Random sampling and reconstruction of concentrated signals in a reproducing kernel space

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Abstract

In this talk, I will discuss random sampling of signals concentrated on a bounded Corkscrew domain Ω of a metric measure space, and reconstructing concentrated signals approximately from their (un)corrupted sampling data taken on a sampling set contained in the domain Ω. This talk is based on a joint paper with Yaxu Li and Jun Xian, ACHA 2021.

Outline

1. Introduction: Sampling and reconstruction
2. Concentrated signals in reproducing kernel spaces
3. Random sampling and stability
4. Finite algorithm to approximate concentrated signals from their random samples
5. Finite algorithm to approximate concentrated signals from their corrupted random samples
6. Numerical simulations

- **Signals** (Analog signals, spatial signals, discrete signals), data on networks, *functions on a domain* Ω

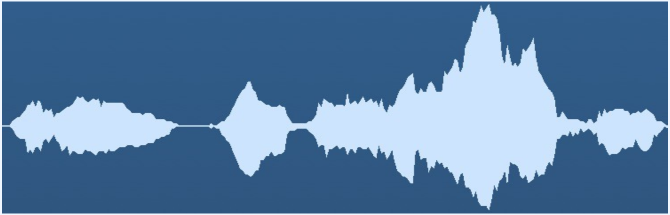


Figure: A Capture Of A Digital Audio Signal In The DAW Logic Pro X[1]

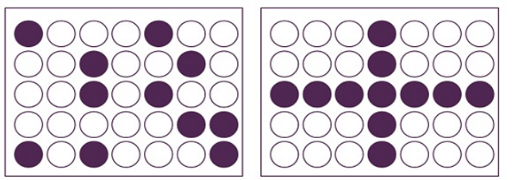
- Signals in real life have certain properties (bandlimited, regularity, finite rate of innovation, sparsity) or reside in some linear spaces (PW, SIS, RKS)
- *Signals in RKS with energy mainly concentrated on a domain*

[1]https://mynewmicrophone.com/microphone-audio-signal/

- **Sampling**: from signals to sampling data

$$f \longmapsto \left(\psi(f)\right)_{\psi\in\Psi},$$

where Ψ (for instance $\{\delta(x_i), x_i \in X\}$) is a family of (non)linear sampling functionals.
Two categories: Non-Probability Sampling (*deterministic sampling*) and Probability Sampling (*random sampling*). The difference is whether the sample selection is based on randomization or not. [2]



Probability Sampling Vs Non-Probability Sampling

[2] https://towardsdatascience.com/sampling-techniques-a4e34111d808

- **Reconstruction** in signal processing, (also known as interpolation and data fitting), attempts to produce a continuous time signal coinciding with the points of the discrete time signal. [3]

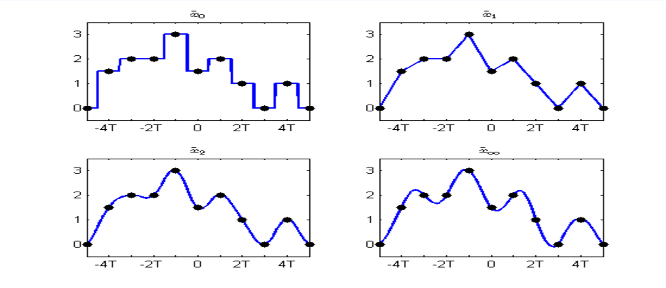


Figure: Cardinal spline interpolations of the discrete time signal of degree $d = 0, 1, 2, \infty$. The interpolations become increasingly smooth and approach the sinc interpolation as the order increases

[3] https://cnx.org/contents/d2CEAGW515.4:gWKrY9L410/Signal-Reconstruction

- Sampling from signals to sampling data $f \longmapsto \big(\psi(f)\big)_{\psi\in\Psi}$.
- Reconstruction from sampling data to signal approximation $\big(\psi(f)\big)_{\psi\in\Psi} \longmapsto \tilde{f}$ is an inverse problem. The inverse problem is **well-posedness**, i.e., $\tilde{f} \approx f$,
  - when the original signal has additional information, such as bandlimited, regularity, finite rate of innovation, sparsity, or residing in some linear spaces (PW, SIS, RKS)
  - when the sampling procedure is appropriate (the sampling family Ψ, such as random sampling with enough samples, deterministic sampling with large density)
- **Reconstruction algorithms**: Iterative (infinite steps) and non-iterative (finite steps)
- In this talk, we consider **random sampling** and **finite reconstruction** of concentrated signals in a reproducing kernel space.

