LOCALIZED STABILITY CERTIFICATES FOR SPATIALLY DISTRIBUTED SYSTEMS OVER SPARSE PROXIMITY GRAPHS

NADER MOTEE AND QIYU SUN

ABSTRACT. In this paper, we focus on localized conditions to verify exponential stability of spatially distributed linear systems whose interconnection structures are defined using a geodesic on proximity coupling graphs. We reformulate the exponential stability condition in the form of a feasibility condition that is amenable to localized implementations. Using finite truncation techniques, we obtain decentralized necessary and sufficient stability certificates. In order to guarantee global stability, it suffices to certify localized conditions over a graph covering, where the computational complexity of the verification of the proposed localized certificate is independent of network size. Several robustness conditions against local matrix perturbations are obtained that are useful for tuning network parameters in a decentralized manner while ensuring global exponential stability.

1. INTRODUCTION

Let us consider (in)finite-dimensional a linear dynamical system over an undirected and unweighted graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ whose dynamics is governed by

$$\frac{d}{dt}\psi(t) = \mathbf{A}\psi(t) \tag{1.1}$$

with initial condition $\psi(0) \in \ell^p$ for some $1 \leq p \leq \infty$, where $\psi(t) = [\psi_i(t)]_{i\in\mathcal{V}}$, the state matrix $\mathbf{A} = [a(i,j)]_{i,j\in\mathcal{V}}$ is time independent and $\ell^p := \ell^p(\mathcal{V}), 1 \leq p \leq \infty$, are the linear spaces of all p-summable sequences on $\mathcal{V}$ with the norm denoted by $\|\cdot\|_p$. The *exponential stability* of linear system (1.1), where one needs to guarantee existence of strictly positive constants E and α such that

$$\|\psi(t)\|_2 \leq E\,e^{-\alpha t}\,\|\psi(0)\|_2 \tag{1.2}$$

for all $t \geq 0$, is one of the fundamental and widely studied subjects in control system literatures, see [9, 11, 20, 24, 41] and references therein. There is rich literature on stability of linear systems over infinite chains. A common assumption in this context is that the network topology enjoys some structural symmetries, e.g., shift or translation invariance, that simplifies the stability analysis significantly [8, 9, 13, 18, 19, 21, 26, 30]. The objective of this paper is to characterize spatially localized certificates, in the form of necessary and sufficient conditions, to verify the exponential stability conditions (1.2) for a general class of spatially-varying linear dynamical networks. This is particularly relevant to the following practical problem in analysis and design of networked systems: how do localized modifications (e.g., adding new or eliminating existing links in the coupling graph

$\mathcal{G}$ or adjusting components of the state matrix $\mathbf{A}$) affect the global exponential stability? It is computationally advantageous to devise a method that allows us to localize and inspect stability only in relevant parts of a network, instead of verifying stability conditions globally. Moreover, localized stability certificates are suitable for decentralized/distributed implementations as they are only required to utilize local information.

The interest in stability verification of distributed and networked control systems dates back to a few decades ago. In the context of infinite-dimensional systems, the existing results in the literature is limited to characterization of stability conditions in the form of global (centralized) conditions [5, 11, 13, 19, 20, 26, 27, 46, 47]. The ongoing research in the context of finite-dimensional systems is mainly focused on developing decentralized stability conditions for some particular classes of dynamical systems [3,23,33,36,45]. The stability conditions for the class of spatially invariant systems are studied in [9], where it is shown that stability conditions in space can be equivalently verified in a proper Fourier domain using standard tools. In [21], the authors use linear matrix inequalities to develop a framework to check stability of a class of spatially invariant systems in a localized fashion. A more general methodology to study stability properties of spatially interconnected systems is proposed in [32] that does not require spatial invariance in the underlying dynamics of the system. In [37], a spatial truncation technique is offered to check stability of a class of spatially decaying systems using covering Lyapunov equations. In [3], the authors consider robust stability analysis of sparsely interconnected networks by modeling couplings among the subsystems with integral quadratic constraints. They show that robust stability analysis of these networks can be performed by solving a set of sparse linear matrix inequalities. The string stability of a platoon of vehicles is studied in [48], where the authors extend the well-known string stability conditions for linear cascaded networks to nonlinear settings. In [45], the problem of designing decentralized control laws using local subsystem models is addressed, where their approach allows decentralized control design in subsystem level using standard robust control techniques. As it is discussed in [45], analysis based on their results may result in quite conservative stability conditions. The authors of [49] propose an approach based on quadratic invariance, where one needs to verify stability conditions in a centralized manner. In [4, 57, 58], a localized and scalable algorithm to solve a class of constrained optimal control problems for discrete-time linear systems is proposed that uses a system level synthesis framework. The authors define some notions of separability that allow parallel implementation of their algorithm. Almost all these previous works deal with synthesizing a linear network using decentralized sufficient conditions. The authors of [26] extend the notion of ℓ_2 stability to ℓ_∞ stability in order to allow the displacements or velocity deviation of vehicles from their equilibria in an infinite chain to remain bounded. In [5, 6], the authors consider a particular family of infinite-dimensional discrete autonomous systems where the state matrix is a Laurent polynomial matrix the shift operator. They provide necessary and sufficient conditions that the exponential ℓ_2 and ℓ_∞ stability for such systems are equivalent, but the ℓ_2 stability is an outlier. In [7], they analyze an infinite chain of kinematic points with nearest neighbor coupling and show that the system will converge to an equilibrium point if the initial perturbations are bounded.

In [35, 38, 39], it is suggested that stability of the class of spatially decaying systems can be verified in a localized manner using spatial truncation techniques. In Section 2, we formally define a class of proximity graphs, introduce a notion of spatial coverings for their corresponding networks, and provide several centralized quantitative characterizations of exponential stability property of spatially distributed linear systems. It is shown that some of these characterizations are more amenable to localized and decentralized verification schemes. Several necessary conditions for exponential stability are obtained in Section 3 that can be verified in a localized manner. In Section 4, it is shown that global stability of a network can be guaranteed by only verifying a set of localized sufficient conditions in vicinity of leading subsystems. We prove that these sufficient conditions are also necessary and almost optimal. The significant feature of our localized verifiable conditions is that they depend only on the spatially localized portions of the state matrix of the system and they are independent of the size of the entire system. The sufficient conditions in Theorem 5.4 provide a reliable tool to re-examine exponential stability of a symmetric linear system on an spatially distributed networks when coupling (e.g., communication) links between some subsystems are lost or added, as it suffices to verify localized stability conditions for affected subsystems. In Section 5, we show that our proposed necessary and sufficient conditions take a more tractable form for symmetric linear networks. It is proven that the global stability threshold of a symmetric linear network can be enhanced by improving the localized stability threshold via adjusting components of properly localized portions of the state matrix. In Section 6, we show how one can design a new symmetric linear network via adjusting coupling weights (i.e., elements of the state matrix) in a localized fashion. In Section 7 we support our theoretical findings by considering two thorough examples, and in Section 8 we conclude the paper with some remarks on the application of our proposed methodology to stability of spatially distributed systems on ℓ^∞ and (non)linear dynamical systems on a spatially distributed network. It should be emphasized that all technical conclusions in this paper hold equally for finite- and infinite-dimensional linear systems.

Preliminary versions of this work were appeared in [40, 42], where the journal version contains new materials, including two extensive case studies in the simulation section, several definitions including materials in Subsection 2.1 and 2.2, proof of all the technical results, example in Section 3, and remarks Section 4 and 5.

1.1. **Notations.** The set of nonnegative integers, nonnegative real numbers and complex numbers with nonnegative real parts are shown by $\mathbb{Z}_+$, $\mathbb{R}_+$ and $\mathbb{C}_+$, respectively. The real and imaginary parts of a complex number $z \in \mathbb{C}$ are represented by $\Re(z)$ and $\Im(z)$ respectively. For a matrix $\mathbf{A}$ with complex entries, we denote its Hermitian by $\mathbf{A}^*$ and we define its Hermitian and skew-Hermitian matrix decomposition by $\mathbf{A} = \mathbf{A}_h + \mathbf{A}_{ah}$, where $\mathbf{A}_h = (\mathbf{A} + \mathbf{A}^*)/2$ and $\mathbf{A}_{ah} = (\mathbf{A} - \mathbf{A}^*)/2$. For a countable set $\mathcal{V}$, let $\ell^p(\mathcal{V}), 1 \leq p \leq \infty$, contain all vectors

$\mathbf{c} = [c(i)]_{i\in\mathcal{V}}$ with bounded norm $\|\mathbf{c}\|_p := \big(\sum_{i\in\mathcal{V}} |c(i)|^p\big)^{1/p}, 1 \le p < \infty$ and $\|\mathbf{c}\|_\infty = \sup_{i\in\mathcal{V}} |c(i)|$. Whenever it is not ambiguous, we simply use abbreviated notation ℓ^p to represent the linear space $\ell^p(\mathcal{V})$. The set $\mathcal{B}^p, 1 \le p \le \infty$, contains all matrices $\mathbf{A}$ on ℓ^p with bounded induced norm $\|\mathbf{A}\|_{\mathcal{B}^p} := \sup_{\|\mathbf{c}\|_p=1} \|\mathbf{A}\mathbf{c}\|_p$. For a set F, we denote its cardinality by $\#F$ and define its characteristic function by χ_F, where $\chi_F(s) = 1$ for $s \in F$ and $\chi_F(s) = 0$ for $s \notin F$.

2. Preliminaries

Our interest in the (in)finite-dimensional linear dynamical system (1.1) is motivated by distributed control on spatially distributed networks (SDNs). In Section 2.1, we define a class of sparse proximity graphs to describe the coupling topology of an SDN, see Assumptions 2.1 and 2.3. The localized stability certificate developed in this paper depends on a covering of the graph with finite overlapping, which follows from Assumptions 2.1 and 2.3 [14]. To consider local stability certificate, we always assume that the state matrix $\mathbf{A}$ in the linear dynamical system (1.1) does not have large bandwidth, which indicates that the linear dynamic subsystems with large geodesic distance having *non-direct* impact on each other's dynamics, see Section 2.2. Our localized stability conditions can be implemented in a decentralized/distributed manner and their computational complexity is independent of network size. Some exponential stability characterizations in the centralized implementation is recalled in Section 2.3.

2.1. **Spatially distributed networks.**

This class consists of networks whose subsystems are distributed over a spatial domain and interconnected to each other through a coupling graph

$$\mathcal{G} := (\mathcal{V}, \mathcal{E}), \tag{2.1}$$

where $\mathcal{V}$ is the set of nodes (also known as vertices) and $\mathcal{E}$ is the set of edges [14, 41]. Every node in the graph $\mathcal{G}$ corresponds to a subsystem and every edge represents a direct coupling (e.g., communication/data transmission) link between those subsystems at the two ends of that edge. For example, two subsystems may communicate with each other via broadcasting their relevant information if their spatial distance is less than their communication range. It is assumed that all subsystems have identical coupling characteristics, e.g., they all use identical communication modules with similar communication range. The resulting coupling graph for this class of networks can be modeled by spatially distributed proximity graphs.

Assumption 2.1. *All coupling graphs are undirected and unweighted.*

The above assumption implies that all coupling links are bidirectional and have identical characteristics.

Definition 2.2. *For an undirected and unweighted graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, we define a geodesic distance $\rho : \mathcal{V} \times \mathcal{V} \to \mathbb{Z}_+ \cup \{+\infty\}$ by imposing: (i) $\rho(i, i) = 0$ for all $i \in \mathcal{V}$; (ii) $\rho(i, j)$ is the number of edges in a shortest path connecting two distinct nodes $i, j \in \mathcal{V}$; and (iii) $\rho(i, j) = +\infty$ if there is no paths connecting distinct nodes $i, j \in \mathcal{V}$.*

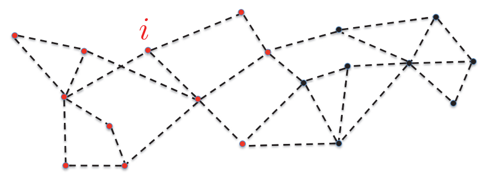

FIGURE 1. Illustration of 2-neighborhood of subsystem i, marked by red color, on a generic coupling graph.

A geodesic distance on a coupling graph can be utilized to assess coupling (e.g., communication) cost between two given subsystems [17]. When two subsystems are not neighbors (i.e., not connected through a direct link), they are still connected with each other via a chain of intermediate subsystems that connect them over a shortest path; however, the coupling cost between two subsystems gets larger as their geodesic distance increases.

For a given coupling graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ equipped with a geodesic distance ρ, the r-neighborhood of subsystem $i \in \mathcal{V}$ is defined by

$$B(i, r) := \big\{ j \in \mathcal{V} \mid \rho(i, j) \le r \big\}.$$

We refer to Figure 1 for an illustrative example. In this paper, we require that the coupling graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ has the following global feature: number of subsystems in the r-neighborhood and $2r$-neighborhood of each subsystem are comparable.

Assumption 2.3. *The counting measure $\mu_{\mathcal{G}} : 2^{\mathcal{V}} \longmapsto \mathbb{Z}_+$ of the coupling graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ is a doubling measure, i.e., there exists a positive number $D_0(\mathcal{G}) \ge 1$ such that*

$$\mu_{\mathcal{G}}(B(i, 2r)) \le D_0(\mathcal{G})\, \mu_{\mathcal{G}}(B(i, r)) \tag{2.2}$$

hold for all $i \in \mathcal{V}$ and $r \ge 0$, where $\mu_{\mathcal{G}}(F) := \# F$ for all $F \subset \mathcal{V}$.

The minimal constant $D_0(\mathcal{G})$ for the inequality (2.2) to hold is known as the doubling constant of the counting measure $\mu_{\mathcal{G}}$ [14, 22]. For a coupling graph $\mathcal{G}$, one can verify that the doubling constant $D_0(\mathcal{G})$ of its counting measure $\mu_{\mathcal{G}}$ dominates its maximal node degree, i.e., $d_{\max}(\mathcal{G}) \le D_0(\mathcal{G})$. This implies that every subsystem in an SDN, whose coupling graph satisfies Assumption 2.3, is coupled with at most $D_0(\mathcal{G})$ other subsystems in that network directly. Shown in the following examples are two classes of coupling graphs satisfying Assumption 2.3.

