ON SYLVESTER EQUATIONS IN BANACH SUBALGEBRAS

QIQUAN FANG, CHANG EON SHIN AND QIYU SUN

In celebration of the 65th birthday of Professor Akram Aldroubi

Abstract. Let $\mathcal{B}$ be a Banach algebra and $\mathcal{A}$ be a Banach subalgebra that admits norm-controlled inversion in $\mathcal{B}$. In this work, we take A, B in the Banach subalgebra $\mathcal{A}$ with the spectra of A and B in the Banach algebra $\mathcal{B}$ being disjoint, and show that the operator Sylvester equation $BX - XA = Q$ has a unique solution $X \in \mathcal{A}$ for every $Q \in \mathcal{A}$. Under the additional assumptions that $\mathcal{B}$ is the operator algebra $\mathcal{B}(H)$ on a Hilbert space H and that A and B are normal in $\mathcal{B}(H)$, an explicit norm estimate for the solution X of the above operator Sylvester equation is provided. In this work, the above conclusion on norm control is also discussed for Banach subalgebras of localized infinite matrices and integral operators.

1. Introduction

Let $m, n \geq 1$ and take matrices $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{m\times m}$. It is well known that the Sylvester equation

$$BX - XA = Q \tag{1.1}$$

has a unique solution $X \in \mathbb{R}^{m\times n}$ for every $Q \in \mathbb{R}^{m\times n}$ if and only if A and B have no common eigenvalues [27, 50]. For the case that both $A = \mathrm{diag}(\lambda_1, \ldots, \lambda_n)$ and $B = \mathrm{diag}(\mu_1, \ldots, \mu_m)$ are diagonal matrices with no common eigenvalues, one may verify that the solution of the above matrix Sylvester equation (1.1) is given by

$$x_{ij} = \frac{q_{ij}}{\mu_i - \lambda_j} \ \text{ for all } 1 \leq i \leq m \text{ and } 1 \leq j \leq n,$$

where $X = (x_{ij})_{1\leq i\leq m, 1\leq j\leq n}$ and $Q = (q_{ij})_{1\leq i\leq m, 1\leq j\leq n}$, see [43, Theorem 3.1] for the explicit expression of the solution X when A and B are diagonalizable.

The Sylvester equation (1.1) appears in block diagonalization of matrices. Roth's theorem states that matrices $\begin{pmatrix} B & Q \\ 0 & A \end{pmatrix}$ and $\begin{pmatrix} B & 0 \\ 0 & A \end{pmatrix}$ are similar if and only if the Sylvester

Date: August 1, 2024.

2020 *Mathematics Subject Classification.* 47G10, 47B38, 47A10, 47B49.

Key words and phrases. Sylvester equation, Lyapunov equation, Banach algebra, norm-controlled inversion, Wiener's lemma.

equation (1.1) has a solution [34]. In particular, we have

$$\begin{pmatrix} B & Q \\ 0 & A \end{pmatrix} = \begin{pmatrix} I & X \\ 0 & I \end{pmatrix}^{-1} \begin{pmatrix} B & 0 \\ 0 & A \end{pmatrix} \begin{pmatrix} I & X \\ 0 & I \end{pmatrix},$$

where I is the identity matrix of appropriate size.

The Sylvester equation (1.1) with assuming $m = n$ and replacing B by $-A^T$ becomes the Lyapunov equation

$$A^T X + XA + Q = 0. \tag{1.2}$$

For the Lyapunov equation (1.2), there exists a unique positive definite solution X when $-A$ and Q are symmetric and positive definite (hence $-A^T$ and A have no common eigenvalues) [2, 9, 28, 29]. In that scenario, the linear dynamical system

$$\dot{z}(t) = Az(t) \tag{1.3}$$

is stable in the sense that $\lim_{t\to+\infty} z(t) = 0$. Define the Lyapunov function by $V(z(t)) = z(t)^T X z(t)$, where X is the positive definite solution of the Lyapunov equation (1.2) with Q replaced by the identity I. Then it follows from (1.2), (1.3) and the positive-definiteness of the matrix X, the quadratic function $V(z)$ satisfies

$$\dot{V}(z(t)) = z(t)^T (A^T X + XA) z(t) = -z(t)^T z(t) \le -\tau V(z(t)),$$

where $\tau > 0$ is an absolute constant. Solving the above differential inequality and using the positive-definiteness of the matrix X, we can find a positive constant C such that

$$0 \le z(t)^T z(t) \le CV(z(t)) \le Ce^{-\tau t} \quad \text{for all} \quad t \ge 0. \tag{1.4}$$

This proves the (exponential) stability of the dynamical system (1.3).

