^d$, the d-dimensional torus $\mathbb{T}^d$ and the vertex set V of a undirected graph $\mathcal{G} = (V, E)$ containing no graph loops or multiple edges that is widely used to describe a spatially distributed network [21]. Let

$$B(x, r) = \{y \in D : \ \rho(x, y) \le r\}$$

be the closed ball with center $x \in D$ and radius $r \ge 0$. In this paper, we always assume the following for the domain D [21, 41, 62].

Assumption 2.1. *The domain D is equipped with a distance ρ and a Borel measure μ so that*

$$B(r) := \sup_{x\in D} \mu\big(B(x, r)\big) < \infty \tag{2.1}$$

and

$$\liminf_{s\to\infty} \inf_{x\in D} \frac{\mu(B(x, s - r))}{\mu(B(x, s))} = 1 \tag{2.2}$$

hold for all $r \ge 0$.

Spatial signals considered in this paper belong to the linear space $V(\Phi)$ in (1.1). Denote the cardinality of a set E by $\#E$. In this paper, we always assume the following three conditions to basis signals $\phi_\lambda, \lambda \in \Lambda$, of the linear space $V(\Phi)$ in (1.1).

Assumption 2.2. *(i) The discrete set Λ has finite density*

$$D_+(\Lambda) := \limsup_{r\to\infty} \sup_{x\in D} \frac{\sharp\big(\Lambda \cap B(x, r)\big)}{\mu\big(B(x, r)\big)} < \infty; \tag{2.3}$$

(ii) the basis signals $\phi_\lambda, \lambda \in \Lambda$, in the generator Φ are nonzero continuous functions being uniformly bounded,

$$\|\Phi\|_\infty := \sup_{\lambda\in\Lambda} \|\phi_\lambda\|_{L^\infty} < \infty, \tag{2.4}$$

and they are supported in balls with center λ and a fixed radius $r_0 > 0$ independent of λ, i.e.,

$$\phi_\lambda(x) = 0 \text{ for all } x \notin B(\lambda, r_0) \text{ and } \lambda \in \Lambda; \tag{2.5}$$

and (iii) any signal in $V(\Phi)$ has a unique parametric representation (1.2).

The prototypical forms of the linear space $V(\Phi)$ in (1.1) are Paley-Wiener space of bandlimited signals [56, 57], the shift-invariant space $V(\phi)$ generated by the shifts of a compactly supported function ϕ [3, 19, 20], twisted shift-invariant spaces generated by (non-)uniform Gabor frame system (or Wilson basis) in the time-frequency analysis [8, 18, 31, 37, 47], and nonuniform spline signals [11, 35, 49]. The linear space $V(\Phi)$ was introduced in [53, 54] to model FRI signals. Following the terminology in [59], signals in the linear space $V(\Phi)$ have rate of innovations $D_+(\Lambda)$ and innovative positions $\lambda \in \Lambda$.

An equivalent statement to the unique parametric representation (1.2) of FRI signals in $V(\Phi)$ in Assumption 2.2 is that the generator Φ has *global linear independence*, i.e., the map

$$c := (c_\lambda)_{\lambda\in\Lambda} \longmapsto c^T\Phi := \sum_{\lambda\in\Lambda} c_\lambda \phi_\lambda \tag{2.6}$$

is one-to-one from the space $\ell(\Lambda)$ of all sequences on Λ to the linear space $V(\Phi)$ [39, 46]. For an open set A, define

$$K_A = \{\lambda \in \Lambda : \ \phi_\lambda \not\equiv 0 \text{ on } A\}. \tag{2.7}$$

A local version of the global linear independence (2.6) is *local linear independence* on a bounded open set $A \subset D$, i.e.,

$$\dim V(\Phi)|_A = \#K_A, \tag{2.8}$$

where $\dim V$ is the dimension for a linear space V and $V|_A$ represents its restriction on a set A. Observe that the restriction of the linear space $V(\Phi)$ on a bounded open set A is generated by $\phi_\lambda, \lambda \in K_A$ (and hence it is finite-dimensional). Then an equivalent formulation of the local linear independence on a bounded open set A is that

$$\sum_{\lambda\in\Lambda} c_\lambda \phi_\lambda(x) = 0 \text{ for all } x \in A \tag{2.9}$$

implies that $c_\lambda = 0$ for all $\lambda \in K_A$ [39, 52].

Set

$$S_\Phi(\lambda, \lambda') := \{x \in D : \ \phi_\lambda(x)\phi_{\lambda'}(x) \neq 0\}, \ \lambda, \lambda' \in \Lambda, \tag{2.10}$$

and use the abbreviation

$$S_\Phi(\lambda) := S_\Phi(\lambda, \lambda)$$

when $\lambda' = \lambda \in \Lambda$. One may verify that the generator Φ has global linear independence (2.6) if it has local linear independence on a family of open sets $T_\theta, \theta \in \Theta$, such that

$$S_\Phi(\lambda, \lambda') \cap \big(\cup_{\theta\in\Theta} T_\theta\big) \neq \emptyset \tag{2.11}$$

for all pairs $(\lambda, \lambda') \in \Lambda \times \Lambda$ with $S_\Phi(\lambda, \lambda') \neq \emptyset$. We remark that a family of open sets $T_\theta, \theta \in \Theta$, satisfying (2.11) is not necessarily a covering of the domain D, however, the converse is true, cf. Corollary 4.3.

3. Phase retrievability and graph connectivity

In this section, we characterize all signals $f \in V(\Phi)$ that can be determined, up to a sign, from their magnitude measurements on the whole domain D, i.e., $\mathcal{M}_f = \{\pm f\}$, see Theorem 3.2.

Given a signal $f = \sum_{\lambda \in \Lambda} c_\lambda \phi_\lambda \in V(\Phi)$, we define an undirected graph

$$\mathcal{G}_f := (V_f, E_f), \tag{3.1}$$

where

$$V_f := \{\lambda \in \Lambda : \ c_\lambda \neq 0\} \tag{3.2}$$

and

$$E_f := \big\{(\lambda, \lambda') \in V_f \times V_f : \ \lambda \neq \lambda' \text{ and } \phi_\lambda \phi_{\lambda'} \not\equiv 0\big\}.$$

For a signal $f \in V(\Phi)$, the graph $\mathcal{G}_f$ in (3.1) is well-defined by (2.6), and it was introduced in [20] when the generator $\Phi = (\phi(\cdot - k))_{k \in \mathbb{Z}^d}$ is obtained from shifts of a compactly supported function ϕ. Its vertex set V_f contains all innovative positions $\lambda \in \Lambda$ with nonzero amplitude c_λ, and its edge set E_f contains all innovative position pairs (λ, λ') in $V_f \times V_f$ with basis signals ϕ_λ and $\phi_{\lambda'}$ having overlapped supports, i.e.,

$$(\lambda, \lambda') \in E_f \text{ if and only if } \lambda, \lambda' \in V_f \text{ and } (\lambda, \lambda') \in E_\Phi, \tag{3.3}$$

where $S_\Phi(\lambda, \lambda'), (\lambda, \lambda') \in \Lambda \times \Lambda$, are given in (2.10) and

$$E_\Phi := \{(\lambda, \lambda') \in \Lambda \times \Lambda : \ S_\Phi(\lambda, \lambda') \neq \emptyset\}. \tag{3.4}$$

To study the phase retrievability of signals in $V(\Phi)$, we recall the local complement property for a linear space of real-valued signals [20].

Definition 3.1. Let A be an open subset of the domain D. We say that a linear space V of real-valued signals on the domain D has *local complement property* on A if for any $A' \subset A$, there does not exist $f, g \in V$ such that $f, g \not\equiv 0$ on A, but $f(x) = 0$ for all $x \in A'$ and $g(y) = 0$ for all $y \in A \backslash A'$.

The local complement property is the complement property in [19] for ideal sampling functionals on a set, cf. the complement property for frames in Hilbert/Banach spaces ([2, 7, 10, 12]). Local complement property is closely related to local phase retrievability. In fact, following the argument in [19], the linear space V has the local complement property on A if and only if all signals in V are *local phase retrieval* on A, i.e., for any $f, g \in V$ satisfying $|g(x)| = |f(x)|, x \in A$, there exists $\delta \in \{-1, 1\}$ such that $g(x) = \delta f(x)$ for all $x \in A$.

In this section, we establish the equivalence between phase retrievability of a nonzero signal $f \in V(\Phi)$ and connectivity of its graph $\mathcal{G}_f$. A similar result is established in [20] for signals residing in a shift-invariant space.

Theorem 3.2. *Let Φ be a family of basis functions satisfying Assumption 2.2, $V(\Phi)$ be the linear space* (1.1) *generated by Φ, and let $\mathcal{T} = \{T_\theta, \theta \in \Theta\}$ be a family of bounded open sets satisfying* (2.11). *Assume that for any $T_\theta \in \mathcal{T}$, Φ has local linear independence on T_θ and $V(\Phi)$ has local complement property on T_θ. Then for a nonzero signal $f \in V(\Phi)$, $\mathcal{M}_f = \{\pm f\}$ if and only if the graph $\mathcal{G}_f$ in* (3.1) *is connected.*

As shown in the next proposition, the local complement property assumption in Theorem 3.2 is satisfied when Φ has local linear independence on all bounded open sets. However, we do not use the above strong assumption in our main theorems, as there are very few families of basis signals available (including those generated by integer shifts of B-splines, scaling/wavelet functions, and box splines), which have local linear independence on all bounded open sets [22, 30, 38, 52].

Proposition 3.3. *Let $\Phi = (\phi_\lambda)_{\lambda\in\Lambda}$ satisfy Assumption 2.2. If Φ has local linear independence on all bounded open sets, then there exist $\mathcal{T} = \{T_\theta, \theta \in \Theta\}$ satisfying* (2.11) *such that $V(\Phi)$ has local complement property on every $T_\theta \in \mathcal{T}$.*

Proof. Define $T_\Phi(\theta) = \cap_{\lambda\in\theta} S_\Phi(\lambda)$ for a set $\theta \subset \Lambda$. We say that $\theta \subset \Lambda$ is maximal if $T_\Phi(\theta) \neq \emptyset$ and $T_\Phi(\theta') = \emptyset$ for all $\theta' \supsetneq \theta$. By (2.3) and (2.5), any maximal set contains finitely many elements. Denote the family of all maximal sets by Θ and define $T_\theta = T_\Phi(\theta), \theta \in \Theta$. Clearly $\mathcal{T} := \{T_\theta, \theta \in \Theta\}$ satisfies (2.11), because any $\theta \subset \Lambda$ with $T_\Phi(\theta) \neq \emptyset$ is a subset of some maximal set in Θ.

Now it remains to prove that $V(\Phi)$ has local complement property on $T_\theta, \theta \in \Theta$. Take an arbitrary $\theta \in \Theta$ and two signals $f, g \in V(\Phi)$ satisfying $|f(x)| = |g(x)|$ for all $x \in T_\theta$. Then

$$(f+g)(x)(f-g)(x) = 0 \text{ for all } x \in T_\theta. \tag{3.5}$$

Write $f + g = \sum_{\lambda\in\Lambda} c_\lambda \phi_\lambda$ and $f - g = \sum_{\lambda\in\Lambda} d_\lambda \phi_\lambda$, and set $B_1 = \{x \in T_\theta : (f+g)(x) \neq 0\}$ and $B_2 = \{x \in T_\theta : (f-g)(x) \neq 0\}$. Then

$$\Big(\sum_{\lambda\in\theta} c_\lambda \phi_\lambda(x)\Big)\Big(\sum_{\lambda\in\theta} d_\lambda \phi_\lambda(x)\Big) = 0 \text{ for all } x \in T_\theta, \tag{3.6}$$

and

$$\phi_\lambda(x) \neq 0 \text{ for all } x \in T_\theta \text{ and } \lambda \in \theta \tag{3.7}$$

by assumption (2.11), (3.5) and the construction of maximal sets. By (3.6), we have that $f - g = 0$ on B_1 if $B_1 \neq \emptyset$, $f + g = 0$ on B_2 if $B_2 \neq \emptyset$, and $f - g = f + g = 0$ on T_θ if $B_1 = B_2 = \emptyset$. This together with (3.7) and the local linear independence on B_1, B_2 and T_θ implies that either $d_\lambda = 0$ for all $\lambda \in \theta$, or $c_\lambda = 0$ for all $\lambda \in \theta$, or $c_\lambda = d_\lambda = 0$ for all $\lambda \in \theta$. Therefore either $f = g$ on T_θ, or $f = -g$ on T_θ, or $f = g = 0$ on T_θ. This completes the proof. ☐

Applying Theorem 3.2 and Proposition 3.3, we have the following corollary, which is established in [20] when the generator Φ is obtained from uniform shifts of a compactly supported function.