Example 2.4. *The first class consists of all (in)finite circulant graphs $\mathcal{C}_G = (\mathcal{V}_G, \mathcal{E}_G)$ associated with an abelian group*

$$\mathcal{V}_G = \Big\{ \prod_{i=1}^{k} g_i^{n_i} \;\Big|\; n_1, \ldots, n_k \in \mathbb{Z} \Big\}$$

generated by $G = \{g_1, \ldots, g_k\}$, where $(\lambda, \lambda') \in \mathcal{E}_G$ if and only if either $\lambda(\lambda')^{-1}$ or $\lambda'\lambda^{-1} \in G$ [10, 25, 28, 34]. This class of graphs enjoys certain shift-invariance in its construction. Let $\Omega = (-N_1, N_1) \times \cdots \times (-N_d, N_d)$ be a (un)bounded domain with $1 \le N_1, \cdots, N_d \le \infty$ and $X = \{x_i\}$ be a relatively-separated subset of the *domain Ω such that the family of open balls $B(x_i, 1/2), x_i \in X$, with radius $1/2$ and center $x_i \in X$ is a covering of the domain Ω with finite overlapping,*

$$1 \le \inf_{x\in\Omega} \sum_{x_i \in X} \chi_{B(x_i,1/2)}(x) \le \sup_{x\in\Omega} \sum_{x_i\in X} \chi_{B(x_i,1/2)}(x) < \infty.$$

The second class includes coupling graphs of spatially distributed networks whose agents $i \in \mathcal{V}$ are located at $x_i \in X$ in the domain Ω and a direct coupling between subsystems equipped on agents i and j exists if the spatial distance of their locations x_i and x_j is no more than one [1, 14, 15, 35, 36, 51].

Definition 2.5. *For an integer $N > 0$, an N-covering of a coupling graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ is a set of indices $\mathcal{V}_N = \{i_m \mid m \ge 1\}$ such that for every subsystem $i \in \mathcal{V}$ there exists at least one i_m such that $i \in B(i_m, N/2)$.*

A simple procedure to identify an N-covering is by the following procedure: (i) taking an arbitrary subsystem $i_1 \in \mathcal{V}$ for every connected component of $\mathcal{G}$, and then (ii) iteratively finding new subsystems $i_m \in \mathcal{V}$ for all $m \ge 2$ such that

$$\rho(i_m, i_1) = \min \left\{ \rho(i, i_1) \;\middle|\; i \notin \bigcup_{m'=1}^{m-1} B(i_{m'}, N/2) \right\}. \tag{2.3}$$

By Assumption 2.3 on the counting measure, the resulting N-covering $\mathcal{V}_N$ from the above algorithm satisfies the following property [14]: every subsystem $i \in \mathcal{V}$ is in the $(N/2)$-neighborhoods of $i_m \in \mathcal{V}_N$ at least once and in the $2N$-neighborhoods of $i_m \in \mathcal{V}_N$ at most $(D_0(\mathcal{G}))^5$ times, i.e.,

$$1 \le \alpha_1 \le \alpha_2 \le \big(D_0(\mathcal{G})\big)^5 \tag{2.4}$$

in which

$$\alpha_1 = \sum_{i_m \in \mathcal{V}_N} \chi_{B(i_m, N/2)}(i), \;\; \alpha_2 = \sum_{i_m \in \mathcal{V}_N} \chi_{B(i_m, 2N)}(i), \tag{2.5}$$

and $D_0(\mathcal{G})$ is the doubling constant in (2.2), cf. Remark 4.3.

The set of all subsystems in $\mathcal{V}_N$ are referred to as leading subsystems of a spatially distributed system. The importance of leading subsystems will become more evident later in the paper, e.g., see results of Theorems 4.1, 4.2 and 5.4, where it is shown that global stability of a network can be inferred by only verifying a set of localized sufficient conditions in vicinity of leading subsystems. Therefore, due to their crucial role, the leading subsystems can be equipped with high performance computational units to enable them verify localized stability conditions using local information.

The set of leading subsystems constructed through the above procedure is neither unique nor optimal, see Figure 2 for an illustrative example. Therefore, in our results, we can safely employ any subset $\tilde{\mathcal{V}}_N \subset \mathcal{V}$ that satisfies inequalities (2.4) as the set of leading subsystems. The number of leading subsystems in an N-covering decreases as N increases. This may impose some trade-offs between the number of leading subsystems and their minimal required on-board computational capabilities: when the number of leading subsystems decreases, they should, in

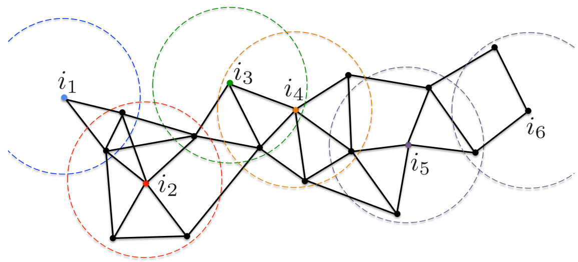

FIGURE 2. A 2-covering of a spatially distributed system with 19 subsystems. This network can also be covered by 1-neighborhood of 6 leading subsystems.

turn, be equipped with more powerful computers to enable them to verify stability conditions for larger covering regions.

Definition 2.6. *For a coupling graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, its counting measure $\mu_{\mathcal{G}}$ has polynomial growth if there exist positive constants $D_1(\mathcal{G})$ and d such that*

$$\mu_{\mathcal{G}}(B(i,r)) \leq D_1(\mathcal{G})(1+r)^d \text{ for all } i \in \mathcal{V} \text{ and } r \geq 0. \tag{2.6}$$

For an SDN with coupling graph $\mathcal{G}$, the smallest constants d and $D_1(\mathcal{G})$ for which the inequality (2.6) holds are so called *Beurling dimension* and *density* of that network, respectively [14]. For an SDN whose subsystems are embedded on a d-dimensional manifold and direct coupling (e.g., communication) link between two subsystems exists only if their spatial locations are within a certain range, the Beurling dimension is equal to the dimension of the manifold, see the second class of coupling graphs in Example 2.4.

We remark that a doubling measure $\mu_{\mathcal{G}}$ has polynomial growth,

$$\mu_{\mathcal{G}}(B(i,r)) \;\leq\; D_0(\mathcal{G})\,(1+r)^{\log_2 D_0(\mathcal{G})}$$

for all $i \in \mathcal{V}$ and $r \geq 0$. However, the Beurling dimension d of the graph $\mathcal{G}$ is usually much smaller than $\log_2 D_0(\mathcal{G})$ in the above estimate. For circulant graphs $\mathcal{C}_G = (\mathcal{V}_G, \mathcal{E}_G)$ in (2.4), one may verify that their Beurling dimensions are at most $\#G$, the cardinality of the generator in G, while their exact value depend on the topological structure of the Abelian group $\mathcal{V}_G$ and the selection of the generator G.

2.2. **State matrices with small bandwidth.** In this paper, we consider (in)finite-dimensional spatially distributed networks whose state matrices have small bandwidth, i.e., subsystems with large geodesic distance have indirect impact on each other's dynamics only.

Definition 2.7. *For a given coupling graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ and a nonnegative integer τ, a matrix $\mathbf{A} = [a(i,j)]_{i,j\in\mathcal{V}}$ is called τ-banded if its entries satisfy*

$$a(i,j) = 0 \text{ if } \rho(i,j) > \tau. \tag{2.7}$$

The set of all τ-banded matrices is represented by $\mathcal{B}_\tau(\mathcal{G})$ or the abbreviated notation $\mathcal{B}_\tau$ whenever it is not ambiguous. For a matrix $\mathbf{A} = [a(i,j)]_{i,j\in\mathcal{V}}$, we define its boundedness norm by $\|\mathbf{A}\|_\infty := \sup_{i,j\in\mathcal{V}} |a(i,j)|$ and its Schur norm by

$$\|\mathbf{A}\|_{\mathcal{S}} := \max\Big\{\sup_{i\in\mathcal{V}}\sum_{j\in\mathcal{V}}|a(i,j)|,\ \sup_{i\in\mathcal{V}}\sum_{j\in\mathcal{V}}|a(i,j)|\Big\}. \tag{2.8}$$

It has been proved that the following hold,

$$\|\mathbf{A}\|_\infty \le \|\mathbf{A}\|_{\mathcal{B}^p} \le \|\mathbf{A}\|_{\mathcal{S}} \le D_1(\mathcal{G})\,(1+\tau)^d\,\|\mathbf{A}\|_\infty \tag{2.9}$$

for all matrices $\mathbf{A} \in \mathcal{B}_\tau \cap \mathcal{B}^p$, where $1 \le p \le \infty$ [14]. Therefore, a matrix with finite bandwidth is bounded on $\ell^p, 1 \le p \le \infty$, if and only if it has bounded entries.

Remark 2.8. *It is emphasized that adding new or eliminating existing edges does not change the set of vertices, however, it will result in a new edge set that will affect the geodesic distance between vertices. Hence, the bandwidth of the state matrix will change accordingly and the double measuring property for the counting measure could be invalid.*

2.3. **Centralized exponential stability conditions.** In this section, we recall several equivalent versions of the exponential stability condition (1.2) of finite and infinite-dimensional linear systems on a spatially distributed proximity graph [11, 12, 20, 52, 59], which will be used in the next two sections to derive localized sufficient and necessary stability conditions.

Theorem 2.9. *Suppose that the state matrix* $\mathbf{A}$ *of the linear system* (1.1) *belongs to* $\mathcal{B}^2$*. Then the exponential stability* (1.2) *is equivalent to each of the following:*

(i) *Spectrum of the state matrix* $\mathbf{A}$ *is strictly contained in the open left-half complex plane, i.e., there exists* $\delta > 0$ *such that*

$$\sigma(\mathbf{A}) \subset \big\{z \in \mathbb{C} \mid \Re(z) \le -\delta\big\}. \tag{2.10}$$

(ii) $z\mathbf{I} - \mathbf{A}$ *is invertible for all* $z \in \mathbb{C}_+$ *and*

$$A_0 := \inf_{z\in\mathbb{C}_+} \|(z\mathbf{I}-\mathbf{A})^{-1}\|_{\mathcal{B}^2}^{-1} > 0. \tag{2.11}$$

(iii) *There exists a positive constant* A_0 *such that*

$$\min\big(\|(z\mathbf{I}-\mathbf{A})\mathbf{c}\|_2, \|(z\mathbf{I}-\mathbf{A}^*)\mathbf{c}\|_2\big) \ge A_0\|\mathbf{c}\|_2 \tag{2.12}$$

for all $z \in \mathbb{C}_+$ *and* $\mathbf{c} \in \ell^2$.

We recall that the solution of the linear system (1.1) is given by

$$\psi(t) = e^{\mathbf{A}t}\psi(0)$$

for all $t \ge 0$. Then, the equivalence between statement (i) of Theorem 2.9 and the exponential stability (1.2) is a restatement of the well-known Hille-Yosida theorem. The equivalent statement (ii) of the exponential stability (1.2) is known as a resolvent growth assumption for the semigroup $e^{\mathbf{A}t}$ for $t \ge 0$ with bounded generator $\mathbf{A}$ on ℓ^2 [52, 59]. The localized stability certificates proposed in this paper are mainly based on the equivalent statement (iii) to the exponential stability (1.2) and its equivalence to statement (ii), which follows from the quantitative version of the conclusion that a bounded operator T on a Hilbert space is invertible if and only if TT^* and T^*T are invertible. For the completeness of this paper, we provide a sketch proof of Theorem 2.9 at the end of this section.

The exponential stability rate α in (1.2), the stability margin δ in (2.10), and constant A_0 in the statements (ii) and (iii) of Theorem 2.9 are closely related. By [59, Theorem 1.5], the exponential stability (1.2) holds for all $\alpha < \delta$ if the spectral set property (2.10) holds. From the proof of Theorem 2.9, it follows that the spectral set property (2.10) is true for all $\delta \le A_0$ if the resolvent growth condition (2.11) is satisfied. Moreover, the constant A_0 in the statements (ii) and (iii) can be chosen to be identical; in that case, we refer to A_0 as *stability threshold* of system (1.1).

By Theorem 2.9, exponential stability of the linear system (1.1) with a state matrix $\mathbf{A}$ can be understood as uniform stability of the family of matrices $z\mathbf{I}-\mathbf{A}$ and $z\mathbf{I}-\mathbf{A}^*$ for $z \in \mathbb{C}_+$ on ℓ^2. We say that a matrix $\mathbf{B} \in \mathcal{B}^2$ is ℓ^2 stable if there exists a positive constant E such that $\|\mathbf{Bc}\|_2 \ge E\|\mathbf{c}\|_2$ hold for all $\mathbf{c} \in \ell^2$. The notion of matrix stability for a matrix is one of the fundamental tools in frame theory, sampling theory, wavelet analysis and many other fields [2, 16, 50, 51, 53, 55, 56], where matrix stability verification in a decentralized/distributed manner has been studied in [14, 54].

Proof of Theorem 2.9. We divide the proof into the following three implications (i) $\Longleftrightarrow$ (ii) $\Longleftrightarrow$ (iii) and prove them one by one. First, we start with (i)$\Longleftrightarrow$(ii). The sufficiency follows as the quantity $\|(z\mathbf{I}-\mathbf{A})^{-1}\|_{\mathcal{B}^2}$ is continuous about z with $\Re(z) \ge 0$ and it tends to zero as $|z| \to +\infty$. For the necessity, we have

$$(w\mathbf{I}-\mathbf{A})^{-1} = ((w-a)\mathbf{I}-\mathbf{A})^{-1}\sum_{n=0}^{\infty}(-a((w-a)\mathbf{I}-\mathbf{A})^{-1})^n \tag{2.13}$$

for all $w \in \mathbb{C}$ with $\Re(w) > -A_0$, where $a = \min\{0, \Re(w)\}$. The Neumann series expansion in (2.13) holds as $|a|\ \|((w-a)\mathbf{I}-\mathbf{A})^{-1}\|_{\mathcal{B}^2} \le |a|/A_0 < 1$. Therefore,

$$\sigma(\mathbf{A}) \subset \{z \in \mathbb{C},\ \Re(z) \le -A_0\}, \tag{2.14}$$

which proves statement (i) with $\delta = A_0$.

Next, we show (ii)$\Longleftrightarrow$(iii). The sufficiency holds as matrices $z\mathbf{I}-\mathbf{A}$ and $z\mathbf{I}-\mathbf{A}^*$, where $\Re(z) \ge 0$, have uniformly bounded inverses. To prove the necessity, let us pick a $z \in \mathbb{C}$ with $\Re(z) \ge 0$. By the ℓ^2-stability property (2.12), it suffices to prove that the range of $z\mathbf{I}-\mathbf{A}$ is the entire ℓ^2 space. Let us suppose, on the contrary, that the range space is not the entire space. As the range space is closed by the ℓ^2-stability property (2.12), the orthogonal complement of the range is nontrivial, i.e., there exists $\mathbf{0} \ne \mathbf{d} \in \ell^2$ such that $\mathbf{d}^*(z\mathbf{I}-\mathbf{A})\mathbf{c} = 0$ for all $\mathbf{c} \in \ell^2$. Thus, $(\bar{z}\mathbf{I}-\mathbf{A}^*)\mathbf{d} = \mathbf{0}$, which together with the ℓ^2-stability property (2.12) for $\bar{z}\mathbf{I}-\mathbf{A}^*$ implies that $\mathbf{d} = \mathbf{0}$. This is a contradiction, which proves our claim. □

3. DECENTRALIZED NECESSARY CONDITIONS

Let $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ be the coupling graph of an SDN that satisfies Assumption 2.1. In this section, we utilize finite truncation techniques to obtain several decentralized necessary conditions for exponential stability of the spatially distributed linear network (1.1). In Section 4, it is shown that these conditions become also sufficient for large value of N.