2. Concentrated signals in reproducing kernel spaces

- **Paley-Wiener space** *PW*:

$$\begin{aligned} PW &= \left\{ f \in L^2(\mathbb{R}) : \operatorname{supp} \hat{f} \subset [-\pi, \pi] \right\} \\ &= \left\{ \sum_{k \in \mathbb{Z}} c(k) \operatorname{sinc}(x - k), \sum_{k \in \mathbb{Z}} |c(k)|^2 < \infty \right\} \end{aligned}$$

  where $\operatorname{sinc}(t) = \frac{\sin \pi t}{\pi t}$ is the cardinal sine function and $\{\operatorname{sinc}(\cdot - k), k \in \mathbb{Z}\}$ is ON.

- **Shift-invariant spaces** [4]

$$V(\phi) = \left\{ \sum_{k \in \mathbb{Z}} c(k) \phi(x - k), \sum_{k \in \mathbb{Z}} |c(k)|^2 < \infty \right\},$$

  where $\{\phi(\cdot - k), k \in \mathbb{Z}\}$ forms a Riesz basis.

[4] A. Aldroubi and K. Grochenig, Nonuniform sampling and reconstruction in shift-invariant spaces. SIAM Review, 2001

- Let $L^p := L^p(X, \rho, \mu), 1 \le p \le \infty$ be the linear space of all p-integrable functions on the metric measure space (X, ρ, μ). Consider the **range space**

$$V_p = \{Tf, f \in L^p\} = \{f \in L^p, Tf = f\}$$

of an idempotent integral operator

$$Tf(x) = \int_X K(x,y)f(y)d\mu(y)$$

- The range space V_p is a **Reproducing Kernel Subspace** of L^p [5] if the integral kernel K has certain off-diagonal decay and Hölder continuity,

$$\|K\|_{\mathcal{S},\theta} = \|K\|_{\mathcal{S}} + \sup_{0<\delta<1} \delta^{-\theta}\|\omega_\delta(K)\|_{\mathcal{S}} < \infty.$$

For instance $|K(x,y)| \le (1+\rho(x,y))^{-N}$ and

$|K(x,y) - K(x',y')| \le (\rho(x,x') + \rho(y,y'))^\theta (1 + \rho(x,y) + \rho(x',y'))^{-N}.$

[5] Z. Nashed and Q. Sun, Sampling and reconstruction of signals in a reproducing kernel subspace of $L^p(R^d)$, JFA, 2010

- The range space $V_p = \{Tf, f \in L^p\} = \{f \in L^p, Tf = f\}$ is a RKS, where
$$Tf(x) = \int_X K(x,y)f(y)d\mu(y)$$
- $PW = \left\{f \in L^2, f = f * \mathrm{sinc}\right\}$ with kernel $K(x,y) = \mathrm{sinc}(x-y)$.
- $SIS = \{f \in L^2, f(x) = \int f(y)\big(\sum_{k\in\mathbb{Z}} \tilde{\phi}(y-k)\phi(x-k)\big)dy$ where $\{\tilde{\phi}(x-k)\}$ is a dual Riesz basis.
- In the Euclidean space setting, $V_p = \{\sum_{\lambda\in\Lambda} c_\lambda\phi_\lambda,\ \sum_\lambda |c_\lambda|^p < \infty\}$, where $\phi_\lambda, \lambda \in \Lambda$ is a p-frame [6]

[6] A Aldroubi, Q Sun, WS Tang, p-Frames and Shift Invariant Subspaces of L_p, JFAA 2021; PG Casazza, D Han, DR Larson, Frames for Banach spaces, Contemporary Mathematics 247, 1999.