Corollary 3.4. *Let Φ be a family of basis functions satisfying Assumption 2.2, and $V(\Phi)$ be the linear space* (1.1) *generated by Φ. If Φ has local linear independence on any bounded open set, then a nonzero signal $f \in V(\Phi)$ satisfies $\mathcal{M}_f = \{\pm f\}$ if and only if the graph $\mathcal{G}_f$ in* (3.1) *is connected.*

3.1. **Proof of Theorem 3.2.** The necessity in Theorem 3.2 holds under a weak assumption on the generator Φ.

Proposition 3.5. *Let* $\Phi := (\phi_\lambda)_{\lambda\in\Lambda}$ *be a family of basis functions satisfying Assumption 2.2,* $V(\Phi)$ *be the linear space* (1.1) *generated by* Φ*, and let* f *be a nonzero signal in* $V(\Phi)$*. If* $\mathcal{M}_f = \{\pm f\}$*, then the graph* $\mathcal{G}_f$ *in* (3.1) *is connected.*

To prove Proposition 3.5, we recall a characterization in [19] on phase retrievability.

Lemma 3.6. *For a nonzero signal* f *in a real-valued linear space* V*,* $\mathcal{M}_f = \{\pm f\}$ *if and only if it is nonseparable, i.e., there does not exist nonzero signals* f_0 *and* $f_1 \in V$ *such that*

$$f = f_0 + f_1 \quad \text{and} \quad f_0 f_1 = 0. \tag{3.8}$$

Proof of Proposition 3.5. Let $f \in V(\Phi)$ be a nonzero signal satisfying $\mathcal{M}_f = \{\pm f\}$, and write $f = \sum_{\lambda\in\Lambda} c_\lambda \phi_\lambda$, where $c_\lambda \in \mathbb{R}, \lambda \in \Lambda$. Suppose, on the contrary, that the graph $\mathcal{G}_f$ is disconnected. Then there exists a nontrivial connected component W such that both W and $V_f\backslash W$ are nontrivial, and no edges exist between vertices in W and in $V_f\backslash W$. Write

$$f = \sum_{k\in V_f} c_\lambda\phi_\lambda = \sum_{\lambda\in W} c_\lambda\phi_\lambda + \sum_{\lambda\in V_f\backslash W} c_\lambda\phi_\lambda =: f_0 + f_1. \tag{3.9}$$

From the global linear independence (2.6) and nontriviality of the sets W and $V_f\backslash W$, we obtain

$$f_0 \not\equiv 0 \quad \text{and} \quad f_1 \not\equiv 0. \tag{3.10}$$

Applying (3.9) and (3.10), and using the characterization in Lemma 3.6, we obtain that

$$f_0(x_0)f_1(x_0) \neq 0$$

for some $x_0 \in D$. This implies the existence of $\lambda \in W$ and $\lambda' \in V_f\backslash W$ such that $c_\lambda\phi_\lambda(x_0) \neq 0$ and $c_{\lambda'}\phi_{\lambda'}(x_0) \neq 0$. Hence (λ, λ') is an edge between $\lambda \in W$ and $\lambda' \in V_f\backslash W$, which contradicts to the construction of the set W. □

Now we prove the sufficiency in Theorem 3.2. Let $f = \sum_{\lambda\in\Lambda} c_\lambda\phi_\lambda \in V(\Phi)$ have its graph $\mathcal{G}_f$ being connected, and take $g = \sum_{\lambda\in\Lambda} d_\lambda\phi_\lambda \in \mathcal{M}_f$. Then for any $\theta \in \Theta$,

$$|g(x)| = |f(x)|, \ x \in T_\theta. \tag{3.11}$$

For any $\theta \in \Theta$, there exists $\delta_\theta \in \{-1, 1\}$ by (3.11) and the local complement property on T_θ such that

$$g(x) = \delta_\theta f(x), \ x \in T_\theta.$$

This together with the local linear independence on T_θ implies that

$$d_\lambda = \delta_\theta c_\lambda \tag{3.12}$$

for all $\lambda \in \Lambda$ with $S_\Phi(\lambda) \cap T_\theta \neq \emptyset$. Using (2.11) and applying (3.12), there exist $\delta_\lambda \in \{-1, 1\}, \lambda \in \Lambda$ such that

$$d_\lambda = \delta_\lambda c_\lambda \tag{3.13}$$

for all $\lambda \in \Lambda$, and

$$\delta_\lambda = \delta_{\lambda'} \tag{3.14}$$

for any edge (λ, λ') in the graph $\mathcal{G}_f$. Combining (3.13) and (3.14), and applying connectivity of the graph $\mathcal{G}_f$, we can find $\delta \in \{-1, 1\}$ such that

$$d_\lambda = c_\lambda = 0 \text{ for all } \lambda \notin V_f \text{ and } d_\lambda = \delta c_\lambda \text{ for all } \lambda \in V_f. \tag{3.15}$$

Thus $g(x) = \delta f(x)$ for all $x \in D$. This completes the proof of the sufficiency.

4. Phase retrievability and landscape decomposition

For a signal $f \in V(\Phi)$, the graph $\mathcal{G}_f$ in (3.1) is not necessarily connected and hence there may exist many signals $g \in V(\Phi)$, other than $\pm f$, belonging to $\mathcal{M}_f$. In this section, we characterize the set $\mathcal{M}_f$ of all signals $g \in V(\Phi)$ that have the same magnitude measurements on the domain D as f has, and then we provide the answer to Question 1.2.

Take $f = \sum_{\lambda \in \Lambda} c_\lambda \phi_\lambda \in V(\Phi)$, let $\mathcal{G}_i = (V_i, E_i), i \in I$, be connected components of the graph $\mathcal{G}_f$, and define

$$f_i = \sum_{\lambda \in V_i} c_\lambda \phi_\lambda, \ i \in I. \tag{4.1}$$

Then (1.4) holds by the definition of $\mathcal{G}_i, i \in I$, and the signal f has the decomposition (1.5), (1.6) and (1.7) by Theorem 3.2. By (1.5) and (1.7), signals $g = \sum_{i \in I} \delta_i f_i$ with $\delta_i \in \{-1, 1\}, i \in I$, have the same magnitude measurements on the domain D as f has. In the following theorem, we show that the converse is also true.

Theorem 4.1. *Let the generator $\Phi := (\phi_\lambda)_{\lambda \in \Lambda}$, the family $\mathcal{T} = \{T_\theta, \theta \in \Theta\}$ of bounded open sets, and the linear space $V(\Phi)$ be as in Theorem 3.2. Take $f \in V(\Phi)$ and let $f_i \in V(\Phi), i \in I$, be as in* (4.1)*. Then $g \in V(\Phi)$ belongs to $\mathcal{M}_f$ if and only if*

$$g = \sum_{i \in I} \delta_i f_i \quad \text{for some } \delta_i \in \{-1, 1\}, i \in I. \tag{4.2}$$

Proof. The sufficiency is obvious. Now the necessity. Let $f, g \in V(\Phi)$ have the same magnitude measurements on the domain D, i.e., $\mathcal{M}_f = \mathcal{M}_g$. Write $f = \sum_{\lambda \in \Lambda} c_\lambda \phi_\lambda$ and $g = \sum_{\lambda \in \Lambda} d_\lambda \phi_\lambda$. Then following the argument used in the sufficiency of Theorem 3.2, we can find $\delta_{\lambda, \lambda'} \in \{-1, 1\}$ for any pair (λ, λ') with $S_\Phi(\lambda, \lambda') \neq \emptyset$ such that

$$(d_\lambda, d_{\lambda'}) = \delta_{\lambda, \lambda'}(c_\lambda, c_{\lambda'}). \tag{4.3}$$

Applying (4.3) with $\lambda' = \lambda$ and recalling that $S_\Phi(\lambda) \neq \emptyset$, we obtain

$$d_\lambda = \delta_\lambda c_\lambda, \ \lambda \in \Lambda, \tag{4.4}$$

for some $\delta_\lambda \in \{-1, 1\}$. This concludes that

$$\delta_\lambda = \delta_{\lambda, \lambda'} = \delta_{\lambda'} \tag{4.5}$$

for any edge (λ, λ') of the graph $\mathcal{G}_f$. Therefore signs δ_λ are the same in any connected component of the graph $\mathcal{G}_f$. This together with (1.4), (4.1) and (4.4) completes the proof. □

The conclusion in Theorem 4.1 can be understood as that the landscape of any signal $g \in \mathcal{M}_f$ is a combination of islands of the original signal f or their reflections. As an application to Theorem 4.1, we have the following result about the cardinality of the set $\mathcal{M}_f$.

Corollary 4.2. *Let the generator Φ, the family $\mathcal{T}$ of bounded open sets, and the linear space $V(\Phi)$ be as in Theorem 3.2. Then for $f \in V(\Phi)$,*

$$\#\mathcal{M}_f = 2^{\# I},$$

where I is given in (1.4).

The union of $T_\theta, \theta \in \Theta$, is not necessarily the whole domain D. Following the argument used in the proof of Theorems 3.2 and 4.1, we have the following corollary.

Corollary 4.3. *Let the generator Φ, the family $\mathcal{T} = \{T_\theta, \theta \in \Theta\}$ of bounded open sets and the linear space $V(\Phi)$ be as in Theorem 4.1. Then*

$$\mathcal{M}_f = \mathcal{M}_{f, D_{\mathcal{T}}} \text{ for all } f \in V(\Phi), \tag{4.6}$$

where $D_{\mathcal{T}} = \cup_{\theta \in \Theta} T_\theta$.

Proof. Let $f, g \in V(\Phi)$ satisfy $|f(x)| = |g(x)|, x \in T_\theta$ for all $\theta \in \Theta$. Write $f = \sum_{i \in I} f_i$ as in (1.5), (1.6) and (1.7). From the argument used in the proof of Theorems 3.2 and 4.1, we have that $g = \sum_{i \in I} \delta_i f_i$ for some $\delta_i \in \{-1, 1\}$. Therefore $|g(x)| = |f(x)|$ for all $x \in D$. □

Take $f = \sum_{\lambda \in \Lambda} c_\lambda \phi_\lambda \in V(\Phi)$, and define $f_i, i \in I$, by (4.1). As discussed in the paragraph just before the statement of Theorem 4.1, the above functions $f_i, i \in I$ form a landscape decomposition of the signal f satisfying (1.5), (1.6) and (1.7). In the next theorem we show the uniqueness of the landscape decomposition satisfying (1.5), (1.6) and (1.7).

Theorem 4.4. *Let the generator Φ and the space $V(\Phi)$ be as in Theorem 4.1. Then for any $f \in V(\Phi)$ there exists a unique decomposition satisfying* (1.5), (1.6) *and* (1.7).

Proof. Write $f = \sum_{\lambda \in \Lambda} c_\lambda \phi_\lambda$ and define $f_i, i \in I$, by (4.1). Suppose that $\{g_j, j \in J\}$ is another decomposition of the signal f satisfying (1.5), (1.6) and (1.7). Then $g_j = \sum_{\lambda \in \Lambda} d_{j,\lambda} \phi_\lambda, j \in J$, are nonzero signals in $V(\Phi)$ such that satisfy

$$f = \sum_{j \in J} g_j, \tag{4.7}$$

$$\mathcal{M}_{g_j} = \{\pm g_j\}, \ j \in J, \tag{4.8}$$

and

$$g_j g_{j'} = 0 \text{ for all distinct } j, j' \in J. \tag{4.9}$$

Then it suffices to find $I_j, j \in J$, such that

$$I_j \text{ only contains exactly one element for any } j \in J, \tag{4.10}$$

$$g_j = \sum_{i \in I_j} f_i, \tag{4.11}$$

and

$$\cup_{j \in J} I_j = I \text{ and } I_j \cap I_{j'} = \emptyset \text{ for all distinct } j, j' \in J, \tag{4.12}$$

as in this case there is an bijective map $\mathcal{P}$ from J and I such that $g_j = f_{\mathcal{P}(j)}, j \in J$.