Definition 3.1. *For every node $i \in \mathcal{V}$ and integer $N \geq 0$, we define the truncation operator $\chi_i^N : \ell^2 \longmapsto \ell^2$ by*

$$\chi_i^N [c(j)]_{j\in\mathcal{V}} := \big[\chi_{B(i,N)}(j)\, c(j)\big]_{j\in\mathcal{V}}. \tag{3.1}$$

The truncation operator χ_i^N localizes a vector to the N-neighborhood of the subsystem $i \in \mathcal{V}$ and its action can be equivalently expressed by a diagonal matrix whose (j, j)-th diagonal entry $\chi_{B(i,N)}(j) := \chi_{[0,N]}(\rho(i, j))$ for all $j \in \mathcal{V}$.

Theorem 3.2. *Let $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ be the undirected and unweighted graph of an SDN. Suppose that the state matrix $\mathbf{A}$ of the linear system* (1.1) *belongs to $\mathcal{B}_\tau \cap \mathcal{B}^2$ for some integer $\tau \geq 0$ and the system is exponentially stable with stability threshold A_0. Then, for all vertices $i \in \mathcal{V}$, positive integers $N \geq 1$ and $\mathbf{d} \in \ell^2$,*

$$\min\big(\|(z\mathbf{I} - \mathbf{A})\chi_i^N \mathbf{d}\|_2, \|(z\mathbf{I} - \mathbf{A}^*)\chi_i^N \mathbf{d}\|_2\big) \geq A_0 \|\chi_i^N \mathbf{d}\|_2, \ \ z \in \mathbb{C}_+, \tag{3.2}$$

hold or equivalently

$$\min\big\{\|\mathbf{A}\chi_i^N \mathbf{d}\|_2^2, \|\mathbf{A}^*\chi_i^N \mathbf{d}\|_2^2\big\} \geq A_0^2 \|\chi_i^N \mathbf{d}\|_2 \,+\, \Phi_i^N(\mathbf{d}) \tag{3.3}$$

hold, where

$$\Phi_i^N(\mathbf{d}) = \frac{|\mathbf{d}^*\chi_i^N \mathbf{A}_{ah}\chi_i^N \mathbf{d}|^2 + \big(\max\big\{0, \mathbf{d}^*\chi_i^N \mathbf{A}_h \chi_i^N \mathbf{d}\big\}\big)^2}{\|\chi_i^N \mathbf{d}\|_2^2}. \tag{3.4}$$

Proof. Condition (3.2) is the localized version of (2.12) in Theorem 2.9 with $\mathbf{d}$ replaced by $\chi_i^N \mathbf{d}$.

For $z \in \mathbb{C}_+$, write $z = a + b\sqrt{-1}$ for $a \geq 0$ and $b \in \mathbb{R}$. Then for any $\mathbf{d} \in \ell^2$ with $\|\chi_i^N \mathbf{d}\|_2 = 1$, we obtain

$$\begin{aligned}
&\inf_{z\in\mathbb{C}_+} \|(z\mathbf{I} - \mathbf{A})\chi_i^N \mathbf{d}\|_2^2 \\
&= \inf_{a\geq 0, b\in\mathbb{R}} a^2 + b^2 - 2a\mathbf{d}^*\chi_i^N \mathbf{A}_h \chi_i^N \mathbf{d} - 2b\sqrt{-1}\mathbf{d}^*\chi_i^N \mathbf{A}_{ah}\chi_i^N \mathbf{d} + \|\mathbf{A}\chi_i^N \mathbf{d}\|_2^2 \\
&= \inf_{a\geq 0} \big(a^2 - 2a\mathbf{d}^*\chi_i^N \mathbf{A}_h \chi_i^N \mathbf{d}\big) + \|\mathbf{A}\chi_i^N \mathbf{d}\|_2^2 - |\mathbf{d}^*\chi_i^N \mathbf{A}_{ah}\chi_i^N \mathbf{d}|^2 \\
&= \|\mathbf{A}\chi_i^N \mathbf{d}\|_2^2 - \Phi_i^N(\mathbf{d}),
\end{aligned} \tag{3.5}$$

and similarly

$$\inf_{z\in\mathbb{C}_+} \|(z\mathbf{I} - \mathbf{A}^*)\chi_i^N \mathbf{d}\|_2^2 = \|\mathbf{A}^*\chi_i^N \mathbf{d}\|_2^2 - \Phi_i^N(\mathbf{d}). \tag{3.6}$$

Therefore combining the above two estimate establishes the equivalence between the requirements in (3.2) and (3.3). □

We remark that the second term in the quantity $\Phi_i^N(\mathbf{d})$ vanishes when $\mathbf{A}_h$ is a negative definite matrix, and the whole quantity $\Phi_i^N(\mathbf{d})$ in (3.4) vanishes when $\mathbf{A}$ is a negative definite Hermitian matrix, cf. Theorem 5.1. The necessary condition (3.3) are spatially localized and can be verified in a decentralized/distributed manner by having access only to local information about the state matrix $\mathbf{A}$, and moreover, the constant in (3.2) can be selected to be the same as the one in (2.12).

For $i \in \mathcal{V}$ and $N \geq 1$, let us define

$$B_N(i) := \inf_{\|\chi_i^N \mathbf{d}\|_2 = 1} \sqrt{\min\left\{\|\mathbf{A}\chi_i^N \mathbf{d}\|_2^2, \|\mathbf{A}^*\chi_i^N \mathbf{d}\|_2^2\right\} - \Phi_i^N(\mathbf{d})}$$

$$= \inf_{z \in \mathbb{C}_+} \inf_{\|\chi_i^N \mathbf{d}\|_2 = 1} \min\left(\|(z\mathbf{I} - \mathbf{A})\chi_i^N \mathbf{d}\|_2^2, \|(z\mathbf{I} - \mathbf{A}^*)\chi_i^N \mathbf{d}\|_2^2\right) \tag{3.7}$$

Using the following relationship $\{\chi_i^N \mathbf{d} \mid \mathbf{d} \in \ell^2\} \subseteq \{\chi_i^{N+1} \mathbf{d} \mid \mathbf{d} \in \ell^2\} \subseteq \ell^2$, we have that the sequences $\{B_N(i)\}_{N=1}^{\infty}, i \in V$, are monotonic,

$$B_N(i) \geq B_{N+1}(i) \geq 0 \text{ for all } N \geq 1 \text{ and } i \in \mathcal{V}$$

and denote their limits by $B_\infty(i), i \in \mathcal{V}$. By Theorem 3.2 it follows that

$$B_N(i) \geq B_\infty(i) \geq A_0 \ \text{ for all } N \geq 1 \text{ and } i \in \mathcal{V}. \tag{3.8}$$

By (3.7), we obtain

$$\min(\|(z\mathbf{I} - \mathbf{A})\chi_i^N \mathbf{d}\|_2, \|(z\mathbf{I} - \mathbf{A}^*)\chi_i^N \mathbf{d}\|_2\big) \geq B_N(i)\|\chi_i^N \mathbf{d}\|_2 \geq B_\infty(i)\|\chi_i^N \mathbf{d}\|_2$$

for all $z \in \mathbb{C}_+$ and $\mathbf{d} \in \ell^2$. Taking limit $N \to \infty$ yields

$$\min(\|(z\mathbf{I} - \mathbf{A})\mathbf{d}\|_2, \|(z\mathbf{I} - \mathbf{A}^*)\mathbf{d}\|_2\big) \geq B_\infty(i)\|\mathbf{d}\|_2 \ \text{ for all } z \in \mathbb{C}_+ \ \text{ and } \ \mathbf{d} \in \ell^2.$$

This together with (3.8) and Theorem 2.9 implies that the sequences $\{B_N(i)\}_{N=1}^{\infty}, i \in \mathcal{V}$ decrease to A_0, the largest positive constant such that (2.12) holds,

$$\lim_{N \to \infty} B_N(i) = A_0 \ \text{ for all } i \in \mathcal{V}. \tag{3.9}$$

Example 3.3. If linear system (1.1) is spatially invariant with a Toeplitz state matrix $\mathbf{A}_0 = \big[p(i-j)\big]_{i,j \in \mathbb{Z}}$, we have

$$\min\left(\|(z\mathbf{I} - \mathbf{A}_0)\mathbf{c}\|_2, \|(z\mathbf{I} - \mathbf{A}_0^*)\mathbf{c}\|_2\right) \ \geq \ \left(\inf_{\xi \in \mathbb{R}} |z - \hat{p}(\xi)|\right) \|\mathbf{c}\|_2 \tag{3.10}$$

for all $z \in \mathbb{C}_+$ and $\mathbf{c} \in \ell^2$, where $\hat{p}(\xi) = \sum_{j \in \mathbb{Z}} p(j) e^{-2\pi j \xi \sqrt{-1}}$. From the above inequality (3.10), we have

$$A_0 = \inf_{z \in \mathbb{C}_+} \inf_{\xi \in \mathbb{R}} \big|z - \hat{p}(\xi)\big| = \inf_{\xi \in \mathbb{R}} \max(-\Re \hat{p}(\xi), 0). \tag{3.11}$$

Also for the above spatially invariant linear system, the local stability threshold $B_N(i)$ in (3.7) is independent of the node index i. If the state matrix $\mathbf{A}_0$ is further assumed to have finite bandwidth $\tau \geq 0$, i.e., $p(j) = 0$ for all $|j| > \tau$, we write $z - \hat{p}(\xi) = \sum_{j \in \mathbb{Z}} p_z(j) e^{-2\pi\sqrt{-1} j \xi}$. Then for every $N \geq \tau$, $i \in \mathbb{Z}$ and $\mathbf{c} =$

$[c(j)]_{j\in\mathbb{Z}} \in \ell^2$,

$$\begin{aligned}
\|(z\mathbf{I}-\mathbf{A}_0)\chi_i^N\mathbf{c}\|_2^2 \ge & \sum_{k=i-N+\tau}^{i+N-\tau}\Big|\sum_{l\in\mathbb{Z}} p_z(k-l)c_{N,i}(l)\Big|^2 \\
& +\frac{1}{2}\sum_{k=i+N-\tau+1}^{i+N}\Big|\sum_{l\in\mathbb{Z}} p_z(k-l)(c_{N,i}(l-2N-1)+c_{N,i}(l))\Big|^2 \\
& +\frac{1}{2}\sum_{k=i-N}^{i-N+\tau-1}\Big|\sum_{l\in\mathbb{Z}} p_z(k-l)(c_{N,i}(l+2N+1)+c_{N,i}(l))\Big|^2 \\
\ge & \frac{1}{2}\sum_{k=i-N}^{i+N}\Big|\sum_{l=i-N}^{i+N} p_z(k-l)\tilde{c}_{N,i}(l)\Big|^2,
\end{aligned}$$

where $\chi_i^N\mathbf{c} = \big[c_{N,i}(j)\big]_{j\in\mathbb{Z}}$, and $\tilde{\mathbf{c}}_{N,i} = \big[\tilde{c}_{N,i}(j)\big]_{j\in\mathbb{Z}}$ is a periodic vector taking the same values with $\chi_i^N\mathbf{c}$ on intervals $[i-N, i+N]$. Thus,

$$\|(z\mathbf{I}-\mathbf{A}_0)\chi_i^N\mathbf{c}\|_2^2 \ \ge\ \frac{\inf_{\xi\in\mathbb{Z}/(2N+1)}|z-\hat{p}(\xi)|^2}{2}\ \|\chi_i^N\mathbf{c}\|_2^2$$

for all $\mathbf{c}\in\ell^2$. Using the same argument, we can show that

$$\|(z\mathbf{I}-\mathbf{A}_0^*)\chi_i^N\mathbf{c}\|_2^2 \ \ge\ \frac{\inf_{\xi\in\mathbb{Z}/(2N+1)}|\bar{z}-\hat{p}(\xi)|^2}{2}\ \|\chi_i^N\mathbf{c}\|_2^2$$

for all $\mathbf{c}\in\ell^2$. As a result, we obtain the following estimate for the local stability threshold of the spatially invariant linear system (1.1),

$$\begin{aligned}
B_N(i) &\ge \frac{\sqrt{2}}{2}\inf_{z\in\mathbb{C}_+}\ \inf_{\xi\in\mathbb{Z}/(2N+1)}|z-\hat{p}(\xi)| \\
&= \frac{\sqrt{2}}{2}\inf_{\xi\in\mathbb{Z}/(2N+1)}\max(-\Re\hat{p}(\xi),0) \qquad (3.12)
\end{aligned}$$

for all $N\ge\tau$ and $i\in\mathbb{Z}$.

4. Decentralized Sufficient Conditions

In this section, we introduce several decentralized sufficient conditions to verify exponential stability of the linear system (1.1) on a spatially distributed proximity graph. Our proposed sufficient conditions are based on the limit property (3.9) and the fact that the localized stability thresholds are uniformly bounded below by the stability threshold, according to (3.8).

Theorem 4.1. *Let the spatially distributed proximity graph $\mathcal{G}=(\mathcal{V},\mathcal{E})$ satisfy Assumption 2.1 and its counting measure $\mu_{\mathcal{G}}$ have the polynomial growth* (2.6). *Suppose that the state matrix $\mathbf{A}$ belongs to $\mathcal{B}_\tau\cap\mathcal{B}^2$ for some $\tau\ge 0$. If there exists a positive integer $N_0\ge\tau$ and a positive number B_{N_0} satisfying*

$$B_{N_0}\ \ge\ 4\tau\sqrt{\alpha_2^*/\alpha_1^*}\ \|\mathbf{A}\|_{\mathcal{S}}\ N_0^{-1} \qquad (4.1)$$

such that

$$\min\Big(\big\|(z\mathbf{I}-\mathbf{A})\chi_{i_m}^{N_0}\mathbf{c}\big\|_2, \big\|(z\mathbf{I}-\mathbf{A}^*)\chi_{i_m}^{N_0}\mathbf{c}\big\|_2\Big) \ \ge\ B_{N_0}\big\|\chi_{i_m}^{N_0}\mathbf{c}\big\|_2 \tag{4.2}$$

hold for all $z\in\mathbb{C}_+, \mathbf{c}\in\ell^2$ and $i_m\in\mathcal{V}_{N_0}$ (the set of leading subsystems according to Definition 2.5), then the linear system (1.1) *with state matrix* $\mathbf{A}$ *is exponentially stable and its stability threshold A_0 satisfies*

$$A_0 \ \ge\ B_{N_0}\sqrt{\alpha_1^*/(4\alpha_2^*)}, \tag{4.3}$$

in which

$$\alpha_1^* := \inf_{i\in\mathcal{V}}\sum_{i_m\in\mathcal{V}_{N_0}}\chi_{B(i_m,N_0/2)}(i) \text{ and } \alpha_2^* := \sup_{i\in\mathcal{V}}\sum_{i_m\in\mathcal{V}_{N_0}}\chi_{B(i_m,2N_0)}(i). \tag{4.4}$$

A detailed proof of Theorem 4.1 is postponed to the end of this section. The sufficient condition (4.2) in Theorem 4.1, in their current form, require verification for all complex numbers $z\in\mathbb{C}_+$. In the following result, following the arguments used to prove (3.5) and (3.6), we obtain an equivalent verifiable condition by eliminating the complex variable and combining these two conditions into one.