- A *Corkscrew domain* Ω if for any $x \in \partial\Omega$ and radius $0 < r \leq \mathrm{diam}\partial\Omega$, there exists $y \in \Omega$ such that $B(y, cr) \subset \Omega \cap B(x, r)$, where $c \in (0, 1)$ is an absolute constant. [7]
- Concentrated signals [8] $V_{p,\Omega,\epsilon} = \{f \in V_p, \|f\|_{p,\Omega^c} \leq \epsilon\|f\|_p\}$

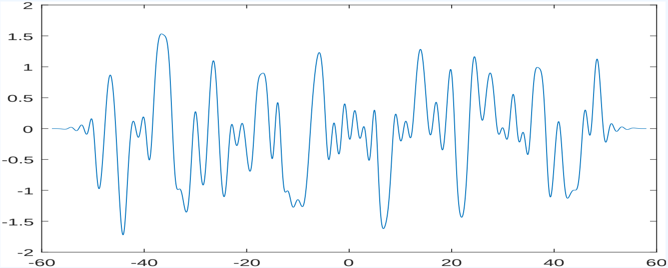


Figure: $\sum_{i=-50}^{50} r_i \exp(-(x - i - \theta_i)^2)$, where $1/2 \leq |r_i| \leq 1$

[7] Matthew Badger, Lipschitz Approximation to Corkscrew Domains https://badger.math.uconn.edu/talk/corkrain.pdf

[8] Used in time-frequency analysis, phase retrieval, and (random) sampling of bandlimited and wavelet signals by Alaifari, Bass, Cheng, Daubechies, Fuhr, Grochenig, Grohs etc

- sampling in linear spaces (PW, SIS, Reproducing kernel), via inverse problem
- sampling of signal with properties, such as smoothness and sparsity, via optimization approach.
- **Random sampling** in Paley-Wiener space and shift-invariant space has been discussed by Bass, Grochenig etc. [9]
- Bass and Grochenig use the spectral approach of the composition projection onto the domain and the space. We use completely different approach.
- The **challenge**: The set $V_{p,\Omega,\epsilon}$ is not a convex set (also not a linear space)

[9] R.F. Bass, K. Gröchenig, Relevant sampling of band-limited functions, Ill. J. Math. 57 (2013) 43–58; R.F. Bass, K. Gröchenig, Random sampling of bandlimited functions, Isr. J. Math. 177 (2010) 1–28

- Concentrated signals f in a reproducing kernel space $V_{p,\Omega,\epsilon} = \{f \in V_p,\ \|f\|_{p,\Omega^c} \le \epsilon \|f\|_p\}$
- Stability of bi-Lipschitz type vs. uniqueness:

Theorem

(Li, S. Xian [10]) If $\Gamma_\Omega \subset \Omega$ is a discrete sampling set with sufficiently small Hausdorff distance $d_H(\Gamma_\Omega, \Omega)$ between Γ_Ω and Ω, then for all $f, g \in V_{p,\Omega,\varepsilon}$,

$$\begin{aligned} &\big(1 - \varepsilon - \|K\|_{\mathcal{S},\theta}(d_H(\Gamma_\Omega, \Omega))^\theta\big)\|f - g\|_p - 2\varepsilon \min(\|f\|_p, \|g\|_p) \\ \le\ & \big\|\big(f(\gamma) - g(\gamma)\big)_{\gamma \in \Gamma_\Omega}\big\|_{p,\mu(\Gamma_\Omega)} \\ \le\ & \big(1 + \|K\|_{\mathcal{S},\theta}(d_H(\Gamma_\Omega, \Omega))^\theta\big)\|f - g\|_p, \end{aligned} \tag{1}$$

where , $I_\gamma, \gamma \in \Gamma_\Omega$, is a Voronoi partition of the domain Ω.