First we prove (4.11) and (4.12). For any distinct $j, j' \in J$ and $(\lambda, \lambda') \in \Lambda \times \Lambda$ with $S_\Phi(\lambda, \lambda') \neq \emptyset$, following the argument used in the sufficiency of Theorem 3.2 with f and g replaced by $g_j \pm g_{j'}$ we obtain from (4.9) that

$$\text{either } (d_{j,\lambda}, d_{j,\lambda'}) = (0, 0) \text{ or } (d_{j',\lambda}, d_{j',\lambda'}) = (0, 0).$$

This together with (4.7) implies that for any $(\lambda, \lambda') \in \Lambda \times \Lambda$ with $S_\Phi(\lambda, \lambda') \neq \emptyset$ there exists $j \in J$ such that

$$d_{j,\lambda} = c_\lambda, \ d_{j,\lambda'} = c_{\lambda'} \tag{4.13}$$

and

$$d_{j',\lambda} = d_{j',\lambda'} = 0 \text{ for all } j' \neq j. \tag{4.14}$$

Observe that $S_\Phi(\lambda) \neq \emptyset, \lambda \in \Lambda$. Applying (4.13) and (4.14) with $\lambda' = \lambda \in \Lambda$, we can find $W_j, j \in J$ such that

$$g_j = \sum_{\lambda \in W_j} c_\lambda \phi_\lambda, \tag{4.15}$$

and

$$\cup_{j \in J} W_j = V_f \text{ and } W_j \cap W_{j'} = \emptyset \text{ for all distinct } j, j' \in J. \tag{4.16}$$

Let $V_i \subset V_f, i \in I$ be as in (4.1). Applying (4.13) and (4.14) with (λ, λ') being an edge in $\mathcal{G}_f$, we obtain that for any $i \in I$ there exists $j \in J$ such that $V_i \subset W_j$. This together with (4.16) implies the existence of a subset I_j of I for every $j \in J$ such that

$$W_j = \cup_{i \in I_j} V_i \text{ for all } j \in J. \tag{4.17}$$

Then the conclusion (4.11) follows from (4.1) and (4.17), and the partition property (4.12) holds by (4.16), (4.17) and the observation that $\cup_{i \in I} V_i = V_f$.

Now we prove (4.10). By (1.5) and (4.11) we have that

$$\mathcal{M}_{g_j} \supset \Big\{ \sum_{i \in I_j} \delta_i f_i, \delta_i \in \{-1, 1\} \Big\},$$

which implies that $\#\mathcal{M}_{g_j} \geq 2^{\# I_j}$. This together with (4.8) proves (4.10). □

5. Phaseless sampling and reconstruction

In this section, we consider phaseless sampling and reconstruction of signals in $V(\Phi)$, and we construct a discrete set Γ such that

$$\mathcal{M}_{f,\Gamma} = \mathcal{M}_f \text{ for all } f \in V(\Phi), \tag{5.1}$$

and its density $D_+(\Gamma)$ is dominated by a multiple of the innovative rate $D_+(\Lambda)$ of signals in $V(\Phi)$.

First, we recall the concept of a (minimal) phase retrievable frame [7, 20, 27, 34, 61].

Definition 5.1. We say that $\mathcal{F} = \{f_m \in \mathbb{R}^n, 1 \leq m \leq M\}$ is a *phase retrievable frame* for $\mathbb{R}^n$ if any vector $v \in \mathbb{R}^n$ is determined, up to a sign, by its measurements $|\langle v, f_m\rangle|, f_m \in \mathcal{F}$, and that $\mathcal{F}$ is a *minimal phase retrievable frame* for $\mathbb{R}^n$ if any true subset of $\mathcal{F}$ is not a phase retrievable frame.

The concept of minimal phase retrievable frame is crucial for us to prove the existence of the phaseless sampling set on which the linear space $V(\Phi)$ has local complement property, cf. [20, Theorem A.4].

Proposition 5.2. *Let the generator $\Phi := (\phi_\lambda)_{\lambda\in\Lambda}$, the family $\mathcal{T} = \{T_\theta, \theta \in \Theta\}$ of bounded open sets, and the linear space $V(\Phi)$ be as in Theorem 3.2. Assume that Φ has local linear independence on open sets $T_\theta, \theta \in \Theta$. Then for any $\theta \in \Theta$, the linear space $V(\Phi)$ generated by Φ has local complement property on T_θ if and only if there exists a finite set $\Gamma_\theta \subset T_\theta$ such that $\{\Phi_\theta(\gamma), \gamma \in \Gamma_\theta\}$ is a minimal phase retrievable frame for $\mathbb{R}^{\# K_\theta}$, where*

$$\Phi_\theta = (\phi_\lambda)_{\lambda\in K_\theta} \text{ and } K_\theta = \{\lambda \in \Lambda : S_\Phi(\lambda) \cap T_\theta \neq \emptyset\}. \tag{5.2}$$

Set

$$R_\Lambda(r) := \sup_{x\in D} \#\big(\Lambda \cap B(x, r)\big),\ r \geq 0. \tag{5.3}$$

We remark that $K_\theta, \theta \in \Theta$ in (5.2) are finite subsets of Λ and their cardinalities are bounded by $R_\Lambda(2r_0)$, see (5.12). In the next theorem, we explicitly construct the phaseless sampling set such that (5.1) holds, and its density is dominated by a multiple of the innovative rate of the signal in $V(\Phi)$.

Theorem 5.3. *Let the domain D satisfy Assumption 2.1, $\Phi := (\phi_\lambda)_{\lambda\in\Lambda}$ be a family of basis functions satisfying Assumption 2.2, $V(\Phi)$ be the linear space* (1.1) *generated by Φ, and $\mathcal{T} = \{T_\theta, \theta \in \Theta\}$ be a family of bounded open sets so that* (2.11) *holds and for every $\theta \in \Theta$, Φ has local linear independence on T_θ and $V(\Phi)$ has local complement property on T_θ. Take discrete sets $\Gamma_\theta \subset T_\theta, \theta \in \Theta$, so that for any $\theta \in \Theta$, $\{\Phi_\theta(\gamma), \gamma \in \Gamma_\theta\}$ forms a minimal phase retrievable frame for $\mathbb{R}^{\# K_\theta}$, and define*

$$\Gamma := \cup_{\theta\in\Theta}\Gamma_\theta, \tag{5.4}$$

where Φ_θ and $K_\theta \subset \Lambda$ is given in (5.2)*. Then* (5.1) *holds for the above discrete set Γ. Moreover if*

$$N_{\mathcal{T}} := \sup_{\lambda\in\Lambda} \#\{\theta :\ T_\theta \cap S_\Phi(\lambda) \neq \emptyset\} < \infty, \tag{5.5}$$

then the set Γ has finite upper density

$$D_+(\Gamma) \leq \frac{R_\Lambda(2r_0)(R_\Lambda(2r_0) + 1)}{2} N_{\mathcal{T}} D_+(\Lambda), \tag{5.6}$$

where r_0 is given in (2.5)*.*

We remark that the existence of discrete sets $\Gamma_\theta, \theta \in \Theta$ in Theorem 5.3 follows from the local complement property on $T_\theta, \theta \in \Theta$, for the linear space $V(\Phi)$, by applying the argument in [20, Theorem A.4].

As an application of Theorem 5.3, we have the following phaseless sampling corollary, which is established in [19, 20] for signals residing in a shift-invariant space generated by a compactly supported function.

Corollary 5.4. *Let $D, \Lambda, \mathcal{T}, \Phi, V(\Phi)$ and Γ be as in Theorem 5.3. Then any signal $f \in V(\Phi)$ with $\mathcal{M}_f = \{\pm f\}$ is determined, up to a sign, from its phaseless samples on the discrete set Γ with finite density.*

In practical applications, the set $\{\Phi_\theta(\gamma), \gamma \in \Gamma_\theta\}, \theta \in \Theta$ is not necessarily required to form a minimal phase retrievable frame for $\mathbb{R}^{\#K_\theta}$. In particular, the set Γ_θ can be chosen such that the density is still dominated by the rate of innovations of signals in $V(\Phi)$ and the set of outer products $\Phi_\theta(\gamma)\Phi_\theta^T(\gamma), \gamma \in \Gamma_\theta$ forms a frame/basis for the linear space of symmetric matrices spanned by outer products $\Phi_\theta(x)(\Phi_\theta(x))^T, x \in T_\theta$.

We finish this section with the proof of Theorem 5.3.

Proof of Theorem 5.3. First we prove (5.1). By (1.11), it suffices to prove

$$\mathcal{M}_{f,\Gamma} \subset \mathcal{M}_f. \tag{5.7}$$

Take $g = \sum_{\lambda\in\Lambda} d_\lambda \phi_\lambda \in \mathcal{M}_{f,\Gamma}$, and write $f = \sum_{\lambda\in\Lambda} c_\lambda\phi_\lambda$. Then for any $\theta \in \Theta$,

$$\Big|\sum_{\lambda\in K_\theta} c_\lambda\phi_\lambda(\gamma)\Big| = |f(\gamma)| = |g(\gamma)| = \Big|\sum_{\lambda\in K_\theta} d_\lambda\phi_\lambda(\gamma)\Big| \ \text{ for all } \gamma \in \Gamma_\theta.$$

This together with the phase retrievable frame property of $\Phi_\theta(\gamma), \gamma \in \Gamma_\theta$, implies that

$$d_\lambda = \delta_\theta c_\lambda,\ \lambda \in K_\theta \tag{5.8}$$

for some $\delta_\theta \in \{-1, 1\}$. Hence for any $\theta \in \Theta$,

$$|g(x)| = |f(x)|,\ x \in T_\theta. \tag{5.9}$$

This together with Corollary 4.3 implies that $g \in \mathcal{M}_f$. This proves (5.7).

To prove (5.6), we claim that for any $\theta \in \Theta$,

$$S_\Phi(\lambda, \lambda') \neq \emptyset \text{ for all } \lambda, \lambda' \in K_\theta. \tag{5.10}$$

Suppose on the contrary that the above claim does not hold, then there exist $\lambda_0, \lambda_0' \in K_\theta$ with $S_\Phi(\lambda_0, \lambda_0') = \emptyset$. Thus $\phi_{\lambda_0} \pm \phi_{\lambda_0'} \in V(\Phi)$ have the same magnitude measurements on T_θ, which contradicts to the local complement property of the space $V(\Phi)$ on $T_\theta, \theta \in \Theta$.

Applying Claim (5.10) and Assumption 2.2, we obtain

$$B(\lambda, r_0) \cap B(\lambda', r_0) \neq \emptyset \text{ for all } \lambda, \lambda' \in K_\theta. \tag{5.11}$$

This implies that

$$\#K_\theta \le R_\Lambda(2r_0),\ \theta \in \Theta. \tag{5.12}$$

Observe that for any $f \in V(\Phi)$, there exists a unique vector $c_\theta = (c_\lambda)_{\lambda\in K_\theta}$ such that

$$|f(x)|^2 = c_\theta^T \Phi_\theta(x)(\Phi_\theta(x))^T c_\theta,\ x \in T_\theta.$$

This together with the minimality of the phase retrievable frame $\{\Phi_\theta(\gamma), \gamma \in \Gamma_\theta\}$ for $\mathbb{R}^{\#K_\theta}$ implies that matrices $\Phi_\theta(\gamma)(\Phi_\theta(\gamma))^T, \gamma \in \Gamma_\theta$ are linearly independent in the linear space of symmetric matrices, which has dimension $\#K_\theta(\#K_\theta + 1)/2$. Hence

$$\#\Gamma_\theta \le \frac{\#K_\theta(\#K_\theta+1)}{2} \le \frac{R_\Lambda(2r_0)(R_\Lambda(2r_0)+1)}{2} \ \text{ for all } \theta \in \Theta, \tag{5.13}$$

where the last inequality follows from (5.12).

By the minimality of the phase retrievable frame $\{\Phi_\theta(\gamma), \gamma \in \Gamma_\theta\}$, we have $\Phi_\theta(\gamma) \neq 0$ for all $\gamma \in \Gamma_\theta$, which implies that

$$\Gamma_\theta \subset \big(\cup_{\lambda\in K_\theta} S_\Phi(\lambda)\big) \cap T_\theta. \tag{5.14}$$

Then for any $x \in D$ and $r \ge 0$, we obtain from (5.5), (5.13), (5.14) and Assumption 2.2 that

$$
\begin{aligned}
\#(\Gamma \cap B(x,r)) &\le \Big(\max_{\theta\in\Theta} \#\Gamma_\theta\Big) \\
&\quad \times \#\big\{\theta \in \Theta : \big(\cup_{\lambda\in K_\theta} S_\Phi(\lambda)\big) \cap T_\theta \cap B(x,r) \ne \emptyset\big\} \\
&\le \frac{R_\Lambda(2r_0)(R_\Lambda(2r_0)+1))}{2}\Big(\max_{\lambda\in\Lambda} \#\{\theta\in\Theta : S_\Phi(\lambda)\cap T_\theta \ne \emptyset\}\Big) \\
&\quad \times \#\big\{\lambda \in \Lambda : \ S_\Phi(\lambda) \cap B(x,r) \ne \emptyset\big\} \\
&\le \frac{R_\Lambda(2r_0)(R_\Lambda(2r_0)+1))}{2} N_{\mathcal{T}} \#\big(\Lambda \cap B(x, r+r_0)\big). \qquad (5.15)
\end{aligned}
$$

This together with (2.2) in Assumption 2.1 and definition of the density (2.3) of a discrete set proves (5.6). □

6. Stable Reconstruction from Phaseless Samples

In this section, we introduce the MAPS algorithm to reconstruct FRI signals in $V_p(\Phi), 1 \le p \le \infty$, approximately from their noisy phaseless samples taken on a discrete set Γ, we show that the MAPS algorithm is theoretically guaranteed to provide a stable reconstruction to the original FRI signal in the magnitude measurements, and we prove that the phaseless sampling operator S_Γ has the bi-Lipschitz property with respect the metric M_p in (1.16).