Theorem 4.2. *Let the spatially distributed proximity graph $\mathcal{G}=(\mathcal{V},\mathcal{E})$ satisfy Assumption 2.1 and its counting measure $\mu_\mathcal{G}$ have the polynomial growth* (2.6) *and the state matrix* $\mathbf{A}$ *belong to $\mathcal{B}_\tau\cap\mathcal{B}^2$ for some $\tau\ge 0$. Then the linear system* (1.1) *with state matrix $\mathbf{A}\in\mathcal{B}_\tau\cap\mathcal{B}^2$ for some $\tau\ge 1$ is exponentially stable if there exists a positive integer $N_0\ge\tau$ and a constant B_{N_0} satisfying* (4.1) *such that*

$$\min\Big\{\|\mathbf{A}\chi_{i_m}^{N_0}\mathbf{d}\|_2^2, \|\mathbf{A}^*\chi_{i_m}^{N_0}\mathbf{d}\|_2^2\Big\} \ge B_{N_0}^2\big\|\chi_{i_m}^{N_0}\mathbf{d}\big\|_2^2+\Phi_{i_m}^{N_0}(\mathbf{d}) \tag{4.5}$$

for all $i_m\in\mathcal{V}_{N_0}$ and vectors $\mathbf{d}\in\ell^2$, where $\Phi_{i_m}^{N_0}$ is defined in Theorem 3.2.

We omit a detailed proof of Theorem 4.2 as the requirements (4.2) and (4.5) are equivalent to each other from the proof of Theorem 3.2. The sufficient conditions in Theorems 4.1 and 4.2 assert that exponential stability can be only verified in neighborhoods of the leading subsystems, i.e., one only needs validate the condition (4.5) for leading subsystems in $\mathcal{V}_{N_0}$, rather than checking them for every single subsystem. This feature drastically reduces time-complexity of the verification process and makes it attractive for real-world applications. Our results also suggest an important design protocol: all leading subsystems of a spatially distributed system should be equipped with high performance computational units to allow them to verify localized stability conditions more reliably and timely.

Remark 4.3. *For the set of leading subsystems $\mathcal{V}_{N_0}$ in Theorem 4.1, the corresponding covering constants α_1,α_2 in* (2.5) *satisfy $1\le\alpha_1\le\alpha_2\le D_1(\mathcal{G})^5$ by* (2.4)*, which follows from Assumption 2.3 on the counting measure on the graph $\mathcal{G}$ [14]. As a result, the right hand side of* (4.1) *tends to zero when $N_0\to\infty$. This implies that the decentralized sufficient condition* (4.2) *for exponential stability of the linear dynamical network* (1.1) *in Theorem 4.1 is also necessary for large N_0 (cf. Theorem 3.2).*

From Theorems 3.2 and 4.1, we conclude that exponential stability of the linear system (1.1) can be verified via a decentralized/distributed manner. Moreover, the global stability threshold A_0 in (2.12) and the local stability threshold B_{N_0} in (4.2) are comparable through the following inequalities

$$A_0 \le B_{N_0} \le 2\sqrt{\alpha_2^*/\alpha_1^*}\, A_0 \tag{4.6}$$

for those integers N_0 satisfying (4.1).

Remark 4.4. *The implication of small N_0 in real-world applications is that the leading subsystems can be equipped with reasonable (bounded) computational powers to verify stability conditions. The requirement* (4.1) *on size of N_0 is conservative, but as shown in Example 4.5, it is almost optimal.*

The next example shows that a linear network may not be exponentially stable even though conditions (4.2) are met with some constant B'_{N_0} that has the same order of N_0^{-1} as the lower bound in (4.1) for large enough N_0.

Example 4.5. Let us consider a spatially invariant system whose state matrix is a bi-infinite Toeplitz matrix $\mathbf{A}_1 = [a_1(i-j)]_{i,j\in\mathbb{Z}}$ with Fourier symbol

$$\sum_{k\in\mathbb{Z}} a_1(k) e^{-2\pi k\sqrt{-1}\xi} = -1 + e^{-2\pi\sqrt{-1}\xi}.$$

It is straightforward to check that $\mathbf{A}_1$ is 1-band matrix with $\|\mathbf{A}_1\|_\infty = 1$ and property $0 \in \sigma(\mathbf{A}_1)$. Therefore, the linear system (1.1) with state matrix $\mathbf{A}_1$ is not exponentially stable. In the following, we show that (4.2) hold for this system with constant $B'_{N_0} = \frac{1}{2}N_0^{-1}$. For every $z \in \mathbb{C}_+, i \in \mathbb{Z}$ and $N_0 \ge 1$, we have

$$\begin{aligned}
&\inf_{\|\chi_i^{N_0}\mathbf{c}\|_2=1} \|(z\mathbf{I}-\mathbf{A}_1)\chi_i^{N_0}\mathbf{c}\|_2^2 \;\ge\; \inf_{\sum_{j=1}^{2N_0+1}|d_j|^2=1} \Big(1+\Re\Big\{\sum_{j=2}^{2N_0+1} d_j\overline{d_{j-1}}\Big\}\Big)^2\\
&= \inf_{\sum_{j=1}^{2N_0+1}|d_j|^2=1} |z+1|^2 + 1 - 2\Re\Big\{(z+1)\sum_{j=2}^{2N_0+1} d_j\overline{d_{j-1}}\Big\}\\
&= \inf_{\sum_{j=1}^{2N_0+1}|e_j|^2=1, e_j\in\mathbb{R}} |e_1|^2 + |e_{2N_0+1}|^2 + \sum_{j=2}^{2N_0+1}|e_j-e_{j-1}|^2 = 4\sin^2\frac{\pi}{4N_0+4},
\end{aligned}$$

where the last equality follows from [31, Lemma 1 of Chapter 9]. Following a similar argument, we have

$$\inf_{\|\chi_i^{N_0}\mathbf{c}\|_2=1} \|(z\mathbf{I}-\mathbf{A}_1^*)\chi_i^{N_0}\mathbf{c}\|_2^2 \ge 2\sin\frac{\pi}{4N_0+4} \ge (2N_0)^{-1}.$$

Thus, the conditions (4.2) hold with constant $B'_{N_0} = (N_0)^{-1}$ for all $N_0 \ge 1$. On the other hand, we observe that the underlying coupling graph of this system with node set $\mathcal{V} = \mathbb{Z}$ has Beurling dimension 1, density 2, and the set of leading subsystems $\mathcal{V}_{N_0} = (N_0+1)\mathbb{Z}$ with covering constants $C_1 = 1$ and $C_2 = 4$. Therefore, the lower bound for the constant B_{N_0} in (4.1) is $2^6 N_0^{-1}$. One observes that although condition (4.2) hold with some constant $B'_{N_0} = 2^{-1}N_0^{-1}$, whose value is smaller than $2^6N_0^{-1}$ but vanishes with the same rate of N_0^{-1}, the linear system is still not exponentially stable. This explains the critical role and near-optimality of the sufficient condition (4.1).

4.1. **Proof of Theorem 4.1.** Let us pick $z \in \mathbb{C}_+$ and $\mathbf{c} \in \ell^2$. By Theorem 2.9, it suffices to prove the uniform stability for matrices $z\mathbf{I} - \mathbf{A}$ and $z\mathbf{I} - \mathbf{A}^*$, i.e.,

$$\|(z\mathbf{I} - \mathbf{A})\mathbf{c}\|_2 \geq \frac{B_{N_0}}{2}\sqrt{\alpha_1^*/\alpha_2^*}\|\mathbf{c}\|_2 \tag{4.7}$$

and

$$\|(z\mathbf{I} - \mathbf{A}^*)\mathbf{c}\|_2 \geq \frac{B_{N_0}}{2}\sqrt{\alpha_1^*/\alpha_2^*}\|\mathbf{c}\|_2. \tag{4.8}$$

For every subsystem $i \in \mathcal{V}$ and integer $N \geq 0$, we define operator Ψ_i^N by

$$\Psi_i^N : \ \ell^2 \ni [c(j)]_{j\in\mathcal{V}} \longmapsto \big[\psi_0(\rho(i,j)/N)c(j)\big]_{j\in\mathcal{V}} \in \ell^2$$

in which $\psi_0(t) = \max(\min(2-2|t|, 1), 0)$ is the trapezoid function. By the local stability assumption (4.2), we have

$$\|(z\mathbf{I} - \mathbf{A})\Psi_{i_m}^{N_0}\mathbf{c}\|_2 \geq B_{N_0}\|\Psi_{i_m}^{N_0}\mathbf{c}\|_2 \tag{4.9}$$

for all $\mathbf{c} \in \ell^2$ and $i_m \in \mathcal{V}_{N_0}$. Let us denote $\mathbf{A} = [a(i,j)]_{i,j\in\mathcal{V}}$. Then, for every $\mathbf{c} = [c(j)]_{j\in\mathcal{V}}$ we obtain

$$\begin{aligned}
&\sum_{i_m\in\mathcal{V}_{N_0}} \big\|(\mathbf{A}\Psi_{i_m}^{N_0} - \Psi_{i_m}^{N_0}\mathbf{A})\mathbf{c}\big\|_2^2 = \sum_{i_m\in\mathcal{V}_{N_0}} \big\|(\mathbf{A}\Psi_{i_m}^{N_0} - \Psi_{i_m}^{N_0}\mathbf{A})\chi_{i_m}^{2N_0}\mathbf{c}\big\|_2^2\\
&\leq \sum_{i_m\in\mathcal{V}_{N_0}}\sum_{i\in\mathcal{V}}\bigg\{\sum_{\rho(i,j)\leq\tau} |a(i,j)|\,\chi_{B(i_m,2N_0)}(j)\,|c(j)|\\
&\qquad\qquad \times\left|\psi_0\left(\frac{\rho(i,i_m)}{N_0}\right) - \psi_0\left(\frac{\rho(j,i_m)}{N_0}\right)\right|\bigg\}^2\\
&\leq 4\sum_{i_m\in\mathcal{V}_{N_0}}\sum_{i\in\mathcal{V}}\bigg\{\sum_{\rho(i,j)\leq\tau}|a(i,j)|\,\frac{\rho(i,j)}{N_0}\,\chi_{B(i_m,2N_0)}(j)\,|c(j)|\bigg\}^2\\
&\leq 4\tau^2N_0^{-2}\|\mathbf{A}\|_{\mathcal{S}}^2\sum_{i_m\in\mathcal{V}_{N_0}}\|\chi_{i_m}^{2N_0}\mathbf{c}\|_2^2 \leq 4\tau^2N_0^{-2}\alpha_2^*\|\mathbf{A}\|_{\mathcal{S}}^2\,\|\mathbf{c}\|_2^2,
\end{aligned} \tag{4.10}$$

in which the second inequality follows from the Lipschitz property for the trapezoid function ψ_0 and the third one holds by the second inequality in (2.9). By combining

(4.9) and (4.10), we get

$$\begin{aligned}
\sqrt{\alpha_2^*}\, \|(z\mathbf{I}-\mathbf{A})\mathbf{c}\|_2 \ &\ge \ \Big(\sum_{i_m\in\mathcal{V}_{N_0}} \big\|\Psi_{i_m}^{N_0}(z\mathbf{I}-\mathbf{A})\mathbf{c}\big\|_2^2\Big)^{1/2}\\
&\ge \Big(\sum_{i_m\in\mathcal{V}_{N_0}} \big\|(z\mathbf{I}-\mathbf{A})\Psi_{i_m}^{N_0}\mathbf{c}\big\|_2^2\Big)^{1/2} - \Big(\sum_{i_m\in\mathcal{V}_{N_0}} \big\|(\mathbf{A}\Psi_{i_m}^{N_0}-\Psi_{i_m}^{N_0}\mathbf{A})\mathbf{c}\big\|_2^2\Big)^{1/2}\\
&\ge B_{N_0}\Big(\sum_{i_m\in\mathcal{V}_{N_0}} \|\Psi_{i_m}^{N_0}\mathbf{c}\|_2^2\Big)^{1/2} - 2\sqrt{\alpha_2^*}\,\tau\, N_0^{-1}\, \|\mathbf{A}\|_{\mathcal{S}}\|\mathbf{c}\|_2\\
&\ge \Big(B_{N_0}\sqrt{\alpha_1^*} - 2\sqrt{\alpha_2^*}\,\tau\, N_0^{-1}\|\mathbf{A}\|_{\mathcal{S}}\Big)\, \|\mathbf{c}\|_2.
\end{aligned}$$

This together with (4.1) proves (4.7). By applying similar arguments, we can establish the lower bound estimate in (4.8).

5. Symmetric Linear Systems

In this section, we consider exponential stability of the following linear system

$$\frac{d}{dt}\psi(t) = \mathbf{B}\psi(t) \tag{5.1}$$

with initial condition $\psi(0) \in \ell^2$, whose state matrix $\mathbf{B} \in \mathcal{B}^2$ is Hermitian. The class of first- and second-order consensus, e.g., platoon of vehicles with translation invariant communication (interconnection) topologies, networks with general nodal dynamics over undirected graphs are examples of networks with Hermitian state matrices [43].

In the following, we present several equivalent conditions for exponential stability of the symmetric linear systems (5.1), which take more simpler forms than those conditions in Theorem 2.9 [11, 12, 20, 52, 59].

Theorem 5.1. *The exponential stability of the linear system* (5.1) *with a Hermitian state matrix* $\mathbf{B}$ *in* $\mathcal{B}^2$ *is equivalent to each of the following statements:*

(i) $\mathbf{B}$ *is strictly negative definite.*

(ii) *There exists a positive constant* A_0 *such that*

$$\|(z\mathbf{I}-\mathbf{B})\mathbf{c}\|_2 \ge A_0\|\mathbf{c}\|_2 \ \text{ for all } z\in\mathbb{C}_+ \text{ and } \mathbf{c}\in\ell^2. \tag{5.2}$$

(iii) *There exists a positive constant* A_0 *such that*

$$\mathbf{c}^*\mathbf{B}\mathbf{c} \le 0 \ \text{ and } \ \|\mathbf{B}\mathbf{c}\|_2 \ge A_0\,\|\mathbf{c}\|_2 \ \text{ for all } \mathbf{c}\in\ell^2. \tag{5.3}$$

The equivalence between the statement (i) of Theorem 5.1 and the exponential stability of the linear system (5.1) is a restatement of the well-known Hille-Yosida Theorem in the self-adjoint operator setting, cf. the statement (i) of Theorem 2.9.