[10] Y Li, Q Sun, J Xian, Random sampling and reconstruction of concentrated signals in a reproducing kernel space, ACHA, 54(2021), 273-302

- With high probability, an original ε-concentrated signal can be "reconstructed" approximately from its random samples corrupted by i.i.d. random noises, when the random sampling size is large enough.

Theorem

If $\{\gamma, \gamma \in \Gamma_\Omega\}$ are i.i.d. random positions drawn on Ω w.r.t. the probability measure $(\mu(\Omega))^{-1} d\mu$, then for any $\tilde{\varepsilon} \in (0, 1-\varepsilon)$, then

$$\begin{aligned} & \big(1-\varepsilon-\tilde{\varepsilon}\big)\|f-g\|_p - 2\varepsilon \min(\|f\|_p, \|g\|_p) \\ \leq & \Big\| \big(f(\gamma)-g(\gamma)\big)_{\gamma\in\Gamma_\Omega} \Big\|_{p,\mu(\Gamma_\Omega)} \leq \big(1+\tilde{\varepsilon}\big)\|f-g\|_p,\ f,g \in V_{p,\Omega,\varepsilon} \end{aligned} \tag{2}$$

hold with probability at least

$$1 - \frac{10^d \mu(\Omega)}{c^d D_1(\mu)(\tilde{\varepsilon}/\|K\|_{\mathcal{S},\theta})^{d/\theta}} \Big(1 - \frac{c^d D_1(\mu)(\tilde{\varepsilon}/\|K\|_{\mathcal{S},\theta})^{d/\theta}}{10^d \mu(\Omega)}\Big)^N,$$

where N is the size of the sampling set Γ_Ω.

4. Reconstruction of concentrated signals from random samples

- Non-uniqueness from i.i.d. random samples $f(\lambda), \lambda \in \Omega$, of concentrated signals f.
- For any $f \in V_p$ we define

$$g_0 = \sum_{\gamma \in \Gamma_\Omega} \mu(I_\gamma) f(\gamma) K(\cdot, \gamma) \in V_p, \tag{3a}$$

and $g_n \in V_p, n \geq 1$, inductively by

$$g_n = g_0 + g_{n-1} - S_\Gamma g_{n-1}, \ n \geq 1, \tag{3b}$$

where the preconstruction operator S_Γ on L^p is given in $S_\Gamma g(x) = \sum_{\gamma \in \Gamma} \mu(T_\gamma)(Tg)(\gamma) K(x, \gamma)$ with $\Gamma = \Gamma_\Omega \cup \Gamma_{\Omega^c}$, and Γ_{Ω^c} being a discrete sampling set of the complement Ω^c of the domain Ω satisfying

$$d_H(\Gamma_{\Omega^c}, \Omega^c) \leq \min\left(\varepsilon^{1/\theta} \|K\|_{\mathcal{S},\theta}^{-1/\theta}, (2\|K\|_{\mathcal{S},\theta}^2)^{-1/\theta}\right).$$

- Finite iterative algorithm $g_n, n \geq 0$, to construct a concentrated signal $\tilde{f}$ to approximate the original signal f:

$$g_n = g_0 + g_{n-1} - S_\Gamma g_{n-1}, \ n \geq 1,$$

with $g_0 = \sum_{\gamma \in \Gamma_\Omega} \mu(I_\gamma) f(\gamma) K(\cdot, \gamma) \in V_p$