Let $\mathcal{T} = \{T_\theta : \ \theta \in \Theta\}$ satisfy (2.11) and $\Gamma = \cup_{\theta\in\Theta}\Gamma_\theta$ with $\Gamma_\theta \subset T_\theta, \theta \in \Theta$ be as in Theorem 5.3. Let $f \in V_p(\Phi), 1 \le p \le \infty$, and

$$
z_\eta(\gamma) = |f(\gamma)| + \eta(\gamma), \ \ \gamma \in \Gamma, \qquad (6.1)
$$

be its samples on a discrete set Γ corrupted by a p-summable noise $\eta = (\eta(\gamma))_{\gamma\in\Gamma}$. A conventional approach to reconstruct the signal f approximately from its noisy phaseless samples (6.1) is to solve the minimization problem

$$
f_\eta = \arg \min_{g \in V_p(\Phi)} \big\|(|g(\gamma)| - z_\eta(\gamma))_{\gamma\in\Gamma}\big\|_{\ell^p}, \qquad (6.2)
$$

which is infinite-dimensional and infeasible. In this section, we propose the following three-step algorithm, MAPS for abbreviation, to construct a signal

$$
g_\eta = \sum_{\lambda\in\Lambda} d_{\eta;\lambda}\phi_\lambda \qquad (6.3)
$$

in $V_p(\Phi)$ from the noisy phaseless samples $z_\eta(\gamma), \gamma \in \Gamma$, which is a good approximation to the original signal f in magnitude measurements, see Theorem 6.5 and Remark 6.6.

MAPS algorithm for phaseless reconstruction

0. Select a phase adjustment threshold value $M_0 \ge 0$ and set $K_\theta = \{\lambda \in \Lambda : S_\Phi(\lambda) \cap T_\theta \neq \emptyset\}$ for $\theta \in \Theta$.
1. For $\theta \in \Theta$, let

$$c_{\eta,\theta} = (c_{\eta,\theta;\lambda})_{\lambda\in\Lambda} \tag{6.4}$$

take zero components except that $(c_{\eta,\theta;\lambda})_{\lambda\in K_\theta}$ is a solution of the local **m**inimization problem

$$\min_{(d_\lambda)_{\lambda\in K_\theta}} \sum_{\gamma\in\Gamma_\theta} \Big|\Big|\sum_{\lambda\in K_\theta} d_\lambda \phi_\lambda(\gamma)\Big| - z_\eta(\gamma)\Big|^2. \tag{6.5}$$

2. **A**djust **p**hases of vectors $c_{\eta,\theta}, \theta \in \Theta$, so that the resulting vectors $\delta_{\eta,\theta} c_{\eta,\theta}$ with $\delta_{\eta,\theta} \in \{-1, 1\}$ have their inner product satisfying

$$\begin{aligned}\langle \delta_{\eta,\theta} c_{\eta,\theta}, \delta_{\eta,\theta'} c_{\eta,\theta'}\rangle &= \delta_{\eta,\theta}\delta_{\eta,\theta'} \sum_{\lambda\in K_\theta\cap K_{\theta'}} c_{\eta,\theta;\lambda} c_{\eta,\theta';\lambda} \\ &\ge -M_0 \times \Big(\sup_{\gamma\in\Gamma_\theta\cup\Gamma_{\theta'}} |\eta(\gamma)|\Big)^2 \qquad (6.6)\end{aligned}$$

for all $\theta, \theta' \in \Theta$, where we set $\sup_{\gamma\in\Gamma_\theta\cup\Gamma_{\theta'}} |\eta(\gamma)| = +\infty$ if $\Gamma_\theta \cup \Gamma_{\theta'} = \emptyset$.
3. **S**ew vectors $\delta_{\eta,\theta} c_{\eta,\theta}, \theta \in \Theta$, together to obtain

$$d_{\eta;\lambda} = \frac{\sum_{\theta\in\Theta} \delta_{\eta,\theta} c_{\eta,\theta;\lambda} \chi_{K_\theta}(\lambda)}{\sum_{\theta\in\Theta} \chi_{K_\theta}(\lambda)},\ \lambda \in \Lambda, \tag{6.7}$$

where χ_E is the indicator function on a set E.

Remark 6.1. The earliest version of the above MAPS algorithm is proposed in [19] to reconstruct phase retrieval signals in a shift-invariant space on the real line from their phaseless samples, where $\Theta = \mathbb{Z}$, $\Gamma_\theta = \Gamma_0 + \theta, \theta \in \Theta$ for some $\Gamma_0 \subset [0, 1]$, and the phase adjustment signs $\delta_{\eta,\theta} \in \{-1, 1\}$ in the second step are selected to satisfy

$$\langle \delta_{\eta,\theta} c_{\eta,\theta}, \delta_{\eta,\theta'} c_{\eta,\theta'}\rangle \ge 0 \ \text{ for all } \theta, \theta' \in \mathbb{Z} \text{ with } \theta' - \theta = 1. \tag{6.8}$$

The MAPS algorithm is modified in [20] to reconstruct phase retrieval signals in a shift-invariant space on high-dimensional Euclidean space $\mathbb{R}^d$ from their phaseless samples, where $\Theta = \mathbb{Z}^d \times \{1, \ldots, M\}$, $\Gamma_{k,m} = \Gamma_m + k, (k, m) \in \Theta$ for some bounded sets $\Gamma_m, 1 \le m \le M$, and the phase adjustment signs $\delta_{\eta,\theta} \in \{-1, 1\}$ in the second step are selected to satisfy

$$\langle \delta_{\eta,\theta} c_{\eta,\theta}, \delta_{\eta,\theta'} c_{\eta,\theta'}\rangle \ge -M_0\Big(\sup_{\gamma\in\Gamma} |\eta(\gamma)|\Big)^2 \ \text{ for all } \theta, \theta' \in \Theta, \tag{6.9}$$

where $\Gamma = \cup_{\theta\in\Theta}\Gamma_\theta$ and M_0 is a phase adjustment threshold constant. Comparing with the phase adjustment requirement (6.9) in the shift-invariant setting, we need a stricter phase adjustment requirement (6.6) in the MAPS algorithm proposed in this paper. The benefit is that as shown in Theorem 6.5, the reconstructed signal g_η obtained from the current MAPS algorithm is an "approximation" to the original signal f without restriction on the noise level and the apriori information on the original signal f, while the reconstructed signal in previous versions of the MAPS algorithm in [19, 20] are shown to be an "approximation" to the original signal f when the original signal f is phase retrieval and noise level η is small.

Remark 6.2. For every $\theta \in \Theta$, the local minimizers $c_{\eta,\theta;\lambda}, \lambda \in K_\Theta$, in the first step of the above MAPS algorithm are determined, up to a sign, from noisy phaseless samples $z_\eta(\gamma), \gamma \in \Gamma_\theta$, by the selection of the sampling set Γ_θ, and they can be found by solving a family of least squares problems,

$$\begin{aligned} & \min_{(d_\lambda)_{\lambda\in K_\theta}} \sum_{\gamma\in\Gamma_\theta} \Big| \Big| \sum_{\lambda\in K_\theta} d_\lambda \phi_\lambda(\gamma) \Big| - z_\eta(\gamma) \Big|^2 \\ = & \min_{\delta_\gamma\in\{-1,1\},\gamma\in\Gamma_\theta} \min_{(d_\lambda)_{\lambda\in K_\theta}} \sum_{\gamma\in\Gamma_\theta} \Big| \sum_{\lambda\in K_\theta} d_\lambda \phi_\lambda(\gamma) - \delta_\gamma z_\eta(\gamma) \Big|^2. \end{aligned} \tag{6.10}$$

The local minimization in the first step is a phase retrieval problem in a finite-dimensional setting with its dimension $\#K_\theta \le R_\Lambda(2r_0)$ by (5.12). The reader may refer to [14, 15, 16, 26, 28, 29, 44, 48, 63] for various algorithms to solve a finite-dimensional phase retrieval problem.

Remark 6.3. For the phase adjustment in the second step, the threshold constant M_0 in (6.6) should be chosen appropriately to guarantee the existence of phase adjustments $\delta_{\eta,\theta} \in \{-1,1\}, \theta \in \Theta$. In Theorem 6.5, we show that such a threshold constant M_0 can be selected to depend only on the stability constant (6.16) to solve the local minimization problem in the first step, see (6.17). For a finite set Θ, define a symmetric symbol matrix $B = (b(\theta,\theta'))_{\theta,\theta'\in\Theta}$ with zero diagonal entries and non-diagonal entries $b(\theta,\theta'), \theta \ne \theta'$ given by

$$b(\theta,\theta') = \begin{cases} 1 & \text{if } \langle c_{\eta,\theta}, c_{\eta,\theta'}\rangle > M_0\big(\sup_{\gamma\in\Gamma_\theta\cup\Gamma_{\theta'}} |\eta(\gamma)|\big)^2 \text{ and } \Gamma_\theta\cup\Gamma_{\theta'} \ne \emptyset, \\ -1 & \text{if } \langle c_{\eta,\theta}, c_{\eta,\theta'}\rangle < -M_0\big(\sup_{\gamma\in\Gamma_\theta\cup\Gamma_{\theta'}} |\eta(\gamma)|\big)^2 \text{ and } \Gamma_\theta\cup\Gamma_{\theta'} \ne \emptyset, \\ 0 & \text{otherwise.} \end{cases} \tag{6.11}$$

Then phase adjustments $\delta_{\eta,\theta} \in \{-1,1\}, \theta \in \Theta$ in the second step can be reformulated as finding a diagonal matrix D with diagonal entries $\delta_{\eta,\theta} \in \{1,-1\}, \theta \in \Theta$ so that DBD has nonnegative entries, cf. [20]. The selection of the above diagonal matrix is not unique. By (6.5), we have

$$\langle c_{\eta,\theta}, c_{\eta,\theta'}\rangle = 0 \ \text{ if } K_\theta \cap K_{\theta'} = \emptyset. \tag{6.12}$$

So we may use the following algorithm to find such a diagonal matrix D.

Phase Adjustment Algorithm

Initial $S_1=\emptyset, S_2=\emptyset, S_3=\Theta$.

Step 1 Stop if $S_3=\emptyset$; otherwise take $\theta\in S_3$, update $S_1=S_1\cup\{\theta\}, S_2=\emptyset$ $S_3=S_3\backslash\{\theta\}$, and select $\delta_{\eta,\theta}\in\{-1,1\}$ and $b(\theta,\theta')=2\delta_{\eta,\theta}b(\theta,\theta')$ and $b(\theta',\theta)=2\delta_{\eta,\theta}b(\theta',\theta)$ for all θ' with $K_\theta\cap K_{\theta'}\neq\emptyset$.

Step 2 If $b(\theta,\theta')=0$ for all θ' with $K_\theta\cap K_{\theta'}\neq\emptyset$, return to Step 1; otherwise update $S_2=\{\theta'\in\Theta$ satisfying $b(\theta,\theta')\neq 0$ and $K_{\theta'}\cap K_\theta\neq\emptyset\}$, $S_1=S_1\cup S_2$ and $S_3=S_3\backslash S_2$.

Step 3 For $\theta'\in S_2$, let $\delta_{\eta,\theta'}=1$ if $b(\theta',\theta'')=2$ for some θ'' satisfying $K_{\theta''}\cap K_{\theta'}\neq\emptyset$ and $\delta_{\eta,\theta'}=-1$ otherwise.

Step 4 Set $K=\cup_{\theta'\in S_2}\{\theta''\in\Theta$ satisfying $b(\theta',\theta'')=\pm1$ and $K_{\theta''}\cap K_{\theta'}\neq\emptyset\}$. Return to Step 1 if $K=\emptyset$; otherwise, redefine $b(\theta',\theta'')=2\delta_{\eta,\theta'}b(\theta',\theta'')$ and $b(\theta'',\theta')=2\delta_{\eta,\theta'}b(\theta'',\theta')$ if $\theta'\in S_2$ and $\theta''\in K$ satisfying $K_{\theta''}\cap K_{\theta'}\neq\emptyset$, update $S_2=K, S_1=S_1\cup S_2, S_3=S_3\backslash S_2$, and then return to Step 3.