The statement (ii) of Theorem 5.1 is the reformulation of the statement (iii) of Theorem 2.9 in the self-adjoint operator setting. The equivalence between the statements (ii) and (iii) of Theorem 5.1 follows from the observation that

$$\begin{aligned}\inf_{z\in\mathbb{C}_+}\|(z\mathbf{I}-\mathbf{B})\mathbf{c}\|_2^2 &= \inf_{a\in\mathbb{R}_+}\inf_{b\in\mathbb{R}}\|(a\mathbf{I}-\mathbf{B})\mathbf{c}\|_2^2+b^2\|\mathbf{c}\|_2^2=\inf_{a\in\mathbb{R}_+}\|(a\mathbf{I}-\mathbf{B})\mathbf{c}\|_2^2\\ &= \inf_{a\in\mathbb{R}_+}a^2\|\mathbf{c}\|_2^2-2a\langle\mathbf{c},\mathbf{B}\mathbf{c}\rangle+\|\mathbf{B}\mathbf{c}\|_2^2=\|\mathbf{B}\mathbf{c}\|_2^2,\ \mathbf{c}\in\ell^2,\end{aligned}$$

for a negative definite Hermitian matrix $\mathbf{B}\in\mathcal{B}^2$. From the above argument, we see that the constant A_0 in the statements (ii) and (iii) can be chosen to be the same.

Example 5.2. When the linear system (1.1) is spatially invariant with a Hermitian Toeplitz state matrix $\mathbf{B}_0=\big[p(i-j)\big]_{i,j\in\mathbb{Z}}$, its Fourier symbol $\hat{p}(\xi)$ becomes real-valued and takes negative values. According to (3.11), stability threshold of the symmetric spatially invariant linear system is equal to $A_0=\inf_{\xi\in\mathbb{R}}-\hat{p}(\xi)$.

Building upon Theorem 5.1, we propose the following necessary conditions that can be verified by evaluating maximum or minimum eigenvalues of some localized matrices, cf. Theorems 3.2.

Theorem 5.3. *Let $\mathcal{G}=(\mathcal{V},\mathcal{E})$ be the coupling graph of a spatially distributed network. Suppose that the symmetric linear system* (5.1) *is exponentially stable with stability threshold A_0, whose state matrix $\mathbf{B}$ is a Hermitian matrix in $\mathcal{B}_\tau\cap\mathcal{B}^2$ for some $\tau\geq 0$. Then, the following localized inequalities*

$$\mathbf{c}^*\chi_i^N\mathbf{B}\chi_i^N\mathbf{c}\leq 0\ \text{ and }\ \mathbf{c}^*\chi_i^N\mathbf{B}^2\chi_i^N\mathbf{c}\geq A_0^2\,\|\chi_i^N\mathbf{c}\|_2^2 \tag{5.4}$$

hold for all $N\geq\tau$, $i\in\mathcal{V}$, and $\mathbf{c}\in\ell^2$.

Proof. We can use similar argument used in the proof of Theorem 3.2 and then we omit the detailed proof. □

Inequalities in (5.4) are localized version of global stability condition (5.3) in Theorem 5.1. For symmetric linear systems, sufficient conditions for the exponential stability take rather simple forms.

Theorem 5.4. *Suppose that all assumptions of Theorem 4.1 hold and state matrix $\mathbf{B}\in\mathcal{B}_\tau\cap\mathcal{B}^2$ is Hermitian for some $\tau\geq 0$. Then, the linear system* (5.1) *with state matrix $\mathbf{B}$ is exponentially stable if there exists a positive integer N_0 and a positive number B_{N_0} satisfying* (4.1) *such that*

$$\mathbf{c}^*\chi_{i_m}^{N_0}\mathbf{B}\chi_{i_m}^{N_0}\mathbf{c}\leq 0\ \text{ and }\ \mathbf{c}^*\chi_{i_m}^{N_0}\mathbf{B}^2\chi_{i_m}^{N_0}\mathbf{c}\geq B_{N_0}^2\,\|\chi_{i_m}^{N_0}\mathbf{c}\|_2^2 \tag{5.5}$$

hold for all $i_m\in\mathcal{V}_{N_0}$ and $\mathbf{c}\in\ell^2$.

Proof. For every $z\in\mathbb{C}_+$ and $\mathbf{c}\in\ell^2$, by Theorem 5.1, it suffices to prove the uniform stability for the family of matrices $z\mathbf{I}-\mathbf{B}$, i.e.,

$$\|(z\mathbf{I}-\mathbf{B})\mathbf{c}\|_2\geq\frac{B_{N_0}}{2}\sqrt{\frac{\alpha_1^*}{\alpha_2^*}}\,\|\mathbf{c}\|_2. \tag{5.6}$$

From sufficient conditions (5.5), it follows that

$$\begin{aligned}\|(z\mathbf{I}-\mathbf{B})\chi_{i_m}^{N_0}\mathbf{c}\|_2^2 &= |z|^2\,\|\chi_{i_m}^{N_0}\mathbf{c}\|_2^2 - 2\,\Re(z)\,\mathbf{c}^*\chi_{i_m}^{N_0}\mathbf{B}\chi_{i_m}^{N_0}\mathbf{c} \;+\; \|\mathbf{B}\chi_{i_m}^{N_0}\mathbf{c}\|^2\\ &\geq B_{N_0}^2\,\|\chi_{i_m}^{N_0}\mathbf{c}\|_2^2\end{aligned}$$

for all $\mathbf{c} \in \ell^2$ and $i_m \in \mathcal{V}_{N_0}$. Applying the above estimate and using similar argument used in the proof of Theorem 4.1, one concludes the inequality (5.6). $\square$

For a given Hermitian matrix $\mathbf{B} = [b(i,j)]_{i,j\in\mathcal{V}}$ in $\mathcal{B}_\tau \cap \mathcal{B}^2$, the sufficient conditions (5.5) in Theorem 5.4 are spatially localized in neighborhoods of each leading subsystem $i_m \in \mathcal{V}_{N_0}$, where each leading subsystem has to only have access to localized portions of state matrix $\mathbf{B}$ determined by truncation operator $\chi_{i_m}^{N_0}$. In particular, the requirement (5.5) holds if the largest eigenvalue of the spatially localized principal submatrix $[b(j,j')]_{j,j'\in B(i_m,N)}$ is non-positive for every subsystem $i_m \in \mathcal{V}_{N_0}$. For a Hermitian matrix $\mathbf{B}$, let us define

$$\tilde{B}_N(i) = \inf_{\|\chi_i^N\mathbf{c}\|_2=1} \|\mathbf{B}\chi_i^N\mathbf{c}\|_2 \tag{5.7}$$

in which $N \geq 1$ and $i \in \mathcal{V}$. The quantity $\tilde{B}_N(i)$ is equal to the square root of the smallest eigenvalue of the spatially localized matrix

$$\chi_i^N\mathbf{B}^2\chi_i^N = \left[\sum_{k\in B(j,\tau)\cap B(j',\tau)} b(j,k)b(k,j')\right]_{j,j'\in B(i,N)} \tag{5.8}$$

that can be evaluated in a decentralized/distributed manner. Then, the constant $\tilde{B}_{N_0}$ in (5.5) can be thought of as the uniform stability threshold for small-scale systems with state matrices $\chi_{i_m}^{N_0}\mathbf{B}^2\chi_{i_m}^{N_0}$ for all $i_m \in \mathcal{V}_{N_0}$.

Using a similar argument that leads to (3.9), one can verify that $\{\tilde{B}_N(i)\}_{N=\tau}^\infty$ is a decreasing sequence that converges to A_0 for every $i \in \mathcal{V}$, i.e.,

$$\lim_{N\to\infty} \tilde{B}_N(i) = A_0 \ \text{ for all } i \in \mathcal{V}. \tag{5.9}$$

Inequalities (4.6) and (5.6) imply that the global stability threshold of a symmetric linear dynamical network can be enhanced by improving the localized stability threshold via adjusting components of properly localized portions of the state matrix.

6. Design of Spatially Distributed Systems

In this section, we consider the problem of coupling weight adjustment between a given pair of subsystems in an exponentially stable symmetric linear dynamical systems (5.1).

The coupling weight between subsystems $k, l \in \mathcal{V}$ can be adjusted in a localized manner via the following class of feedback control laws

$$u(t) \;=\; w\,\mathbf{E}_{kl}\,\psi(t) \tag{6.1}$$

that modifies the dynamics of (5.1) as follows:

$$\frac{d}{dt}\psi(t) = (\mathbf{B} + w\mathbf{E}_{kl})\,\psi(t) \tag{6.2}$$

where w is a scalar feedback gain, $\mathbf{E}_{kl} = [e(i,j)]_{i,j\in\mathcal{V}}$, and

$$e(i,j) = \begin{cases} 1 & \text{if } (i,j) \in \{(k,l),(l,k)\} \\ 0 & \text{otherwise.} \end{cases}$$

The conclusion of Theorem 5.4 plays a critical role in computing an admissible range of values for the scalar w such that the resulting closed-loop network (6.2) remains exponentially stable with stability threshold equal or greater than the original network. From a network design perspective, when an existing coupling between subsystems k and l with coupling weight $b(k,l)$, as an element of state matrix $\mathbf{B} = [b(i,j)]_{i,j\in\mathcal{V}}$, is nonzero, the local weight adjustment law (6.1) will strengthen the existing coupling when $wb(k,l) > 0$ and weaken the existing coupling (and possibly zero it out) whenever $wb(k,l) < 0$.

To state the following main result of this section, we let $\mathcal{B}_\tau(M)$ denote the set of all band matrices $\mathbf{B} \in \mathcal{B}_\tau$ with bounded entries $\|\mathbf{B}\|_\infty < M$, where $\tau \in \mathbb{Z}_+$ and $M \in \mathbb{R}_+$.

Theorem 6.1. *Suppose that the coupling graph $\mathcal{G} = (\mathcal{V},\mathcal{E})$ of the linear control network* (5.1) *satisfies Assumption 2.1, its counting measure $\mu_{\mathcal{G}}$ enjoys the polynomial growth property* (2.6)*, and the state matrix $\mathbf{B}$ is a strictly negative definite matrix in $\mathcal{B}_\tau(M) \cap \mathcal{B}^2$. A positive integer N_0 exists such that*

$$\tilde{B}_{N_0} := \inf_{i_m\in\mathcal{V}_{N_0}} \tilde{B}_{N_0}(i_m) \geq 4M\sqrt{\frac{\alpha_2^*}{\alpha_1^*}}\, D_1(\mathcal{G})\,\tau(\tau+1)^d\, N_0^{-1} \tag{6.3}$$

in which $\tilde{B}_N(i)$ is defined by (5.7)*. For every pair of subsystems $k,l \in \mathcal{V}$ with $\rho(k,l) \leq \tau$, let us define the following quantities*

$$\eta_{kl} = \inf_{\substack{\rho(k,i_m)\leq N_0 \\ \rho(l,i_m)\leq N_0}} \; \inf_{\|\mathbf{P}_{i_m}^{N_0}\mathbf{c}\|_2=1} \Re\big((\mathbf{P}_{i_m}^{N_0}\,\mathbf{c})^*\,\mathbf{E}_{kl}\,\mathbf{B}\,\mathbf{P}_{i_m}^{N_0}\,\mathbf{c}\big) \tag{6.4}$$

and

$$\beta_{kl} = \sup_{\substack{\rho(k,i_m)\leq N_0 \\ \rho(l,i_m)\leq N_0}} \; \sup_{\|\mathbf{P}_{i_m}^{N_0}\mathbf{c}\|_2=1} \Re\big((\mathbf{P}_{i_m}^{N_0}\,\mathbf{c})^*\mathbf{E}_{kl}\mathbf{B}\mathbf{P}_{i_m}^{N_0}\,\mathbf{c}\big), \tag{6.5}$$

where $\mathbf{P}_{i_m}^{N_0}$ is the projection matrix onto the eigenspace of the localized matrix $\chi_{i_m}^{N_0}\mathbf{B}^2\chi_{i_m}^{N_0}$ corresponding to its smallest eigenvalue, which is equal to $\big(\tilde{B}_{N_0}(i_m)\big)^2$. Then, the following design rules hold:

(i) When $\eta_{kl} > 0$, there exists $\epsilon_0 > 0$ such that for all w in $(0,\epsilon_0)$ the resulting closed-loop network (6.2) *is exponentially stable with the state matrix $\mathbf{B} + w\mathbf{E}_{kl}$ that still belongs to $\mathcal{B}_\tau(M)$.*

(ii) When $\beta_{kl} < 0$, there exists $\epsilon_1 > 0$ such that for all w in $(-\epsilon_1, 0)$ the resulting closed-loop network (6.2) *is exponentially stable with the state matrix $\mathbf{B} + w\mathbf{E}_{kl}$ that still belongs to $\mathcal{B}_\tau(M)$.*

A detailed proof of the above theorem will be given at the end of this section. When the smallest eigenvalue is simple with normalized eigenvector $\mathbf{q}_{i_m}^{N_0}$, the project matrix is given by $\mathbf{P}_{i_m}^{N_0} = \mathbf{q}_{i_m}^{N_0}(\mathbf{q}_{i_m}^{N_0})^*$. In this case, the norm constraint $\|\mathbf{P}_{i_m}^{N_0}\mathbf{c}\|_2 = 1$ implies that $\mathbf{P}_{i_m}^{N_0}\mathbf{c} = \mathbf{q}_{i_m}^{N_0}$. This gives us the following closed-form solutions

$$\inf_{\|\mathbf{P}_{i_m}^{N_0}\mathbf{c}\|_2=1} \Re\big((\mathbf{P}_{i_m}^{N_0}\mathbf{c})^*\, \mathbf{E}_{kl}\mathbf{B}\, \mathbf{P}_{i_m}^{N_0}\mathbf{c}\big) = \Re\left((\mathbf{q}_{i_m}^{N_0})^*\, \mathbf{E}_{kl}\mathbf{B}\, \mathbf{q}_{i_m}^{N_0}\right)$$

and

$$\sup_{\|\mathbf{P}_{i_m}^{N_0}\mathbf{c}\|_2=1} \Re\big((\mathbf{P}_{i_m}^{N_0}\mathbf{c})^*\mathbf{E}_{kl}\mathbf{B}\mathbf{P}_{i_m}^{N_0}\mathbf{c}\big) = \Re\left((\mathbf{q}_{i_m}^{N_0})^*\, \mathbf{E}_{kl}\mathbf{B}\, \mathbf{q}_{i_m}^{N_0}\right)$$

that are useful in calculating quantities η_{kl} and β_{kl}. Computation of quantities η_{kl} and β_{kl} only involve those entries $b(i,j)$ of the state matrix $\mathbf{B}$ whose indices satisfy $i,j \in B(k, 2N_0+\tau)\cap B(l, 2N_0+\tau)$. Therefore, the requirements $\eta_{kl} > 0$ and $\beta_{kl} < 0$ in Theorem 6.1 can be verified by utilizing localized information about the state matrix $\mathbf{B}$ in neighborhoods of subsystems $k, l \in \mathcal{V}$.

The design parameter N_0 determines the size of neighborhoods required to compute quantities η_{kl} and β_{kl} using localized matrices. Suppose that N_0 and $\tilde{B}_{N_0}$ are chosen properly to satisfy the inequality (6.3). According to (2.9), (4.1), and (4.3), stability threshold of all networks with state matrices in $\mathcal{B}_\tau(M)$ is lower bounded by $\tilde{A}_{N_0} := \tilde{B}_{N_0}\sqrt{\alpha_1^*/(4\alpha_2^*)}$. Theorem 6.1 shows that state matrix of the resulting closed-loop networks still belong to $\mathcal{B}_\tau(M)$, which implies that their stability threshold is guaranteed to be greater than $\tilde{A}_{N_0}$.