Theorem

(Li, S. Xian, ACHA 2021) Suppose that $\{\gamma, \gamma \in \Gamma_\Omega\}$ are i.i.d. random positions drawn on Ω with respect to probability measure $(\mu(\Omega))^{-1} d\mu$, and denote the size of Γ_Ω by N. Then for $n + 1 \geq \frac{\ln(1/\varepsilon) - \ln \|K\|_{\mathcal{S},\theta}}{\ln 2}$,, the following reconstruction error estimates

$$\|g_n - f\|_p \leq 8 \|K\|_{\mathcal{S},\theta} \varepsilon \|f\|_p, \ f \in V_{p,\Omega,\varepsilon}, \tag{4}$$

hold with probability at least

$$1 - \frac{10^d (2\|K\|^2_{\mathcal{S},\theta})^{d/\theta} \mu(\Omega)}{c^d D_1(\mu)} \Big(1 - \frac{c^d D_1(\mu)}{10^d (2\|K\|^2_{\mathcal{S},\theta})^{d/\theta} \mu(\Omega)}\Big)^N. \tag{5}$$

5. Signal reconstruction from noisy random samples

- Consider i.i.d. random samples $f(\gamma), \gamma \in \Gamma_\Omega \subset \Omega$, of concentrated signals f corrupted by i.i. d. random noises, $\xi(\gamma)$, and define the iterative algorithm

$$\tilde{g}_0 = \sum_{\gamma\in\Gamma_\Omega} \mu(I_\gamma)(f(\gamma) + \xi(\gamma))K(\cdot, \gamma) \in V_p, \tag{6a}$$

and $g_n \in V_p, n \geq 1$, inductively by

$$\tilde{g}_n = \tilde{g}_0 + \tilde{g}_{n-1} - S_\Gamma \tilde{g}_{n-1},\ n \geq 1, \tag{6b}$$

where the preconstruction operator S_Γ on L^p is given by $S_\Gamma g(x) = \sum_{\gamma\in\Gamma} \mu(I_\gamma)(Tg)(\gamma)K(x, \gamma)$ with $\Gamma = \Gamma_\Omega \cup \Gamma_{\Omega^c}$, and Γ_{Ω^c} being a discrete sampling set of the complement Ω^c of the domain Ω satisfying

$$d_H(\Gamma_{\Omega^c}, \Omega^c) \leq \min\left(\varepsilon^{1/\theta} \|K\|_{\mathcal{S},\theta}^{-1/\theta}, (2\|K\|_{\mathcal{S},\theta}^2)^{-1/\theta}\right).$$

Theorem

Suppose that $\tau \in (0, 1/2)$ *and* $\xi(\gamma), \gamma \in \Gamma_\Omega$*, are i.i.d. random variables with mean zero and variance* σ^2*, i.e.,* $\mathbb{E}(\xi(\gamma)) = 0$ *and* $\mathrm{Var}(\xi(\gamma)) = \sigma^2,\ \gamma \in \Gamma_\Omega$*. Let* $f \in V_{p,\Omega,\varepsilon}$ *and set*

$$\tilde{\delta}_1 = \min\left((2\|K\|^2_{\mathcal{S},\theta})^{-1/\theta}, \Big(\frac{\tau\varepsilon^2\sigma^{-2}\|f\|_p^2}{D_2(\mu)(D_1(\mu))^{2/p-1}} \Big)^{1/d} \right). \tag{7}$$

If the size N of the random sampling set Γ_Ω *satisfies*

$$N \geq \frac{10^d \mu(\Omega)}{c^d D_1(\mu)\tilde{\delta}_1^d} \ln \frac{10^d \mu(\Omega)}{c^d D_1(\mu)\tau\tilde{\delta}_1^d}, \tag{8}$$

then for any integer $n + 1 \geq \frac{\ln(1/\varepsilon) - \ln \|K\|_{\mathcal{S},\theta}}{\ln 2}$*,*

$$\|\tilde{g}_n - f\|_\infty \leq 10(D_1(\mu))^{-1/p}\|K\|^2_{\mathcal{S},\theta}\varepsilon\|f\|_p \tag{9}$$

hold with probability at least $1 - 2\tau$*.*

- In our simulations, we consider the following family of signals

$$f_{L,\alpha} = \sum_{i=-L}^{L} r_i (1+|i|)^{-\alpha} \exp(-(x-i-\theta_i)^2) \tag{10}$$