Output $\delta_{\eta,\theta}, \theta\in\Theta$.

Remark 6.4. We remark that complexity of the proposed MAPS algorithm depends almost linearly on the size $N=\#\Lambda_0$ of the set of innovative positions Λ_0 for the original signal $f=\sum_{\lambda\in\Lambda}c_\lambda\phi_\lambda=\sum_{\lambda\in\Lambda_0}c_\lambda\phi_\lambda\in V(\Phi)$, where component vector $(c_\lambda)_{\lambda\in\Lambda}$ is supported on $\Lambda_0\subset\Lambda$. Define $\Theta_0=\{\theta\in\Theta:\ K_\theta\cap\Lambda_0\neq\emptyset\}$. Then

$$\#\Theta_0=\#\big(\cup_{\lambda\in\Lambda_0}\{\theta\in\Theta:\ \lambda\in K_\theta\}\big)\le N_{\mathcal T}\#\Lambda_0=N_{\mathcal T}N \tag{6.13}$$

by (5.5). By (6.7), in the first step of the proposed MAPS algorithm, it suffices to solve local minimization problems (6.5) with $\theta\in\Theta_0$. Observe that for each $\theta\in\Theta_0$ the number of additions and multiplications required to find the local minimizer $c_{\eta,\theta}$ in the first step is $O(1)$ by (5.12) and (5.13). This together with (6.13) implies that the total number of additions and multiplications required in the first step is $O(N)$. Let $B=(b(\theta,\theta'))_{\theta,\theta'\in\Theta_0}$ be the symmetric symbol matrix in Remark 6.3. For each $\theta,\theta'\in\Theta_0$, the number of additions and multiplications required to evaluate the inner product $\langle c_{\eta,\theta},c_{\eta,\theta'}\rangle$ and the supremum $\sup_{\gamma\in\Gamma_\theta\cup\Gamma_{\theta'}}|\eta(\gamma)|$ are $O(1)$ by (5.12) (5.13), and so is $O(1)$ for evaluating every entry $b(\theta,\theta')$ of the matrix B. By (6.7) and (6.12), we have that

$$b(\theta,\theta')=0 \text{ if } K_\theta\cap K_{\theta'}=\emptyset,$$

and for any $\theta\in\Theta$, we obtain from (5.5) and (5.12) that

$$\begin{aligned}\#\{\theta'\in\Theta:\ K_\theta\cap K_{\theta'}\neq\emptyset\}&\le\#\big(\cup_{\lambda\in K_\theta}\{\theta'\in\Theta:\ \lambda\in K_{\theta'}\}\big)\\ &\le N_{\mathcal T}\#K_\theta\le N_{\mathcal T}R_\Lambda(2r_0).\end{aligned} \tag{6.14}$$

Hence the number of nonzero entries in each row of the symmetric matrix B is at most $N_{\mathcal T}R_\Lambda(2r_0)$, and the total number of additions and multiplications required to define the symmetric matrix B is $O(\#\Theta_0)=O(N)$, where the last equality follows from (6.13). By Remark 6.3, the phase adjustment in the second step of the MAPS algorithm reduces to finding a diagonal matrix D with diagonal entries $\delta_{\eta,\theta}\in\{1,-1\},\theta\in\Theta$ so that DBD has nonnegative entries. We observe that the total number of additions and multiplications to find such a diagonal matrix

D by applying the Phase Adjustment Algorithm in Remark 6.3 to the above symmetric matrix B with Θ replaced by Θ_0 is $O(N)$. From the above argument about the computational cost to evaluate the symmetric matrix B and to find the diagonal matrix D, we see that the total number of additions and multiplications required in the second step is $O(N)$. For any $\lambda \in \Lambda_0$, the number of additions and multiplications required to evaluate $d_{\eta;\lambda}$ is $O(1)$ by (5.5), and hence the total number of additions and multiplications required in the third step of the proposed MAPS algorithm is $O(N)$. Combining the above arguments, we conclude that the total number of additions and multiplications required in the proposed MAPS algorithm to reconstruct an "approximation" g_η to the original signal f is about $O(N)$.

For a phase retrievable frame $\mathcal{F} = \{f_m \in \mathbb{R}^n, 1 \le m \le M\}$, we use

$$\|\mathcal{F}\|_{\mathrm{P}} = \min_{T\subset\{1,\cdots,M\}} \max\Big(\inf_{\|v\|_2=1} \Big(\sum_{m\in T} |\langle v, f_m\rangle|^2\Big)^{1/2}, \inf_{\|v\|_2=1} \Big(\sum_{m\notin T} |\langle v, f_m\rangle|^2\Big)^{1/2}\Big) \tag{6.15}$$

to describe the stability of reconstructing a vector v from its phaseless frame measurements $|\langle v, f_m\rangle|, 1 \le m \le M$, cf. [2, 10] for the σ-strong complement property. In the next theorem, we show that the reconstructed signal g_η approximates the original signal f in the new induced metric M_p in (1.16).

Theorem 6.5. *Let the domain D, the generator $\Phi := (\phi_\lambda)_{\lambda\in\Lambda}$ and the family $\mathcal{T} = \{T_\theta, \theta \in \Theta\}$ of bounded open sets be as in Theorem 5.3, and let $V_p(\Phi), 1 \le p \le \infty$ be as in (1.12). Assume that the sampling set $\Gamma = \cup_{\theta\in\Theta}\Gamma_\theta$ is chosen so that $\Gamma_\theta \subset T_\theta, \theta \in \Theta$, and $\Phi_{\theta,\Gamma_\theta} = \{\Phi_\theta(\gamma), \gamma \in \Gamma_\theta\}, \theta \in \Theta$, are phase retrievable frames, and*

$$\sup_{\theta\in\Theta} \#\Gamma_\theta (\|\Phi_{\theta,\Gamma_\theta}\|_P)^{-2} < \infty. \tag{6.16}$$

Select M_0 in (6.6) by

$$M_0 = 24 \sup_{\theta\in\Theta} \#\Gamma_\theta \big(\|\Phi_{\theta,\Gamma_\theta}\|_P\big)^{-2}, \tag{6.17}$$

and denote the reconstructed signal via the MAPS algorithm (6.3)–(6.7) by g_η, where noisy phaseless samples $z_\eta(\gamma), \gamma \in \Gamma$ in (6.1) are generated from a signal $f \in V_p(\Phi)$ and a p-summable noise $\eta = (\eta(\gamma))_{\gamma\in\Gamma} \in \ell^p, 1 \le p \le \infty$. Then

$$M_p(g_\eta, f) \le 6\sqrt{6} C_0 \Big(\max_{\theta\in\Theta} \sqrt{\#\Gamma_\theta} \big(\|\Phi_{\theta,\Gamma_\theta}\|_P\big)^{-1}\Big) \|\Phi\|_\infty \|\eta\|_{\ell^p}, \tag{6.18}$$

where the metric $M_p(g_\eta, f)$ is defined in (1.16), $C_0 = (R_\Lambda(r_0))^{1-1/p} (N_{\mathcal{T}})^{1/p} (B(4r_0))^{2/p}$, and $r_0, R_\Lambda(r_0), N_{\mathcal{T}}, B(4r_0)$ and $\|\Phi\|_\infty$ are constants given in (2.5), (5.3), (5.5), (2.1) and (2.4) respectively.

We postpone the proof of Theorem 6.5 to the end of this section.

By (1.17) and Theorem 6.5, the reconstructed signal g_η from the proposed MAPS algorithm provides an approximation to the original signal in magnitude measurements,

$$\big\||g_\eta|-|f|\big\|_{L^p} \le 12\sqrt{6}C_0\Big(\max_{\theta\in\Theta}\sqrt{\#\Gamma_\theta}\big(\|\Phi_{\theta,\Gamma_\theta}\|_P\big)^{-1}\Big)\|\Phi\|_\infty\|\eta\|_{\ell^p}. \tag{6.19}$$

In the next remark, we show that the estimation (6.19) in magnitude measurements is suboptimal in the sense that the quantity $C_0\big(\max_{\theta\in\Theta}\sqrt{\#\Gamma_\theta}\big(\|\Phi_{\theta,\Gamma_\theta}\|_P\big)^{-1}\big)\|\Phi\|_\infty$ in (6.19) cannot be replaced by a sufficiently small constant.

Remark 6.6. Take $\lambda_0\in\Lambda$ so that $\|\phi_{\lambda_0}\|_{L^p}\ge\delta_0\|\Phi\|_\infty$ for some $\delta_0>0$. Then for any signal $f\in V_p\Phi), 1\le p<\infty$ and $\epsilon\ge0$, we have

$$\begin{aligned}\big\|\big(|f(\gamma)\pm\epsilon\phi_{\lambda_0}(\gamma)|-|f(\gamma)|\big)_{\gamma\in\Gamma}\big\|_{\ell^p} &\le \epsilon\|\Phi\|_\infty\Big(\sum_{\gamma\in\Gamma\cap S_\Phi(\lambda_0)}1\Big)^{1/p}\\ &\le \Big(\frac{N_{\mathcal T}R_\Lambda(2r_0)(R_\Lambda(2r_0)+1)}{2}\Big)^{1/p}\|\Phi\|_\infty\epsilon \end{aligned}\tag{6.20}$$

by (5.5) and (5.13), and

$$\begin{aligned}&\max\Big(\big\||f+\epsilon\phi_{\lambda_0}|-|f|\big\|_{L^p},\big\||f-\epsilon\phi_{\lambda_0}|-|f|\big\|_{L^p}\Big)\\ &\ge\frac12\Big\|\max\Big(\big||f+\epsilon\phi_{\lambda_0}|-|f|\big|,\big||f-\epsilon\phi_{\lambda_0}|-|f|\big|\Big)\Big\|_{L^p}\\ &\ge\frac12\|\epsilon\phi_{\lambda_0}\|_{L^p}\ge\frac{\delta_0}{2}\|\Phi\|_\infty\epsilon.\end{aligned}\tag{6.21}$$

By (6.19), (6.20) and (6.21), we conclude that the reconstructed signal g_η from the proposed MAPS algorithm is a suboptimal approximation to the original signal f in magnitude measurements.

Take a signal $g\in V_p(\Phi), 1\le p\le\infty$. For the noise $\eta=(\eta(\gamma))_{\gamma\in\Gamma}$ in (6.1) given by $\eta(\gamma)=|g(\gamma)|-|f(\gamma)|, \gamma\in\Gamma$, one may verify that the signal g could be reconstructed from the MAPS algorithm. Therefore it follows from Theorem 6.5 that

$$M_p(f,g)\le 6\sqrt{6}C_0\Big(\max_{\theta\in\Theta}\sqrt{\#\Gamma_\theta}\big(\|\Phi_{\theta,\Gamma_\theta}\|_P\big)^{-1}\Big)\|\Phi\|_\infty D_p(S_\Gamma f,S_\Gamma g)\ \text{for all}\ f,g\in V_p(\Phi). \tag{6.22}$$

In the following theorem, we show that metric D_p on the sampling data set is dominated by the metric M_p in the signal space $V_p(\Phi)$, provided that the family Φ of basis signals forms a Riesz basis for the signal space $V_p(\Phi)$, i.e., there exist positive constants $A_p(\Phi)$ and $B_p(\Phi)$ such that in the sense that

$$A_p(\Phi)\|(c_\lambda)_{\lambda\in\Lambda}\|_{\ell^p}\le\Big\|\sum_{\lambda\in\Lambda}c_\lambda\phi_\lambda\Big\|_{L^p}\le B_p(\Phi)\|(c_\lambda)_{\lambda\in\Lambda}\|_{\ell^p}\ \text{for all}\ (c_\lambda)_{\lambda\in\Lambda}\in\ell^p. \tag{6.23}$$

Therefore the phaseless sampling operator S_Γ has the bi-Lipschitz property on the signal space $V_p(\Phi)$.