In summary, the result of Theorem 6.1 asserts that: a properly chosen positive feedback gain w when $\eta_{kl} > 0$, and a properly chosen negative feedback gain w when $\beta_{kl} < 0$, will both result in exponentially stable closed-loop networks with guaranteed stability thresholds.

Remark 6.2. The conclusions of Theorem 6.1 will remain true if matrix $\mathbf{E}_{kl}$ is replaced by rotation matrix $\mathbf{R}_{kl}(\theta)$ whose (k,l)-th and (l,k)-th entries are $\sin\theta$, (k,k)-th entry is $\cos\theta$, and (l,l)-th entry is $-\cos\theta$ for some $0 \le \theta \le \pi$. Moreover, one can establish similar results when the matrix $\mathbf{E}_{kl}$ in Theorem 6.1 is replaced by $\mathbf{L}_{ij}$ where $\mathbf{L}_{ij} = \mathbf{e}_i\mathbf{e}_i^* + \mathbf{e}_j\mathbf{e}_j^* - \mathbf{E}_{ij}$ and $\mathbf{e}_i$'s are the standard basis for ℓ^2. This is particularly useful when the state matrix $\mathbf{B}$ is a graph Laplacian.

Proof of Theorem 6.1. According to our assumptions, the state matrix $\mathbf{B}$ is a strictly negative definite matrix in $\mathcal{B}_\tau(M)\cap\mathcal{B}^2$. By Theorem 5.4, it suffices to find proper positive or negative weight adjustment w such that

$$\mathbf{c}^*\,\chi_{i_m}^{N_0}\left(\mathbf{B} + w\mathbf{E}_{kl}\right)\chi_{i_m}^{N_0}\,\mathbf{c} \le 0 \tag{6.6}$$

and

$$\left\|(\mathbf{B} + w\mathbf{E}_{kl})\,\chi_{i_m}^{N_0}\,\mathbf{c}\right\|_2^2 \;\ge\; \left(\tilde{B}_{N_0}\right)^2 \left\|\chi_{i_m}^{N_0}\,\mathbf{c}\right\|_2^2 \tag{6.7}$$

hold for all $i_m \in \mathcal{V}_{N_0}$ and $\mathbf{c} \in \ell^2$, where $\tilde{B}_{N_0}$ is the constant in (5.5).

Since $\mathbf{B} \in \mathcal{B}_\tau(M)$ and $\rho(k,l) \le \tau$, we have $|b(k,l)| < M$. Hence, $\mathbf{B} + w\,\mathbf{E}_{kl} \in \mathcal{B}_\tau(M)$ if

$$|w| \le M - |b(k,l)|. \tag{6.8}$$

We observe that (4.1) holds according to (2.9), (6.3), and the assumption $\mathbf{B} \in \mathcal{B}_\tau(M)$. Therefore, $\mathbf{B}$ is strictly negative definite by Theorem 4.1. Moreover, it follows from (5.6), (6.3) and Theorem 5.1 that

$$\mathbf{c}^*\mathbf{B}\mathbf{c} \le -\frac{B_{N_0}}{2}\sqrt{\frac{\alpha_1^*}{\alpha_2^*}}\,\|\mathbf{c}\|_2^2 \le -2MD_1(\mathcal{G})\,\tau(\tau+1)^d\,N_0^{-1}\,\|\mathbf{c}\|_2^2 \tag{6.9}$$

for all $\mathbf{c} \in \ell^2$. Direct calculations reveal that $\left|\mathbf{c}^*\mathbf{E}_{kl}\mathbf{c}\right| \le \|\mathbf{c}\|_2^2$ for all $\mathbf{c} \in \ell^2$. This together with (6.9) implies that $\mathbf{B} + w\mathbf{E}_{kl}$ is negative definite matrices and, as a result, inequality (6.6) holds for all w satisfying

$$|w| < 2MD_1(\mathcal{G})\,\tau(\tau+1)^d\,N_0^{-1}. \tag{6.10}$$

Now, suppose that $i_m \in \mathcal{V}_{N_0}$ such that $k,l \in B(i_m, N_0)$, where k and l are indices of $\mathbf{E}_{kl}$. From the definition of the projection matrix $\mathbf{P}_{i_m}^{N_0}$, it follows that there exists $\tilde{C}_{N_0}(i_m) > B_{N_0}(i_m)$ such that

$$\|\mathbf{B}\chi_{i_m}^{N_0}\mathbf{c}\|_2^2 \;\ge\; \tilde{B}_{N_0}(i_m)^2\,\|\mathbf{P}_{i_m}\mathbf{c}\|_2^2 + \;\tilde{C}_{N_0}(i_m)^2\,\|\chi_{i_m}^{N_0}\mathbf{c} - \mathbf{P}_{i_m}\mathbf{c}\|_2^2 \tag{6.11}$$

for all $\mathbf{c} \in \ell^2$. In fact, the second smallest eigenvalue of the matrix $\chi_{i_m}^{N_0}\mathbf{B}^2\chi_{i_m}^{N_0}$, if it exists, can be employed as the constant $\tilde{C}_{N_0}(i_m)$ in (6.11). Let us choose $\mathbf{c}_1 = \mathbf{P}_{i_m}\mathbf{c}$ and $\mathbf{c}_2 = \chi_{i_m}^{N_0}\mathbf{c} - \mathbf{P}_{i_m}\mathbf{c}$ for $\mathbf{c} \in \ell^2$. Then,

$$\|\chi_{i_m}^{N_0}\mathbf{c}\|_2^2 \;=\; \|\mathbf{c}_1\|_2^2 \;+\; \|\mathbf{c}_2\|_2^2.$$

For any positive weight w, we obtain

$$\begin{aligned}
&\|(\mathbf{B} + w\mathbf{E}_{kl})\chi_{i_m}^{N_0}\mathbf{c}\|_2^2 \\
&= \mathbf{c}^*\chi_{i_m}^{N_0}\mathbf{B}^2\chi_{i_m}^{N_0}\mathbf{c} + 2w\,\Re\big(\mathbf{c}^*\chi_{i_m}^{N_0}\mathbf{E}_{kl}\mathbf{B}\chi_{i_m}^{N_0}\mathbf{c}\big) + w^2\|\mathbf{E}_{kl}\chi_{i_m}^{N_0}\mathbf{c}\|_2^2 \\
&\ge \mathbf{c}^*\chi_{i_m}^{N_0}\mathbf{B}^2\chi_{i_m}^{N_0}\mathbf{c} + 2w\,\Re\big(\mathbf{c}^*\chi_{i_m}^{N_0}\mathbf{E}_{kl}\mathbf{B}\chi_{i_m}^{N_0}\mathbf{c}\big) \\
&\ge \tilde{B}_{N_0}(i_m)^2\,\|\mathbf{c}_1\|_2^2 \;+\; \tilde{C}_{N_0}(i_m)^2\,\|\mathbf{c}_2\|_2^2 \;+\; 2w\,\Re\big(\mathbf{c}_1^*\mathbf{E}_{kl}\mathbf{B}\mathbf{c}_1\big) \\
&\quad - 2w\,\|\mathbf{E}_{kl}\mathbf{c}_2\|_2\|\mathbf{B}\chi_{i_m}^{N_0}\mathbf{c}\|_2 \;-\; 2w\,\|\mathbf{E}_{kl}\mathbf{c}_1\|_2\,\|\mathbf{B}\mathbf{c}_2\|_2 \\
&\ge \Big(\tilde{B}_{N_0}(i_m)^2 \;+\; 2w\eta_{kl}\Big)\,\|\mathbf{c}_1\|_2^2 + \;\Big(\tilde{C}_{N_0}(i_m)^2 \;-\; 2wMD_1(\mathcal{G})(\sigma+1)^d\Big)\,\|\mathbf{c}_2\|_2 \\
&\quad - 4wMD_1(\mathcal{G})(\sigma+1)^d\,\|\mathbf{c}_1\|_2\,\|\mathbf{c}_2\|_2 \\
&\ge \tilde{B}_{N_0}(i_m)^2\,\|\chi_{i_m}^{N_0}\mathbf{c}\|_2^2 \;+\; \Big(\tilde{C}_{N_0}(i_m)^2 \;-\; \tilde{B}_{N_0}(i_m)^2\Big)\,\|\mathbf{c}_2\|_2^2 \\
&\quad - 2wMD_1(\mathcal{G})(\sigma+1)^d\,\left(\frac{D_1(\mathcal{G})(\sigma+1)^d}{\eta_{ij}} + 1\right)\,\|\mathbf{c}_2\|_2^2,
\end{aligned}$$

in which the second inequality holds according to (6.11) and the third inequality follows from (2.9) and the observation

$$\|\mathbf{E}_{kl}\mathbf{c}\|_2 \le \|\mathbf{c}\|_2 \;\text{ for all } \mathbf{c} \in \ell^2.$$

Hence, the inequality (6.7) holds when

$$0 < w < \frac{\eta_{kl}\big(\tilde{C}_{N_0}(i_m)^2 \ - \ \tilde{B}_{N_0}(i_m)^2\big)}{2MD_1(\mathcal{G})(\sigma+1)^d\big(MD_1(\mathcal{G})(\sigma+1)^d+\eta_{kl}\big)}. \tag{6.12}$$

This together with (6.8) and (6.10) proves the first conclusion (6.6) with ϵ_0 given by

$$\begin{aligned}\epsilon_0 \ \ = \ \ \min\Big\{ & M-|b(k,l)|,\ 2MD_1(\mathcal{G})\sigma(\sigma+1)^d N_0^{-1}, \\ & \frac{\eta_{ij}\big(\tilde{C}_{N_0}(i_m)^2 \ - \ \tilde{B}_{N_0}(i_m)^2\big)}{2MD_1(\mathcal{G})(\sigma+1)^d\big(MD_1(\mathcal{G})(\sigma+1)^d+\eta_{ij}\big)}\Big\}.\end{aligned}$$

Similar to the arguments used in proof of the first conclusion (6.6), for negative weight w, one can obtain

$$\begin{aligned}\|(\mathbf{B}+w\mathbf{E}_{kl})\chi_{i_m}^{N_0}\mathbf{c}\|_2^2 \ge & \ \big(\tilde{B}_{N_0}(i_m)^2 \ + \ 2w\beta_{kl}\big)\,\|\mathbf{c}_1\|_2^2 \\ & +\big(\tilde{C}_{N_0}(i_m)^2+2wMD_1(\mathcal{G})(\sigma+1)^d\big)\,\|\mathbf{c}_2\|_2 \\ & +4wMD_1(\mathcal{G})(\sigma+1)^d\,\|\mathbf{c}_1\|_2\,\|\mathbf{c}_2\|_2 \\ \ge & \ \tilde{B}_{N_0}(i_m)^2\,\|\mathbf{c}_1\|_2^2 \ + \ \tilde{C}_{N_0}(i_m)^2\,\|\mathbf{c}_2\|_2^2 \\ & +2wMD_1(\mathcal{G})(\sigma+1)^d\left(1-\frac{D_1(\mathcal{G})(\sigma+1)^d}{\beta_{kl}}\right)\|\mathbf{c}_2\|_2^2.\end{aligned}$$

Therefore, the second conclusion (6.7) holds by letting

$$\begin{aligned}\epsilon_1 \ \ = \ \ \min\Big\{ & M-|b(k,l)|, 2MD_1(\mathcal{G})\sigma(\sigma+1)^d N_0^{-1}, \\ & \frac{-\beta_{ij}\big((\tilde{C}_{N_0}(i_m))^2-(\tilde{B}_{N_0}(i_m))^2\big)}{2MD_1(\mathcal{G})(\sigma+1)^d\big(MD_1(\mathcal{G})(\sigma+1)^d-\beta_{ij}\big)}\Big\}.\end{aligned}$$

7. NUMERICAL SIMULATIONS

In this section, we interpret and illustrate some of the key concepts that are developed in the previous sections.

7.1. **Finite-dimensional linear networks with randomly and uniformly generated spatial locations.** We consider a linear dynamical system consisting of 500 subsystems which are randomly and uniformly distributed over a square-shape region of size 100×100 square meter. Let us denote spatial location of subsystem $i\in\{1,2,\ldots,500\}$ by $x_i\in[0,100]\times[0,100]$. The coupling graph of this network is defined as follows: there is an undirected coupling link between subsystems i and j, i.e., $\{i,j\}\in\mathcal{E}$, only if the Euclidean distance between subsystems i and j, i.e., $\|x_i-x_j\|_2$, is less than or equal to 10 meter; otherwise, there is no coupling link between the two subsystems. The above coupling graph, denoted by $\mathcal{G}=(\mathcal{V},\mathcal{E})$, is known as a random geometric graph in the engineering community [29,44]. Figure 3 depicts a sample coupling graph obtained according to the above procedure, and a 2-covering of the underlying connected graph, where the graph has diameter 18,

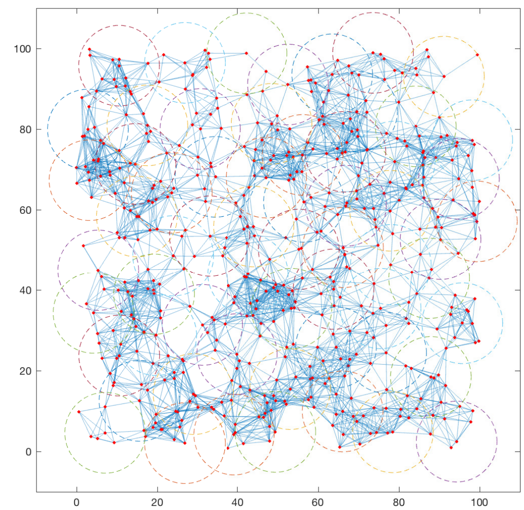

FIGURE 3. The red dots in the figure show spatial locations of 500 subsystems that are randomly and uniformly distributed in a square-shape region of size 100×100. The blue links represent the coupling graph among these subsystems as discussed in Section 7. The colored circles represent a 2-covering of the coupling graph.

Beurling dimension 2 and density 8.1389, and the number of leading subsystems is 62 (i.e., there are 62 localized regions).