concentrated on the interval $\Omega_L = [-L, L]$, where $L \geq 1, \alpha \geq 0$, and random variables $r_i, -L \leq i \leq L$, are independently selected in $[-1,1]\backslash(-1/2,1/2)$ with uniform distribution, and $\theta_i \in [-1/10, 1/10], i \in \mathbb{Z}$, are randomly selected.
- The sampling data of a concentrated signal f on the sampling set $\Gamma_{N,L} = \{\gamma_k \subset [-L, L], 1 \leq k \leq N\}$ being corrupted by i.i.d random noises $\xi(\gamma_k) \in [-\delta, \delta], \gamma_k \in \Gamma_{N,L}$, with uniform distribution,

$$\tilde{f}_{\gamma_k} = f(\gamma_k) + \xi(\gamma_k),\ \gamma_k \in \Gamma_{N,L}. \tag{11}$$

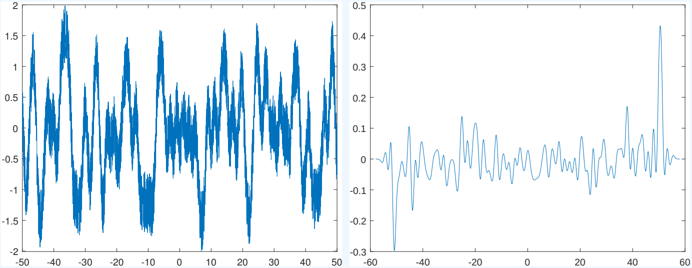


Figure: Plotted on the left is noisy sampling data with $N = 2L^2 = 5000$ and $\alpha = 0$, and the right is the difference $\tilde{g}^{(n)}_{N,L,\alpha} - f_{L,\alpha}$ between the reconstructed signal $\tilde{g}^{(n)}_{N,L,\alpha}$ at the sixth iteration ($n = 6$) from noisy sampling data and the original signal $f_{L,\alpha}$. The relative approximation error $\|g^{(n)}_{N,L,\alpha} - f_{L,\alpha}\|_2/\|f_{L,\alpha}\|_2$ and the concentration ratio $\|f_{L,\alpha}\|_{2,\Omega_L^c}/\|f_{L,\alpha}\|_2$ are 0.1036 and 0.0886.

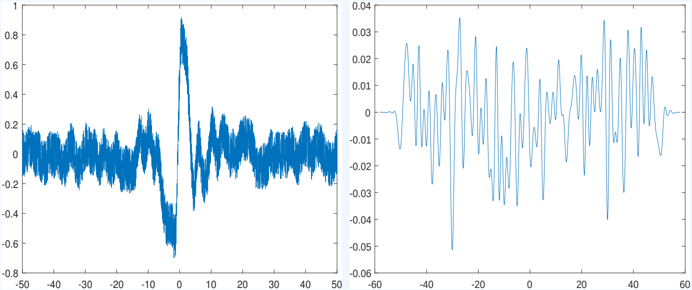


Figure: Plotted on the left is the noisy sampling data with $N = 2L^2 = 5000$ and $\alpha = 0.8$. Shown on the right is the difference $\tilde{g}^{(n)}_{N,L,\alpha} - f_{L,\alpha}$ between the reconstructed signal $\tilde{g}^{(n)}_{N,L,\alpha}$ at the sixth iteration ($n = 6$) from noisy sampling data in the middle figures and the original signal $f_{L,\alpha}$. The relative approximation error $\|g^{(n)}_{N,L,\alpha} - f_{L,\alpha}\|_2 / \|f_{L,\alpha}\|_2$ and the concentration ratio $\|f_{L,\alpha}\|_{2,\Omega_L^c} / \|f_{L,\alpha}\|_2$ are 0.0948 and 0.0145.

7. Summary

- concentrated signals (nonconvex) vs signals with regularity, sparsity or residing in linear space
- random vs deterministic sampling
- Exact reconstruction vs approximation
- random sampling on networks and feature learning from sampling data?

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