Theorem 6.7. *Let the domain D, the generator Φ, the family $\mathcal T$ of bounded open sets, the phaseless sampling set Γ, and the linear space $V_p(\Phi), 1\le p\le\infty$ be as in Theorem 6.5. Assume that Φ forms a Riesz basis for the signal space $V_p(\Phi)$ with lower and upper Riesz bounds denoted* *by $A_p(\Phi)$ and $B_p(\Phi)$ respectively. Then the nonlinear sampling operator S_Γ in* (1.13) *has the following bi-Lipschitz property, i.e., there exist positive constants A_1 and A_2 such that*

$$A_1 M_p(f,g) \le D_p(S_\Gamma f, S_\Gamma g) \le A_2 M_p(f,g) \text{ for all } f,g \in V_p(\Phi), \tag{6.24}$$

where metrics M_p and D_p are given in (1.16) *and* (1.15) *respectively.*

Proof. The first inequality in (6.24) follows from (6.22). Then it suffices to prove the second inequality in (6.24). For any $f,g \in V_p(\Phi)$ and $\tilde f, \tilde g \in V_p(\Phi)$ with $\mathcal{M}_{\tilde f} = \mathcal{M}_{\tilde g}$, one may verify that

$$||f(\gamma)| - |g(\gamma)|| \le |f(\gamma) - \tilde f(\gamma)| + |g(\gamma) - \tilde g(\gamma)| \text{ for all } \gamma \in \Gamma.$$

Hence

$$\|S_\Gamma f - S_\Gamma g\|_{\ell^p} \le 2 \inf_{\tilde f, \tilde g \in V_p(\Phi) \text{ with } \mathcal{M}_{\tilde f} = \mathcal{M}_{\tilde g}} \max\left(\big\|((f-\tilde f)(\gamma))_{\gamma\in\Gamma}\big\|_{\ell^p}, \big\|((g-\tilde g)(\gamma))_{\gamma\in\Gamma}\big\|_{\ell^p}\right).$$

By (1.16) it suffices to prove that

$$\|h\|_{\ell^p} \le \frac{\|\Phi\|_\infty R_\Lambda(r_0)}{A_p(\Phi)} \Big(\frac{N_{\mathcal{T}} R_\Lambda(2r_0)(R_\Lambda(2r_0)+1)}{2R_\Lambda(r_0)}\Big)^{1/p} \|h\|_{L^p} \text{ for all } h \in V_p(\Phi). \tag{6.25}$$

For $p=\infty$, we immediately have

$$\|h\|_{\ell^\infty} \le \|h\|_{L^\infty} \text{ for all } h \in V_\infty(\Phi). \tag{6.26}$$

For $1 \le p < \infty$, we write $h = \sum_{\lambda\in\Lambda} c_\lambda \phi_\lambda$. Then

$$\begin{aligned}
\|h\|_{\ell^p} &= \Big(\sum_{\gamma\in\Gamma}\Big|\sum_{\lambda\in\Lambda} c_\lambda \phi_\lambda(\gamma)\Big|^p\Big)^{1/p} \le \|\Phi\|_\infty \Big(\sum_{\gamma\in\Gamma}\Big|\sum_{\lambda\in\Lambda} c_\lambda \chi_{S_\Phi(\lambda)}(\gamma)\Big|^p\Big)^{1/p} \\
&\le \|\Phi\|_\infty \Big(\sum_{\gamma\in\Gamma}\Big(\sum_{\lambda\in\Lambda} |c_\lambda|^p \chi_{S_\Phi(\lambda)}(\gamma)\Big) \times \Big(\sum_{\lambda\in\Lambda} \chi_{S_\Phi(\lambda)}(\gamma)\Big)^{p-1}\Big)^{1/p} \\
&\le \|\Phi\|_\infty \Big(\sum_{\lambda\in\Lambda} |c_\lambda|^p \sum_{\gamma\in\Gamma} \chi_{S_\Phi(\lambda)}(\gamma)\Big)^{1/p} \times \sup_{\gamma\in\Gamma}\Big(\sum_{\lambda\in\Lambda} \chi_{B(\gamma,r_0)}(\lambda)\Big)^{1-1/p} \\
&\le \|\Phi\|_\infty (R_\Lambda(r_0))^{1-1/p} \Big(\sup_{\lambda\in\Lambda}\sum_{\theta\in\Theta}\sum_{\gamma\in\Gamma_\theta} \chi_{S_\Phi(\lambda)}(\gamma)\Big)^{1/p} \|(c_\lambda)_{\lambda\in\Lambda}\|_{\ell^p} \\
&\le \|\Phi\|_\infty (R_\Lambda(r_0))^{1-1/p} \Big(\sup_{\theta\in\Theta} \#\Gamma_\theta\Big)^{1/p} \Big(\sup_{\lambda\in\Lambda} \#\{\theta:\ T_\theta \cap S_\Phi(\lambda) \ne \emptyset\}\Big)^{1/p} \|(c_\lambda)_{\lambda\in\Lambda}\|_{\ell^p} \\
&\le \frac{\|\Phi\|_\infty R_\Lambda(r_0)}{A_p(\Phi)} \Big(\frac{N_{\mathcal{T}} R_\Lambda(2r_0)(R_\Lambda(2r_0)+1)}{2R_\Lambda(r_0)}\Big)^{1/p} \|h\|_{L^p}, \qquad (6.27)
\end{aligned}$$

where the third inequality follows from Assumption 2.2, the fourth one is true by (5.3) and the last one holds by (5.5), (5.13) and (6.23). Combing (6.26) and (6.27) proves (6.25), and hence completes the proof. □

We finish this section with the proof of Theorem 6.5.

Proof of Theorem 6.5. By (1.16), it suffices to find $f_\eta, h_\eta \in V(\Phi)\cap L^p$ with the same magnitude measurements on the whole domain,

$$\mathcal{M}_{h_\eta} = \mathcal{M}_{f_\eta}, \tag{6.28}$$

such that

$$\|f_\eta - f\|_{L^p} \le 4\sqrt{6}C_0\Big(\max_{\theta\in\Theta}\sqrt{\#\Gamma_\theta}\big(\|\Phi_{\theta,\Gamma_\theta}\|_P\big)^{-1}\Big)\|\Phi\|_\infty\|\eta\|_{\ell^p} \tag{6.29}$$

and

$$\|g_\eta - h_\eta\|_{L^p} \le 6\sqrt{6}C_0\Big(\max_{\theta\in\Theta}\sqrt{\#\Gamma_\theta}\big(\|\Phi_{\theta,\Gamma_\theta}\|_P\big)^{-1}\Big)\|\Phi\|_\infty\|\eta\|_{\ell^p}. \tag{6.30}$$

Take $\theta\in\Theta$ and define

$$g_{\eta,\theta} = \sum_{\lambda\in\Lambda} c_{\eta,\theta;\lambda}\phi_\lambda, \tag{6.31}$$

where $c_{\eta,\theta;\lambda}, \lambda\in\Lambda$, are given in (6.4). By (6.4) and the definitions of the sets K_θ and $\Gamma_\theta, \theta\in\Theta$, we have

$$g_{\eta,\theta}(\gamma)\pm f(\gamma) = \sum_{\lambda\in K_\theta}(c_{\eta,\theta;\lambda}\pm c_\lambda)\phi_\lambda(\gamma),\ \ \gamma\in\Gamma_\theta. \tag{6.32}$$

Then there exists a subset $\Gamma'_\theta\subset\Gamma_\theta$ such that

$$\begin{aligned}
&\Big(\sum_{\gamma\in\Gamma'_\theta}\Big|\sum_{\lambda\in K_\theta}(c_{\eta,\theta;\lambda}-c_\lambda)\phi_\lambda(\gamma)\Big|^2\Big)^{\frac12} + \Big(\sum_{\gamma\in\Gamma_\theta\backslash\Gamma'_\theta}\Big|\sum_{\lambda\in K_\theta}(c_{\eta,\theta;\lambda}+c_\lambda)\phi_\lambda(\gamma)\Big|^2\Big)^{\frac12}\\
&= \Big(\sum_{\gamma\in\Gamma'_\theta}\big|g_{\eta,\theta}(\gamma)-f(\gamma)\big|^2\Big)^{\frac12} + \Big(\sum_{\gamma\in\Gamma_\theta\backslash\Gamma'_\theta}\big|g_{\eta,\theta}(\gamma)+f(\gamma)\big|^2\Big)^{\frac12}\\
&= \Big(\sum_{\gamma\in\Gamma'_\theta}\big||g_{\eta,\theta}(\gamma)|-|f(\gamma)|\big|^2\Big)^{\frac12} + \Big(\sum_{\gamma\in\Gamma_\theta\backslash\Gamma'_\theta}\big||g_{\eta,\theta}(\gamma)|-|f(\gamma)|\big|^2\Big)^{\frac12}\\
&\le \sqrt{2}\Big(\sum_{\gamma\in\Gamma_\theta}\big||g_{\eta,\theta}(\gamma)|-|f(\gamma)|\big|^2\Big)^{\frac12}\\
&\le \sqrt{2}\Big(\sum_{\gamma\in\Gamma_\theta}\big||g_{\eta,\theta}(\gamma)|-z_\eta(\gamma)\big|^2\Big)^{\frac12} + \sqrt{2}\Big(\sum_{\gamma\in\Gamma_\theta}\big||f(\gamma)|-z_\eta(\gamma)\big|^2\Big)^{\frac12}\\
&\le 2\sqrt{2}\Big(\sum_{\gamma\in\Gamma_\theta}\big||f(\gamma)|-z_\eta(\gamma)\big|^2\Big)^{\frac12} \le 2\sqrt{2}\sqrt{\#\Gamma_\theta}\Big(\sup_{\gamma\in\Gamma_\theta}|\eta(\gamma)|\Big),
\end{aligned} \tag{6.33}$$

where the third inequality follows from (6.5) and the last inequality holds by (6.1). By (6.15) and the phase retrievable frame assumption for $\Phi_{\theta,\Gamma_\theta}$, we have

$$\|\Phi_{\theta,\Gamma_\theta}\|_{\rm P} \le \max\Bigg(\frac{\Big(\sum_{\gamma\in\Gamma'_\theta}\Big|\sum_{\lambda\in K_\theta}(c_{\eta,\theta;\lambda}-c_\lambda)\phi_\lambda(\gamma)\Big|^2\Big)^{\frac12}}{\Big(\sum_{\lambda\in K_\theta}|c_{\eta,\theta;\lambda}-c_\lambda|^2\Big)^{\frac12}},$$

$$\frac{\Big(\sum_{\gamma\in\Gamma_\theta\backslash\Gamma'_\theta}\Big|\sum_{\lambda\in K_\theta}(c_{\eta,\theta;\lambda}+c_\lambda)\phi_\lambda(\gamma)\Big|^2\Big)^{\frac12}}{\Big(\sum_{\lambda\in K_\theta}|c_{\eta,\theta;\lambda}+c_\lambda|^2\Big)^{\frac12}}\Bigg)$$

$$\le \frac{\Big(\sum_{\gamma\in\Gamma'_\theta}\Big|\sum_{\lambda\in K_\theta}(c_{\eta,\theta;\lambda}-c_\lambda)\phi_\lambda(\gamma)\Big|^2\Big)^{\frac12}+\Big(\sum_{\gamma\in\Gamma_\theta\backslash\Gamma'_\theta}\Big|\sum_{\lambda\in K_\theta}(c_{\eta,\theta;\lambda}+c_\lambda)\phi_\lambda(\gamma)\Big|^2\Big)^{\frac12}}{\min\Bigg(\Big(\sum_{\lambda\in K_\theta}|c_{\eta,\theta;\lambda}+c_\lambda|^2\Big)^{\frac12},\Big(\sum_{\lambda\in K_\theta}|c_{\eta,\theta;\lambda}-c_\lambda|^2\Big)^{\frac12}\Bigg)}. \tag{6.34}$$

Combining (6.33) and (6.34) yields

$$\Big(\sum_{\lambda\in K_\theta}|c_{\eta,\theta;\lambda}-\tilde\delta_{\eta,\theta}c_\lambda|^2\Big)^{1/2}\le 2\sqrt2\sqrt{\#\Gamma_\theta}\big(\|\Phi_{\theta,\Gamma_\theta}\|_{\rm P}\big)^{-1}\Big(\sup_{\gamma\in\Gamma_\theta}|\eta(\gamma)|\Big) \tag{6.35}$$

for some $\tilde\delta_{\eta,\theta}\in\{-1,1\}$.