We assume that state variable of each subsystem is scalar, i.e., $\psi_i \in \mathbb{R}$ for all $i \in \{1, 2, \dots, 500\}$. We utilize shortest path on coupling graph $\mathcal{G}$ as geodesic distance $\rho(i,j)$ to define state matrices of this class of linear networks. In our simulations, the bandwidth is set to $\tau = 1$. We construct the state matrix $\mathbf{A} = [a(i,j)] \in \mathbb{R}^{500\times 500}$ of our linear networks using the coupling graph of Figure 3,

$$a(i,j) = \begin{cases} -1 & \text{if } j = i \\ 0.05\mathrm{sgn}(\zeta)\, e^{-\alpha\|x_i - x_j\|^{\beta}} & \text{if } 0 < \rho(j,i) \leq \tau \\ 0 & \text{if } \rho(j,i) > \tau, \end{cases} \tag{7.1}$$

where decay parameters are $\alpha = 0.05$ and $\beta = 0.9$ [35, 41]. In order to show that our methodology works for a broad class of systems, sign of each entry $a(i,j)$ is chosen randomly using $\mathrm{sgn}(\zeta)$, where ζ is a Bernoulli random variable taking values from $\{+1, -1\}$ with probably $1/2$. After executing these steps, we adopted one sample matrix $\mathbf{A}$ for our simulation purposes. The resulting linear dynamical network is time-invariant, whose dynamics is governed by (1.1) with state vector $\psi \in \mathbb{R}^{500}$. The value of the Schur norm (2.8) of the state matrix is $\|\mathbf{A}\|_{\mathcal{S}} = 2.0057$ and all eigenvalues of $\mathbf{A}$ are located in the region $\{z \in \mathbb{C} \mid \Re z \leq -\delta\}$ on the left-hand-side of the imaginary axis, where δ in Theorem 2.9 is equal to 0.7702.

According to Definition 2.5, the number of leading agents in a N_0-covering of the network decreases as N_0 increases. We applied the algorithm described in Definition 2.5 in order to find N_0-coverings. This algorithm may not be optimal, but it works well for both finite and infinite graphs, see Table 1 for a sample execution of this algorithm on $\mathcal{G}$. We note that the number $\#\mathcal{V}_{N_0}$ of leading subsystems is

TABLE 1. Leading subsystems in a covering

N_0	1	3	5	7	9	11	13	15	17	18
$\#\mathcal{V}_{N_0}$	500	55	23	14	9	7	6	3	3	1

equal to the network size when $N_0 = 1$, and it is equal to 1 when N_0 is equal to the graph diameter.

The parameter B_{N_0} in the inequality (4.2) can be interpreted as an estimate for the global stability threshold A_0; cf. (3.2). The quantity in the right hand side of the inequality (4.1), which is a lower bound for the localized stability threshold B_{N_0}, decays as radius of N_0-covering increases, see Table 2.

TABLE 2. Local stability threshold

N_0	1	3	5	7	9	11	13	15	17	18
B_{N_0}	66.6	16.7	7.7	4.3	2.7	1.9	1.5	0.92	0.81	0.45

We observe that for $N_0 = 18$, the lower bound for B_{N_0} is around 0.45. This is compatible with the estimate in (4.6) as 0.45 is a lower bound A_0 in (2.11) for the global stability threshold. We highlight that for the state matrix $\mathbf{A}$ defined by (7.1), the quantities δ and A_0 in Theorem 2.9 satisfy $0.77 \approx \delta > A_0 \approx 0.45$.

7.2. **Infinite-dimensional linear networks with uniformly random couplings.** Let us consider the following class of infinite-dimensional spatially distributed systems

$$\frac{d}{dt}\psi(t) = (\mathbf{A}_0 + \mathbf{E}_{\epsilon,\tau})\psi(t) \tag{7.2}$$

that consist of subsystems with state variables ψ_i with spatial indices $i \in \mathbb{Z}$. The spatially invariant matrix $\mathbf{A}_0 = [p(i-j)]_{i,j\in\mathbb{Z}}$ is the Toeplitz state matrix in Example 3.3 in which $p(k), k \in \mathbb{Z}$ is a symmetric real-valued sequence satisfying $p(-k) = p(k)$ for all $k \in \mathbb{Z}$, $p(k) = 0$ for all $k \notin [-\tau, \tau]$, and $P_0 := -\sup_{\xi\in\mathbb{R}} \Re\hat{p}(\xi) > 0$, where $\hat{p}(\xi) = \sum_{j\in\mathbb{Z}} p(j)\exp(-2\pi\sqrt{-1}\xi)$. For a given parameter $\epsilon > 0$, the second term in the state matrix $\mathbf{E}_{\epsilon,\tau} = [e_{\epsilon,\tau}(i,j))]_{i,j\in\mathbb{Z}}$ is a symmetric matrix whose elements are independent random variables drawn from uniform distribution $U(-\epsilon,\epsilon)$ if $i \le j \le i+\tau$ and zeros if $j \ge i+\tau$ for every $i \in \mathbb{Z}$. One may verify that

$$\|\mathbf{A}_0 + \mathbf{E}_{\epsilon,\tau}\|_{\mathcal{S}} \le \|\mathbf{A}_0\|_{\mathcal{S}} + \|\mathbf{E}_{\epsilon,\tau}\|_{\mathcal{S}} \le \sum_{|k|\le\tau} |p(k)| + (2\tau+1)\epsilon \tag{7.3}$$

and

$$\|\mathbf{E}_{\epsilon,\tau}\mathbf{c}\|_2 \le \|\mathbf{E}_{\epsilon,\tau}\|_{\mathcal{S}}\|\mathbf{c}\|_2 \le (2\tau+1)\epsilon\|\mathbf{c}\|_2 \ \text{ for all } c \in \ell^2. \tag{7.4}$$

From (3.11) and (7.4), we obtain that the stability threshold (2.12) satisfies

$$A_0 \geq P_0 - (2\tau + 1)\epsilon. \tag{7.5}$$

Hence, the infinite-dimensional system (7.2) is exponentially stability if

$$\epsilon < \frac{P_0}{2\tau + 1}.$$

The underlying graph of linear dynamical network (7.2) is the circulant graph in Example 2.4 generated by $G = \{\pm 1\}$, which has Beurling dimension 1 and density 2. Moreover, $\mathcal{V}_{N_0} = N_0\mathbb{Z}$ can be selected as the set of leading subsystems for every given $N_0 \geq 1$ and the corresponding lower and upper covering numbers α_1^* and α_2^* in (4.4) become $\alpha_1^* = 1$ and $\alpha_2^* = 5$, respectively. According to Theorem 5.4, the exponential stability of (7.2) can be verified in a distributed manner by finding N_0 such that

$$\mathbf{c}^* \chi_{N_0 k}^{N_0} (\mathbf{A}_0 + \mathbf{E}_{\epsilon,\tau}) \chi_{N_0 k}^{N_0} \mathbf{c} \leq 0 \tag{7.6}$$

and

(7.7)

$$\mathbf{c}^* \chi_{N_0 k}^{N_0} (\mathbf{A}_0 + \mathbf{E}_{\epsilon,\tau})^2 \chi_{N_0 k}^{N_0} \mathbf{c} \geq \left(4\sqrt{5}\tau \Big(\sum_{|k|\leq\tau} |p(k)| + (2\tau+1)\epsilon \Big) N_0^{-1} \right)^2 \|\chi_{N_0 k}^{N_0} \mathbf{c}\|_2^2$$

for all $k \in \mathbb{Z}$ and $\mathbf{c} \in \ell^2$. One observes that matrices $\chi_{N_0 k}^{N_0}(\mathbf{A}_0 + \mathbf{E}_{\epsilon,\tau})\chi_{N_0 k}^{N_0}$ and $\chi_{N_0 k}^{N_0}(\mathbf{A}_0 + \mathbf{E}_{\epsilon,\tau})^2\chi_{N_0 k}^{N_0}$ for all $k \in \mathbb{Z}$ have the following structures

$$\mathbf{A}_{\epsilon,\tau,N_0} := \big[p(i-j) + e_{\epsilon,\tau}(i,j) \big]_{1 \leq i,j \leq 2N_0+1}$$

and

$$\mathbf{H}_{\epsilon,\tau,N_0} := \Big[\sum_{|k-i|\leq\tau} (p(i-k) + e_{\epsilon,\tau}(i,k))(p(k-j) + e_{\epsilon,\tau}(k,j)) \Big]_{1 \leq i,j \leq 2N_0+1},$$

respectively. Let us denote the smallest and largest eigenvalues of a symmetric matrix $\mathbf{A}$ by $\lambda_{\min}(\mathbf{A})$ and $\lambda_{\max}(\mathbf{A})$, respectively. Thus, the requirements (7.6) and (7.7) are satisfied if and only if the following two statements that

$$\lambda_{\max}(\mathbf{A}_{\epsilon,\tau,N_0}) \leq 0 \tag{7.8}$$

and

$$\lambda_{\min}(\mathbf{H}_{\epsilon,\tau,N_0}) \geq 80\tau^2 \Big(\sum_{|k|\leq\tau} |p(k)| + (2\tau+1)\epsilon \Big)^2 N_0^{-2} \tag{7.9}$$

hold for all $e_{\epsilon,\tau}(i,j)$ such that $e_{\epsilon,\tau}(i,j) = e_{\epsilon,\tau}(j,i)$ for all $1 \leq i,j \leq 2N_0 + \tau$. We recall that $e_{\epsilon,\tau}(i,j)$ for all $1 \leq i \leq j \leq i + \tau$ are independent random variables with uniform distribution $U(-\epsilon,\epsilon)$ and $e_{\epsilon,\tau}(i,j) = 0$ for all $i + \tau < j \leq i$.

In the next step, we examine exponential stability of the infinite-dimensional linear systems (7.2) with an $\mathbf{A}_0$ whose symbol is given by $\hat{p}(\xi) = -2 + \cos \pi\xi$. We provide numerical simulations to verify (7.8) and (7.9). For this system, we have that $\tau = 1$ and $P_0 = 1$. Hence, one can conclude from (7.5) that the system is exponential stability if $0 \leq \epsilon < P_0/(2\tau + 1) = 1/3$. We remark that the above conclusion about exponential stability can not be extended to linear system (7.2) with $\epsilon = 1/3$. For instance, system (7.2) with state matrix symbol $\hat{p}(\xi) = -2 + \cos\pi\xi + (\exp(-2\pi\sqrt{-1}\xi) + 1 + \exp(2\pi\sqrt{-1}\xi))/3 = 5(-1 + \cos\pi\xi)/3$ is not exponentially stable by Theorem 2.9 and the fact that the spectrum of the state matrix is the interval $[-10/3, 0] = \{\hat{p}(\xi) \mid \xi \in \mathbb{R}\}$, which contains the origin. Let us denote the matrices involved in (7.6) and (7.7) by $\mathbf{A}_{\epsilon,N_0} = \mathbf{A}_0 + \mathbf{E}_{\epsilon,\tau}$ and $\mathbf{H}_{\epsilon,N_0} = (\mathbf{A}_0 + \mathbf{E}_{\epsilon,\tau})^2$. Our numerical simulations indicate that the requirement (7.8) is satisfied for state matrices $\mathbf{A}_{\epsilon,N_0}$ with $\epsilon = i/60$, $0 \le i \le 18$, and $1 \le N_0 \le 40$ over 1000 independent trials. The left hand side of Figure 4 depicts behavior of the minimum of the smallest eigenvalue of matrices $\mathbf{H}_{\epsilon,N_0}$ over 1000 independent trials, where the z-coordinate $z(\epsilon, N_0)$ is the minimum of the following quantity over 1000 trials

(7.10)
$$\min\left(\frac{\lambda_{\min}(\mathbf{H}_{\epsilon,N_0})N_0^2}{80\tau^2\big(\sum_{|k|\le\tau}|p(k)| + (2\tau+1)\epsilon\big)^2}, 1\right) = \min\left(\frac{\lambda_{\min}(\mathbf{H}_{\epsilon,N_0})N_0^2}{80(3+3\epsilon)^2}, 1\right),$$

where the last equality holds as $\tau = 1$ and $\sum_{|k|\le\tau}|p(k)| = 3$ in our simulations. Suppose that $N_0(\epsilon)$ is the minimum integer such that $z(\epsilon, N_0) = 1$ for all $N_0 \ge N_0(\epsilon)$. Therefore, the numerical verification of the requirement (7.8), which is equivalent to the exponential stability of the infinite-dimensional systems (7.2), reduces to finding integers $N_0(\epsilon)$. This is illustrated in the right hand side of Figure 4. This simulation demonstrates that our theoretical results can be applied to verify exponential stability of infinite-dimensional linear systems using a series of finite-dimensional conditions.

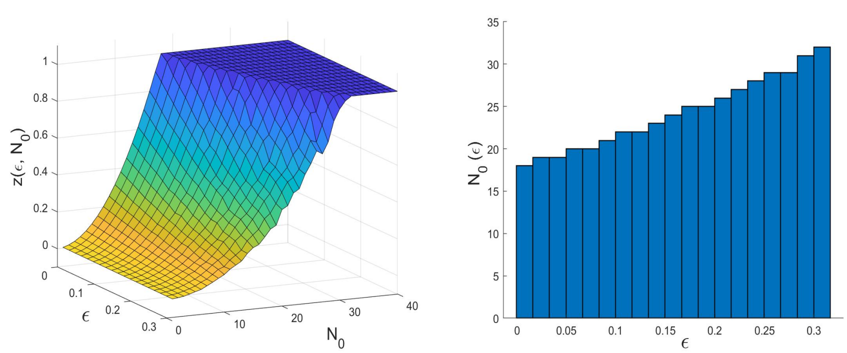

FIGURE 4. The plot on the left shows the behavior of the minimum of the smallest eigenvalue $\lambda_{\min}(\mathbf{H}_{\epsilon,N_0})$ over 1000 independent trials, where the x-axis is $\epsilon = i/60$ for all $0 \le i \le 18$, the y-axis is $1 \le N_0 \le 40$, and the z-axis is $z(\epsilon, N_0)$. The plot on the right depicts the minimal integers $N_0(\epsilon)$ such that $z(\epsilon, N_0) = 1$ for all $N_0 \ge N_0(\epsilon)$, $\epsilon = i/60$, and $0 \le i \le 18$.

8. FINAL REMARKS

This work proposes a decentralized framework to verify exponential stability of linear dynamical network that are defined over spatial proximity connectivity graphs. Several necessary and sufficient conditions have been formulated that allow us to re-examine exponential stability of a given finite- and infinite-dimensional linear system using spatially localized certificates. There are several related problems and areas that can benefit from our proposed methodology.

Stability of Spatially Distributed Systems on ℓ^∞: In real-world applications, the dynamics of each subsystem depends on the state variables, control inputs, and exogenous disturbance inputs of its neighboring subsystems. Thus, it is reasonable to consider networks with governing dynamics

$$\frac{d}{dt}\psi(t) = \mathbf{A}\psi(t) + \boldsymbol{\xi}(t), \tag{8.1}$$

with initial condition $\psi(0) \in \ell^\infty$, that is driven by a time-dependent bounded control or exogenous noise $\boldsymbol{\xi}(t) = [\xi_i(t)]_{i\in\mathcal{V}}$. Suppose that the control system is exponentially stable on ℓ^∞, i.e., there exist strictly positive constants C and α such that

$$\|e^{\mathbf{A}t}\psi(0)\|_\infty \le Ce^{-\alpha t}\|\psi(0)\|_\infty,\ t \ge 0, \tag{8.2}$$

then we have

$$\begin{aligned}
\|\psi(t)\|_\infty &\le \left\| \int_0^t e^{\mathbf{A}(t-s)}\boldsymbol{\xi}(s)ds + e^{\mathbf{A}t}\psi(0) \right\|_\infty \\
&\le Ce^{-\alpha t}\|\psi(0)\|_\infty + C\int_0^t e^{-\alpha(t-s)}\|\boldsymbol{\xi}(s)\|_\infty ds \\
&\le Ce^{-\alpha t}\|\psi(0)\|_\infty + C\alpha^{-1}\sup_{0\le t\le s}\|\boldsymbol{\xi}(s)\|_\infty.
\end{aligned}$$

This implies that control system (8.1) with bounded input has bounded state $\psi(t)$ for all $t \ge 0$. It is proven in [41] that if the linear system (8.1) is exponentially stable on ℓ^2, i.e., the inequality (1.2) holds, then it is also exponential stable on ℓ^∞, i.e., the inequality (8.2) holds. When the state matrix $\mathbf{A}$ belongs to $\mathcal{B}_\tau\cap\mathcal{B}^2$, we have that the constant C in (8.2) will depend only on the constants E, α given in (1.2), Beurling dimension d, Beurling density $D_1(\mathcal{G})$ and doubling constant $D_0(\mathcal{G})$ of the graph $\mathcal{G}$, bandwidth τ, and the value of $\|\mathbf{A}\|_\infty$. This suggests that our proposed methodology in this paper can be applied to the control system (8.1) driven by input (which can be a feedback control law or exogenous noise) to infer global stability in a decentralized manner.

Stability of Spatially Distributed Nonlinear Systems: Our methodology can be extended to verify local stability of equilibria of spatially distributed systems with nonlinear dynamics. Let us consider a nonlinear system of the form

$$\frac{d}{dt}\psi = F(\psi),\ \ t \ge 0, \tag{8.3}$$

where $\psi(t) = [\psi_i(t)]_{i\in\mathcal{V}} \in \ell^2$ and $F : \ell^2 \to \ell^2$ satisfies $F(0) = 0$. Following Definition 2.7, we say that the nonlinear system (8.3) is τ-banded over the coupling graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ if $F(\psi) = [F_i(\Psi_i)]_{i\in\mathcal{V}}$ for some continuously differentiable functions F_i on $\mathbb{C}^q$, where $q = \#(B(i,\tau)\cap\mathcal{V})$ and $\Psi_i = [\psi_j]_{j\in B(i,\tau)\cap\mathcal{V}}$ for all $i \in \mathcal{V}$. Let us define the gradient $\mathbf{A}(\psi^*) = \nabla_\psi F(\psi^*)$ (with respect to ψ at working point ψ^*) of the nonlinear system (8.3) with (i,j)-th entry $a(i,j)$ given by

$$a(i,j) = \begin{cases} \frac{\partial F_i}{\partial \psi_j}(\psi^*) & \text{if } \rho(i,j) \le \tau \\ 0 & \text{if } \rho(i,j) > \tau, \end{cases}$$

where ρ is a geodesic distance on $\mathcal{G}$. By assuming that ψ^* is a hyperbolic equilibrium and the initial state $\psi(0)$ is **close** to ψ^*, our proposed localized stability certificates can be applied to the nonlinear dynamic system (8.3) with the linearized state matrix $\mathbf{A}(\psi^*)$ to infer stability in a spatially localized manner. The scope of this research problem is beyond this paper.

Acknowledgement The authors would like to thank the editor and reviewers for their valuable comments and suggestions for the improvement of this manuscript.

References

[1] I. F. Akyildiz, W. Su, Y. Sankarasubramaniam and E. Cayirci, "Wireless sensor networks: a survey," *Comput. Netw.*, Vol. 38, pp. 393–422, 2002.

[2] A. Aldroubi and K. Gröchenig, "Nonuniform sampling and reconstruction in shift-invariant spaces," *SIAM Review*, Vol. 43, pp. 585–620, 2001.

[3] M. S. Andersen, S. K. Pakazad, A. Hansson, and A. Rantzer, "Robust stability analysis of sparsely interconnected uncertain systems," *IEEE Trans. Autom. Control*, Vol. 59, no. 8, pp. 2151-2156, 2014.

[4] J. Anderson, J.C. Doyle, S. H. Low, and N. Matni, "System level synthesis," *Annual Reviews in Control*, Vol. 47, pp.364–393, 2019.

[5] C. D. Athalye, D. Pal, and H. K. Pillai, "ℓ_∞-stability analysis of discrete autonomous systems described by Laurent polynomial matrix operators," *Systems & Control Letters*, Vol. 93, pp.13-20, 2016.

[6] C. D. Athalye, D. Pal and H. K. Pillai, "Comparison Between Different Notions of Stability for Laurent Systems," *IEEE Trans. Autom. Control*, Vol. 66, pp. 768-772, Feb. 2021.

[7] C. D. Athalye, D. Pal, and H. K. Pillai, "Behavior of n Infinite chains of kinematic points with the immediate-neighbors interaction dynamics," *IEEE Trans. Autom. Control*, Vol. 65, pp. 2929-2940, 2019.

[8] B. Bamieh, M. R. Jovanovic, P. Mitra, and S. Patterson, "Coherence in large-scale networks: Dimension-dependent limitations of local feedback," *IEEE Trans. Autom. Control*, Vol. 57, pp. 2235-2249, 2012.

[9] B. Bamieh, F. Paganini, and M. A. Dahleh. "Distributed control of spatially-invariant systems," *IEEE Trans. Automatic Control*, Vol. 47, pp. 1091–1107, 2002.

[10] L. Bartholdi and A. Erschler, "Growth of permutational extensions," *Invent. Math.*, Vol. 189, pp. 431–455, 2012.

[11] A. Bensoussan, G. Da Prato, M. C. Delfour, and S. K. Mitter, *Representation and Control of Infinite Dimensional Systems,* Springer Science & Business Media, 2007.

[12] D. S. Bernstein, *Matrix Mathematics Theory, Facts, and Formulas*, Second Edition, Princeton University Press, 2009.

[13] J. W. Bunce, "Stabilizability of linear systems defined over C^*-algebras," *Math. Systems Theory*, Vol. 18, pp. 237–250, 1985.

[14] C. Cheng, Y. Jiang, and Q. Sun, “Spatially distributed sampling and reconstruction”, *Appl. Comput. Harmon. Anal.*, Vol. 47, pp. 109-148, 2019.
[15] C. Chong and S. Kumar, “Sensor networks: evolution, opportunities, and challenges,” *Proc. IEEE*, Vol. 91, pp. 1247–1256, 2003.
[16] O. Christensen, *An Introduction to Frames and Riesz Bases*, Applied and Numerical Harmonic Analysis, Birkhauser, 2003.
[17] F. R. K. Chung, *Spectral Graph Theory*, American Mathematical Society, 1997.
[18] R. Curtain, O. V. Iftime, and H. Zwart, “System theoretic properties of a class of spatially invariant systems,” *Automatica*, Vol. 45, pp. 1619-1627, 2009.
[19] R. Curtain, O. V. Iftime, and H. Zwart, “A comparison between LQR control for a long string of SISO systems and LQR control of the infinite spatially invariant version,” *Automatica*, Vol. 46, pp. 1604-1615, 2010.
[20] R. F. Curtain and H. J. Zwart, *An Introduction to Infinite-Dimensional Linear Systems Theory*, Springer, 1995.
[21] R. D’Andrea and G. E. Dullerud, “Distributed control design for spatially interconnected systems,” *IEEE Trans. Autom. Control*, Vol. 48, pp. 1478–1495, 2003.
[22] D. Deng and Y. Han, *Harmonic Analysis on Spaces of Homogeneous Type*, Springer, 2008.
[23] F. Deroo, M. Meinel, M. Ulbrich, and S. Hirche, “Distributed stability tests for large-scale systems with limited model information,” *IEEE Trans. Control Netw. Syst.*, Vol. 2, pp. 298–309, 2015.
[24] G. E. Dullerud and R. D’Andrea, “Distributed control of heterogeneous systems,” *IEEE Trans. Autom. Control*, Vol. 49, pp. 2113–2128, 2004.
[25] Q. Fang, C. E. Shin, and Q. Sun, “Polynomial control on weighted stability bounds and inversion norms of localized matrices on simple graphs,” *J. Fourier Anal. Appl.*, accepted, 2021.
[26] A. Feintuch and B. Francis, “Infinite chains of kinematic points,” *Automatica*, Vol. 48, pp. 901-908, 2012.
[27] P. A. Fuhrmann and U. Helmke. *The mathematics of networks of linear systems*, New York: Springer, Vol. 150, 2015.
[28] K. Gröchenig, J. L. Romero, D. Rottensteiner and J. T. van Velthoven, “Balian-Low type theorems on homogeneous groups,” *Anal. Math.*, Vol. 46, pp. 483–515, 2020.
[29] J. Jiang, C. Cheng, and Q. Sun, Nonsubsampled graph filter banks: Theory and distributed algorithms, *IEEE Trans. Signal Process.*, Vol. 67, pp. 3938–3953, 2019.
[30] M. R. Jovanovic and B. Bamieh, “On the ill-posedness of certain vehicular platoon control problems,” *IEEE Transactions on Automatic Control*, Vol. 50, pp. 1307-1321, 2005.
[31] D. Kincaid and W. Cheney, *Numerical Analysis: Mathematics of Scientific Computing,* Brooks/Cole, Third edition, 2002.
[32] C. Langbort, R. S. Chandra, and R. D’Andrea, “Distributed control design for systems interconnected over an arbitrary graph,” *IEEE Trans. Autom. Cont.*, Vol. 49, pp. 1502–1519, 2004.
[33] I. Lestas and G. Vinnicombe, “Scalable decentralized robust stability certificates for networks of interconnected heterogeneous dynamical systems,” *IEEE Trans. Autom. Control*, Vol. 51, pp. 1613–1625, 2006.
[34] V. Losert, On the structure of groups with polynomial growth II, *J. London Math. Soc.*, Vol. 63, pp. 640–654, 2001.
[35] N. Motee and A. Jadbabaie, “Optimal control of spatially distributed systems,” *IEEE Trans. Autom. Control*, Vol. 53, pp. 1616–1629, 2008.
[36] N. Motee and A. Jadbabaie, “Distributed multi-parametric quadratic programming,” *IEEE Trans. Autom. Control*, Vol. 54, pp. 2279–2289, 2009.
[37] N. Motee and A. Jadbabaie, “Approximation methods and spatial interpolation in distributed control systems,” in *Proc. American Control Conference,* pp. 860–865, 2009.
[38] N. Motee and Q. Sun, “Measuring sparsity in spatially interconnected systems,” in *Proc. IEEE 52nd Conference Decision and Control*, pp. 1520–1525, 2013.
[39] N. Motee and Q. Sun, “Sparsity measures for spatially decaying systems,” in *Proc. American Control Conference*, pp. 5459–5464, 2014.

[40] N. Motee and Q. Sun, "Localized stability certificates for spatially distributed systems with sparse graph topologies," In *Proceedings of the 55th IEEE Conference on Decision and Control*, Las Vegas, NV, pp. 1388–1393, 2016.
[41] N. Motee and Q. Sun, "Sparsity and spatial localization measures for spatially distributed systems," *SIAM J. Control Optim.*, Vol. 55, pp. 200–235, 2017.
[42] N. Motee and Q. Sun, "Localized stability analysis and design of symmetric spatially distributed systems over sparse proximity graphs," In *The 23rd International Symposium on Mathematical Theory of Networks and Systems,* Hong Kong, pp. 185–188, 2018.
[43] H. K. Mousavi and N. Motee, "Explicit characterization of performance of a class of networked linear control systems," *IEEE Transactions on Control of Network Systems*, Vol. 7, p. 1688–1699, 2020.
[44] P. Nathanael, J. Paratte, D. Shuman, L. Martin, V. Kalofolias, P. Vandergheynst, and D. K. Hammond, GSPBOX: A toolbox for signal processing on graphs, *arXiv:1408.5781*, 2014.
[45] R. Pates and G. Vinnicombe, "Scalable design of heterogeneous networks," *IEEE Trans. Autom. Control*, Vol. 62, pp. 2318–2333, 2017.
[46] L. Paunonen and D. Seifert, "Asymptotic behaviour in the robot rendezvous problem," *Automatica*, 79, pp. 127-130, 2017.
[47] L. Paunonen and D. Seifert, "Asymptotics for infinite systems of differential equations," *SIAM Journal on Control and Optimization*, Vol. 55, pp. 1153-1178, 2017.
[48] J. Ploeg, N. Van De Wouw, and H. Nijmeijer, "L_p string stability of cascaded systems: application to vehicle platooning," *IEEE Trans. Contr. Syst. Tech.*, Vol. 22, pp. 786–793, 2014.
[49] M. Rotkowitz and S. Lall, "A characterization of convex problems in decentralized control," *IEEE Trans. Autom. Control*, Vol. 51, pp. 274–286, 2006.
[50] C. E. Shin and Q. Sun, "Stability of localized operators," *J. Funct. Anal.*, Vol. 256, pp. 2417–2439, 2009.
[51] C. E. Shin and Q. Sun, "Polynomial control on stability, inversion and powers of matrices on simple graphs," *J. Funct. Anal.*, Vol. 276, pp. 148–182, 2019.
[52] M. Slemrod, "Asymptotic behavior of C_0 semi-groups as determined by the spectrum of the generator," *Indiana Univ. Math. J.*, Vol. 25, pp. 783–792, 1976
[53] Q. Sun, "Nonuniform average sampling and reconstruction of signals with finite rate of innovation," *SIAM J. Math. Anal.*, Vol. 38, pp. 1389–1422, 2006.
[54] Q. Sun, "Stability criterion for convolution-dominated infinite matrices," *Proc. Amer. Math. Soc.*, Vol. 138, pp. 3933–3943, 2010.
[55] Q. Sun and J. Xian, ' Rate of innovation for (non-)periodic signals and optimal lower stability bound for filtering", *J. Fourier Anal. Appl.*, Vol. 20, pp. 119–134, 2014.
[56] M. Unser, "Sampling–50 years after Shannon," *Proc. IEEE*, Vol. 88 (4), pp. 569–587, 2000.
[57] Y. S. Wang, N. Matni and J. C. Doyle, "Separable and localized system level synthesis for large-scale systems," *IEEE Trans. Autom. Control*. Vol. 63, pp. 4234–4249, 2018.
[58] Y. S. Wang, N. Matni, and J. C. Doyle, "A system-level approach to controller synthesis," *IEEE Transactions on Automatic Control*, Vol. 64, pp.4079–4093, 2019.
[59] G. Weiss, "The resolvent growth assumption for semigroup on Hilbert space," *J. Math. Anal. Appl.*, Vol. 145, pp. 154–170, 1990.

MOTEE: DEPARTMENT OF MECHANICAL ENGINEERING AND MECHANICS, PACKARD LABORATORY, LEHIGH UNIVERSITY, BETHLEHEM, PA 18015. EMAIL: motee@lehigh.edu

SUN: DEPARTMENT OF MATHEMATICS, UNIVERSITY OF CENTRAL FLORIDA, ORLANDO, FL 32816. EMAIL: qiyu.sun@ucf.edu