Let $\tilde\delta_{\eta,\theta},\theta\in\Theta$, be as in (6.35). Then for any $\theta,\theta'\in\Theta$, we have

$$\begin{aligned}
\langle\tilde\delta_{\eta,\theta}c_{\eta,\theta},\tilde\delta_{\eta,\theta'}c_{\eta,\theta'}\rangle &= \sum_{\lambda\in K_\theta\cap K_{\theta'}}\tilde\delta_{\eta,\theta}\tilde\delta_{\eta,\theta'}c_{\eta,\theta;\lambda}c_{\eta,\theta';\lambda}\\
&\ge \sum_{\lambda\in K_\theta\cap K_{\theta'}}|c_\lambda|^2-\sum_{\lambda\in K_\theta\cap K_{\theta'}}|c_\lambda||\tilde\delta_{\eta,\theta}c_{\eta,\theta;\lambda}-c_\lambda|\\
&\quad -\sum_{\lambda\in K_\theta\cap K_{\theta'}}|\tilde\delta_{\eta,\theta'}c_{\eta,\theta';\lambda}-c_\lambda||c_\lambda|\\
&\quad -\sum_{\lambda\in K_\theta\cap K_{\theta'}}|\tilde\delta_{\eta,\theta}c_{\eta,\theta;\lambda}-c_\lambda||\tilde\delta_{\eta,\theta'}c_{\eta,\theta';\lambda}-c_\lambda|\\
&\ge \frac12\sum_{\lambda\in K_\theta\cap K_{\theta'}}|c_\lambda|^2-\frac32\sum_{\lambda\in K_\theta\cap K_{\theta'}}\Big(|\tilde\delta_{\eta,\theta}c_{\eta,\theta;\lambda}-c_\lambda|^2+|\tilde\delta_{\eta,\theta'}c_{\eta,\theta';\lambda}-c_\lambda|^2\Big). \qquad (6.36)
\end{aligned}$$

This together with (6.17) and (6.35) implies

$$\begin{aligned}
\langle\tilde\delta_{\eta,\theta}c_{\eta,\theta},\tilde\delta_{\eta,\theta'}c_{\eta,\theta'}\rangle &\ge -\frac32\sum_{\lambda\in K_\theta\cap K_{\theta'}}\Big(|\tilde\delta_{\eta,\theta}c_{\eta,\theta;\lambda}-c_\lambda|^2+|\tilde\delta_{\eta,\theta'}c_{\eta,\theta';\lambda}-c_\lambda|^2\Big)\\
&\ge -M_0\Big(\sup_{\gamma\in\Gamma_\theta\cup\Gamma_{\theta'}}|\eta(\gamma)|\Big)^2 \qquad (6.37)
\end{aligned}$$

for all $\theta,\theta'\in\Theta$. This proves that phases of $c_{\eta,\theta},\theta\in\Theta$, in (6.4) can be adjusted so that (6.6) holds.

Let $\delta_{\eta,\theta} \in \{-1, 1\}, \theta \in \Theta$, be signs in (6.6) used for the phase adjustment of vectors $c_{\eta,\theta}, \theta \in \Theta$, in (6.4). We remark that the above signs are not necessarily the ones in (6.35), however as shown in (6.48) below they are related. Define

$$f_\eta = \sum_{\lambda \in \Lambda_{f,\eta}} c_\lambda \phi_\lambda, \tag{6.38}$$

where $\Lambda_{f,\eta}$ contains all $\lambda \in \Lambda$ such that

$$|c_\lambda| > 2\sqrt{M_0}\Big(\sup_{\lambda \in K_\theta} \sup_{\gamma \in \Gamma_\theta} |\eta(\gamma)|\Big). \tag{6.39}$$

Then for $x \in D$, we obtain from (2.5) and (5.3) that

$$\begin{aligned} |f(x) - f_\eta(x)| &\le 2\sqrt{M_0} \sum_{\lambda \in \Lambda} \Big(\sup_{\lambda \in K_\theta} \sup_{\gamma \in \Gamma_\theta} |\eta(\gamma)|\Big)|\phi_\lambda(x)| \\ &\le 2\sqrt{M_0} R_\Lambda(r_0) \|\Phi\|_\infty \Big(\sup_{\lambda \in B(x,r_0)} \sup_{\lambda \in K_\theta} \sup_{\gamma \in \Gamma_\theta} |\eta(\gamma)|\Big). \end{aligned} \tag{6.40}$$

By (5.11) and the phase retrievability of frame on $\Phi_{\theta,\Gamma_\theta}, \theta \in \Theta$, we have that

$$\gamma \in B(x, 4r_0) \tag{6.41}$$

for all $\gamma \in \Gamma_\theta, \theta \in \Theta$ with $\phi_{\lambda'}(\gamma) \neq 0$ for some $\lambda' \in K_\theta$. Therefore it follows from (6.40) and (6.41) that

$$\begin{aligned} \sup_{x \in D} |f(x) - f_\eta(x)| &\le 2\sqrt{M_0} R_\Lambda(r_0) \|\Phi\|_\infty \Big(\sup_{\lambda \in B(x,r_0), \lambda \in K_\theta, \gamma \in \Gamma_\theta} |\eta(\gamma)|\Big) \\ &\le 2\sqrt{M_0} R_\Lambda(r_0) \|\Phi\|_\infty \sup_{\gamma \in \Gamma_\theta, \theta \in \Theta} |\eta(\gamma)| \end{aligned} \tag{6.42}$$

for $p = \infty$, and

$$\begin{aligned} & \Big(\int_{x \in D} |f(x) - f_\eta(x)|^p d\mu(x)\Big)^{1/p} \\ & \le 2\sqrt{M_0}\bigg(\int_{x \in D} \Big(\sum_{\lambda \in \Lambda} \Big(\sup_{\lambda \in K_\theta} \sup_{\gamma \in \Gamma_\theta} |\eta(\gamma)|\Big)|\phi_\lambda(x)|\Big)^p d\mu(x)\bigg)^{1/p} \\ & \le 2\sqrt{M_0}\bigg(\int_{x \in D} \Big(\sum_{\lambda \in \Lambda} \Big(\sup_{\lambda \in K_\theta} \sup_{\gamma \in \Gamma_\theta} |\eta(\gamma)|^p\Big)|\phi_\lambda(x)|\Big) \times \Big(\sum_{\lambda \in \Lambda} |\phi_\lambda(x)|\Big)^{p-1} d\mu(x)\bigg)^{1/p} \\ & \le 2\sqrt{M_0}\|\Phi\|_\infty (R_\Lambda(r_0))^{1-1/p}\bigg(\int_{x \in D} \sum_{\lambda \in B(x,r_0)} \Big(\sup_{\lambda \in K_\theta} \sup_{\gamma \in \Gamma_\theta} |\eta(\gamma)|^p\Big) d\mu(x)\bigg)^{1/p} \\ & \le 2\sqrt{M_0}\|\Phi\|_\infty (R_\Lambda(r_0))^{1-1/p} (B(4r_0))^{1/p}\Big(\sum_{\lambda \in \Lambda} \sum_{\lambda \in K_\theta} \sum_{\gamma \in \Gamma_\theta} |\eta(\gamma)|^p\Big)^{1/p} \\ & \le 2\sqrt{M_0}\|\Phi\|_\infty (R_\Lambda(r_0))^{1-1/p} (N_{\mathcal{T}})^{1/p} (B(4r_0))^{2/p} \|\eta\|_{\ell^p} \end{aligned} \tag{6.43}$$

for $1 \le p < \infty$. This proves (6.29).

By (6.17), (6.35), (6.36), (6.38) and (6.39), we obtain that

$$V_{f_\eta} = \Lambda_{f,\eta}$$

and

$$\langle \tilde{\delta}_{\eta,\theta} c_{\eta,\theta}, \tilde{\delta}_{\eta,\theta'} c_{\eta,\theta'} \rangle > M_0 \Big(\sup_{\gamma \in \Gamma_\theta \cup \Gamma_{\theta'}} |\eta(\gamma)| \Big)^2 \tag{6.44}$$

for all $\theta, \theta' \in \Theta$ with $K_\theta \cap K_{\theta'} \cap V_{f_\eta} \neq \emptyset$. This together with (6.6) implies that

$$\delta_{\eta,\theta} \tilde{\delta}_{\eta,\theta} = \delta_{\eta,\theta'} \tilde{\delta}_{\eta,\theta'}$$

hold for all pairs (θ, θ') satisfying $K_\theta \cap K_{\theta'} \cap V_{f_\eta} \neq \emptyset$. Hence for $\lambda \in V_{f_\eta}$ there exists $\delta_\lambda \in \{-1, 1\}$ such that

$$\delta_{\eta,\theta} \tilde{\delta}_{\eta,\theta} = \delta_\lambda \tag{6.45}$$

for all $\theta \in \Theta$ satisfying $\lambda \in K_\theta$. Decompose the graph $\mathcal{G}_{f_\eta}$ into the union of connected components $(V_{\eta,i}, E_{\eta,i}), i \in I_\eta$, and the signal f_η as in (1.5), (1.6) and (1.7),

$$f_\eta = \sum_{i \in I_\eta} \sum_{\lambda \in V_{\eta,i}} c_\lambda \phi_\lambda. \tag{6.46}$$

Observe that for any edge (λ, λ') of V_{f_η}, there exists $\theta_0 \in \Theta$ such that $\lambda, \lambda' \in K_{\theta_0}$ by (2.11). Hence

$$\delta_\lambda = \delta_{\eta,\theta_0} \tilde{\delta}_{\eta,\theta_0} = \delta_{\lambda'}. \tag{6.47}$$

Combining (6.45) and (6.47), there exists $\delta_i, i \in I_\eta$, such that

$$\delta_{\eta,\theta} \tilde{\delta}_{\eta,\theta} = \delta_i \tag{6.48}$$

for all $\theta \in \Theta$ satisfying $K_\theta \cap V_{\eta,i} \neq \emptyset$. Set

$$h_\eta = \sum_{i \in I_\eta} \delta_i \sum_{\lambda \in V_{\eta,i}} c_\lambda \phi_\lambda.$$

Then f_η and h_η have the same magnitude measurements on the whole domain by (1.5), which proves (6.28).

For all $\lambda \notin V_{f_\eta}$, we obtain from (6.35) that

$$|d_{\eta;\lambda}| \le \frac{\sum_{K_\theta \ni \lambda} (|\delta_{\eta,\theta} c_{\eta,\theta;\lambda} - \delta_{\eta,\theta} \tilde{\delta}_{\eta,\theta} c_\lambda| + |c_\lambda|)}{\sum_{K_\theta \ni \lambda} 1} \le 3\sqrt{M_0} \Big(\sup_{\lambda \in K_\theta} \sup_{\gamma \in \Gamma_\theta} |\eta(\gamma)| \Big). \tag{6.49}$$

For any $\lambda \in V_{\eta,i}, i \in I_\eta$, we get

$$\begin{aligned} |d_{\eta;\lambda} - \delta_i c_\lambda| &\le \frac{\sum_{K_\theta \ni \lambda} |\delta_{\eta,\theta} c_{\eta,\theta;\lambda} - \delta_i c_\lambda|}{\sum_{K_\theta \ni \lambda} 1} = \frac{\sum_{K_\theta \ni \lambda} |c_{\eta,\theta;\lambda} - \tilde{\delta}_{\eta,\theta} c_\lambda|}{\sum_{K_\theta \ni \lambda} 1} \\ &\le \sqrt{M_0} \Big(\sup_{\lambda \in K_\theta} \sup_{\gamma \in \Gamma_\theta} |\eta(\gamma)| \Big). \end{aligned} \tag{6.50}$$

Combining (6.49) and (6.50), and applying similar argument used in the proof of (6.42) and (6.43), we can prove (6.30). □

7. Numerical Simulations

In this section, we present some numerical results to demonstrate the performance of the MAPS algorithm proposed in the last section, where signals are one-dimensional non-uniform cubic splines and two-dimensional piecewise affine functions on a triangulation.

Denote the positive part of a real number x by $x_+ = \max(x, 0)$. In the first simulation, we consider phaseless sampling and reconstruction of cubic spline signals f on the interval $[a, b]$ with non-uniform knots $a = t_0 < t_1 < \ldots < t_N = b$, see the top left plot in Figure 1, where $a = 0, b = 100$ and $N = 100$. Those signals have the following parametric representation

$$f(x) = \sum_{n=0}^{N-4} c_n B_n(x),\ x \in [a, b], \tag{7.1}$$

where

$$B_n(x) = (t_{n+4} - t_n) \sum_{l=0}^{4} \frac{(x - t_{n+l})_+^3}{\prod_{0\le j\le 4, j\ne l}(t_{n+l} - t_{n+j})},\quad 0 \le n \le N - 4$$

are cubic B-splines with knots $t_{n+l}, 0 \le l \le 4$ [58, 60]. In our simulations, we assume that

$$c_n \in [-1, 1],\ 0 \le n \le N - 4,$$

are randomly selected, and

$$t_n = a + (n + \epsilon_n)\frac{b - a}{N},\ 1 \le n \le N - 1$$

for some $\epsilon_n, 1 \le n \le N - 1$, being randomly selected in $[-0.2, 0.2]$. Then cubic spline signals in the first simulation have $(b - a)/N$ as their rate of innovations.

Consider the scenario that phaseless samples of the signal f in (7.1) on a discrete set Γ are corrupted by a bounded random noise,

$$z_\eta(\gamma) = |f(\gamma)| + \eta(\gamma),\ \gamma \in \Gamma, \tag{7.2}$$

where $\eta(\gamma), \gamma \in \Gamma$, are randomly selected in the interval $[-\eta, \eta]$ for some $\eta \ge 0$,

$$\Gamma := \cup_{n=0}^{N-1}\Gamma_n := \bigcup_{n=0}^{N-1} \Big\{t_n + k\frac{t_{n+1} - t_n}{K + 1} \in (t_n, t_{n+1}),\ 1 \le k \le K\Big\}, \tag{7.3}$$

and $K \ge 7$ is a positive integer. We remark that the proposed MAPS algorithm is not applicable for $1 \le K \le 6$.

Denote by g_η the reconstructed signal from the above noisy phaseless samples via the proposed MAPS algorithm. Performance of the proposed MAPS algorithm depends on the noise level η and also the oversampling rate K, the ratio between the density $K(b - a)/N$ of the sampling set Γ in (7.3) and the rate $(b - a)/N$ of innovations of signals in $V(\Phi)$. Denote by

$$E_{\eta,K} := \big\| |g_\eta| - |f| \big\|_{L^\infty}$$

the maximal reconstruction error in magnitude measurements between the original signal f and the reconstructed signal g_η for different noise levels η and oversampling rates K. Plotted on the bottom right of Figure 1 are averages of the maximal reconstruction error $E_{\eta,K}$ in 200 trials against the noise level η and oversampling rate K. We observe that the maximal reconstruction

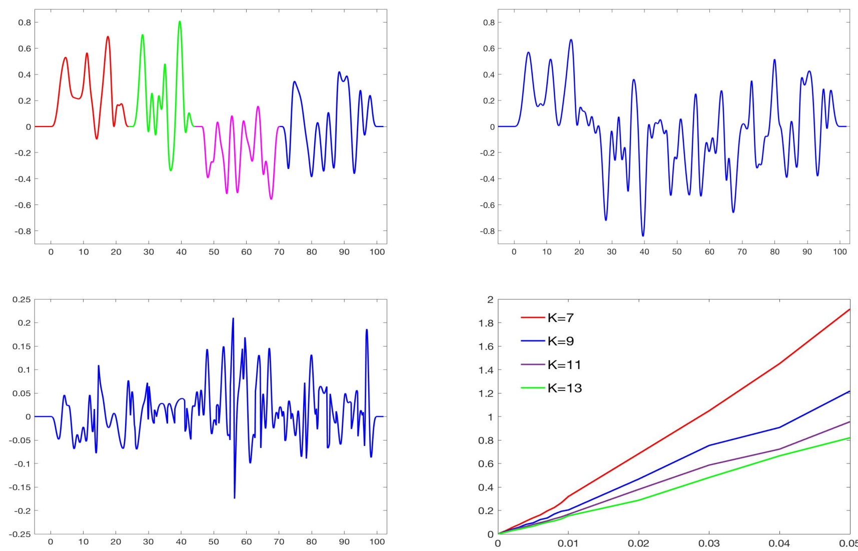

FIGURE 1. Plotted on the top left is a non-uniform cubic spline signal f_o, while on the top right is the signal g_η reconstructed via the proposed MAPS algorithm, which provide good approximation to the original signal f_o on the intervals $[0, 24.1323)$, $[44.0290, 69.8080)$ and $[82.0449, 100]$, and reflection $-f_o$ of the original signal on intervals $[24.1323, 44.0290)$ and $[69.8080, 82.0449)$. On the bottom left is the difference $|g_\eta| - |f_o|$ between magnitude measurements of the reconstructed signal g_η on the top right and the original signal f_o plotted on the top left. On the bottom right is the average of maximal reconstruction error $E_{\eta,K}$ in 200 trials with respect to different noise levels η and oversampling rates K.

error $E_{\eta,K}$ depends almost linearly on the noise level η, and the stability constant in (6.19) and Theorem 6.5 measured by $\sup_{0\le\eta\le 0.05} E_{\eta,K}/\eta$ decreases as the oversampling rate $K \le 7$ increases. This demonstrates the approximation property in Theorem 6.5. Presented on the top left is a non-uniform cubic spline signal f_o that has four "islands" in the decomposition (1.5), (1.6) and (1.7), and on the right is the reconstructed signal g_η via the proposed MAPS algorithm, where $\eta = 0.01, K = 9$ and the maximal error $\||g_\eta| - |f_o|\|_{L^\infty}$ in magnitude measurements is 0.2104.

Let D be a triangulation composed by the triangles $T_\theta, \theta \in \Theta$, and denote the set of all inner nodes of the triangulation by Λ. In the second simulation, we consider piecewise affine signals

$$f(x, y) = \sum_{\lambda\in\Lambda} c_\lambda \phi_\lambda(x, y) \tag{7.4}$$

on the triangulation D, where the basis signals $\phi_\lambda, \lambda \in \Lambda$ are piecewise affine on triangles $T_\theta, \theta \in \Theta$ with $\phi_\lambda(\lambda) = 1$ and $\phi_\lambda(\lambda') = 0$ for all other nodes $\lambda' \neq \lambda$, see the left plot in Figure 2. From the definition of basis signals $\phi_\lambda, \lambda \in \Lambda$, a signal f of the form (7.4) has the following interpolation property,

$$f(x, y) = \sum_{\lambda \in \Lambda} f(\lambda)\phi_\lambda(x, y).$$

In the simulation, phaseless samples of a piecewise affine signal f on a discrete set $\Gamma = \cup_{\theta \in \Theta} \Gamma_\theta$ are corrupted by the bounded random noise,

$$z_\eta(\gamma) = |f(\gamma)| + \eta(\gamma), \ \ \gamma \in \Gamma, \tag{7.5}$$

where $\eta(\gamma), \gamma \in \Gamma$, are randomly selected in the interval $[-\eta, \eta]$ for some $\eta \geq 0$ and for every $\theta \in \Theta$, the set Γ_θ contains 7 points randomly selected inside T_θ. Shown in the middle of Figure 2 is a signal g_η reconstructed from the noisy phaseless samples (7.5) via the proposed MAPS algorithm, where $\eta = 0.01$, the original piecewise affine signal f is plotted on the left of Figure 2, and the maximal reconstruction error $\| |g_\eta| - |f| \|_{L^\infty}$ in magnitude measurements between the original signal f and the reconstructed signal g_η is 0.0360.

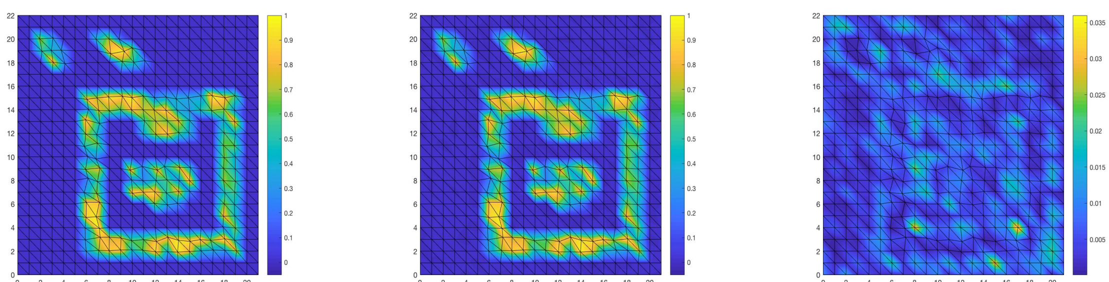

FIGURE 2. Plotted on the left is a piecewise affine signal f on a triangulation which has four "islands" in the decomposition (1.5), (1.6) and (1.7). Shown in the middle is a reconstructed signal g_η via the MAPS algorithm, while on the right is the difference $||g_\eta| - |f||$ between magnitude measurements of the reconstructed signal g_η and the original signal f plotted on the left.

In the simulation, we consider the performance of the proposed MAPS algorithm to construct piecewise affine approximation when the original signal f of the form (7.4) has evaluations $f(\lambda), \lambda \in \Lambda$ on their inner nodes being randomly selected in $[-1, 1]$. Denote by g_η the reconstructed signal from the noisy phaseless samples (7.5) via the proposed MAPS algorithm and let $E_\eta := \| |g_\eta| - |f| \|_{L^\infty}$ be the maximal reconstruction error in magnitude measurements between the original signal f and the reconstructed signal g_η for different noise levels η. Shown in Table 1 is the average of maximal reconstruction error E_η in 200 trials. This confirms the conclusion in Theorem 6.5 that the maximal reconstruction error depends almost linearly on the noise level $\eta \geq 0$.

Table 1. Maximal reconstruction error via the MAPS algorithm

η	0.04	0.03	0.02	0.01	0.008	0.004	0.002	0.001
E_η	0.1878	0.1366	0.0791	0.0305	0.0226	0.0101	0.0050	0.0025

Appendix A. Density of phaseless sampling sets

In the appendix, we introduce a necessary condition on a discrete set Γ such that $\mathcal{M}_{f,\Gamma} = \mathcal{M}_f$ for all $f \in V(\Phi)$. We show that the density of such a discrete set Γ is no less than the innovative rate of signals in $V(\Phi)$, see Theorem A.1 and Corollary A.2.

Theorem A.1. *Let the domain D, the generator $\Phi := (\phi_\lambda)_{\lambda\in\Lambda}$, the family $\mathcal{T} = \{T_\theta, \theta \in \Theta\}$ of open sets and the linear space $V(\Phi)$ be as in Theorem 5.3, and let $\Gamma \subset D$. If $\mathcal{M}_{f,\Gamma} = \mathcal{M}_f$ for all $f \in V(\Phi)$ with $\mathcal{M}_f = \{\pm f\}$, then*

$$D_+(\Gamma) \ge D_+(\Lambda). \tag{A.1}$$

Proof. Take $x_0 \in D$ and $r \ge r_0$. By (2.2) and (2.3), it suffices to prove that

$$\#(\Gamma \cap B(x_0, r)) \ge \#(\Lambda \cap B(x_0, r - r_0)). \tag{A.2}$$

Assume, on the contrary, that (A.2) does not hold. Then we can find a nonzero vector $(d_\lambda)_{\lambda\in\Lambda\cap B(x_0,r-r_0)}$ such that

$$\sum_{\lambda\in\Lambda\cap B(x_0,r-r_0)} d_\lambda\phi_\lambda(\gamma) = 0,\ \gamma \in \Gamma \cap B(x_0, r). \tag{A.3}$$

Recall that $\phi_\lambda, \lambda \in \Lambda$, are supported in $B(\lambda, r_0)$ by Assumption 2.2. Hence

$$\sum_{\lambda\in\Lambda\cap B(x_0,r-r_0)} d_\lambda\phi_\lambda(\gamma) = 0,\ \gamma \in \Gamma\backslash B(x_0, r). \tag{A.4}$$

Therefore the set

$$W = \Big\{f := \sum_{\lambda\in\Lambda\cap B(x_0,r-r_0)} c_\lambda\phi_\lambda :\ f(\gamma) = 0,\ \gamma \in \Gamma\Big\} \subset V(\Phi)$$

contains nonzero signals. Take a nonzero signal $f \in W$. By Theorem 4.4, $f = \sum_{i\in I} f_i$ for some nonzero signals $f_i \in V(\Phi), i \in I$, such that $\mathcal{M}_{f_i} = \{\pm f_i\}, i \in I$, and $f_i f_{i'} = 0$ for all distinct $i, i' \in I$. This together with $f \in W$ implies that $f_i(\gamma) = 0$ for all $\gamma \in \Gamma$ and $i \in I$. Hence $0 \in \mathcal{M}_{f_i,\Gamma}, i \in I$, which contradicts with $\mathcal{M}_{f_i,\Gamma} = \mathcal{M}_{f_i} = \{\pm f_i\}, i \in I$. □

From the above argument, we have the following result without the assumption on the family $\mathcal{T}$ of open sets in Theorem A.1.

Corollary A.2. *Let the domain D and the generator $\Phi = (\phi_\lambda)_{\lambda\in\Lambda}$ satisfy Assumptions 2.1 and 2.2 respectively, and define the linear space $V(\Phi)$ by (1.1). If Γ is a discrete set with $\mathcal{M}_{f,\Gamma} = \mathcal{M}_f$ for all $f \in V(\Phi)$, then $D_+(\Gamma) \ge D_+(\Lambda)$.*

We finish this appendix with a remark that the lower bound in (A.1) can be reached when the generator $\Phi = (\phi_\lambda)_{\lambda\in\Lambda}$ satisfies that

$$S_\Phi(\lambda, \lambda') = \emptyset \text{ for all distinct } \lambda, \lambda' \in \Lambda. \tag{A.5}$$

As in this case, a signal $f \in V(\Phi)$ is nonseparable if and only if $f = c_\lambda \phi_\lambda$ for some $\lambda \in \Lambda$. Thus the set $\Gamma = \{a(\lambda), \lambda \in \Lambda\}$ is a phaseless sampling set whose upper density is the same as the rate of innovation, where $a(\lambda), \lambda \in \Lambda$, are chosen so that $\phi_\lambda(a(\lambda)) \neq 0$.

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