Let $\mathcal{B}$ be a Banach algebra. Given A and $B \in \mathcal{B}$, we define the Sylvester operator $T_{A,B}$ on $\mathcal{B}$ by

$$T_{A,B}(X) = BX - XA, \quad X \in \mathcal{B}. \tag{1.5}$$

As the family of all matrices $A \in \mathbb{R}^{n\times n}$ forms a Banach algebra, the Sylvester equation (1.1) has been extended in the Banach algebra setting. Sylvester-Rosenblum theorem states that the operator Sylvester equation

$$T_{A,B}(X) = Q \tag{1.6}$$

has a unique solution in $\mathcal{B}$ for every $Q \in \mathcal{B}$ (hence the homogenous Sylvester equation $BX = XA$ has zero as its unique solution) if the spectra $\sigma_{\mathcal{B}}(A)$ and $\sigma_{\mathcal{B}}(B)$ of A and B in $\mathcal{B}$ are disjoint [7, 33, 36].

Let H be a complex Hilbert space and denote the $C^\star$-algebra of all linear operators on H by $\mathcal{B}(H)$. We say that a linear subspace M of the Hilbert space H is *invariant* under the operator $A \in \mathcal{B}(H)$ if $Ax \in M$ for every $x \in M$, and *hyperinvariant* if it is invariant under every operator $B \in \mathcal{B}(H)$ which commutes with A. One of the most famous problems in functional analysis is whether every operator on an infinite-dimensional Hilbert space have a non-trivial invariant subspace. The Sylvester-Rosenblum Theorem provides a sufficient condition on $A \in \mathcal{B}(H)$ so that its invariant space M is also hyperinvariant. Given $A \in \mathcal{B}(H)$, its invariant subspace $M \subset H$ can be described by

$$(I - P)AP = 0, \tag{1.7}$$

where P is the projection operator from H onto M. For any $B \in \mathcal{B}(H)$ commutating with A, we have

$$\begin{aligned} &((I-P)A(I-P) + \mu_1 P)(I-P)BP = (I-P)A(I-P)BP = (I-P)ABP \\ &= (I-P)BAP = (I-P)BPAP = (I-P)BP(PAP + \mu_2(I-P)), \end{aligned} \tag{1.8}$$

where $\mu_1, \mu_2 \in \mathcal{C}$ and the second and fourth equalities follow from (1.7). By (1.8) with μ_1 and μ_2 appropriately chosen and the Sylvester-Rosenblum theorem for homogenous operator Sylvester equations, we conclude that $(I - P)BP = 0$ and hence M is hyperinvariant, provided that the spectra of restrictions of the operator A onto the invariant subspace M and its orthogonal complement $M^\perp$ are disjoint [7, 31].

The Sylvester-Rosenblum theorem on the operator Sylvester equation (1.6) also plays a crucial role to establish spectral theorem for normal operator on a Hilbert space, see [7, Section 6]. The Sylvester equation appears in many mathematical fields and engineering applications, including linear algebra, functional analysis, ordinary differential equation, control theory, and signal processing. For historical remarks and recent advances on Sylvester equations, the reader may refer to [5, 7, 12, 36] and references therein.

In this work, we solve the operator Sylvester equation (1.6) in an inverse-closed Banach subalgebra and consider norm control of its unique solution.

2. Main Results

Given a Banach algebra $\mathcal{B}$, we say that its subalgebra $\mathcal{A}$ sharing the same identity I with $\mathcal{B}$ is *inverse-closed* in $\mathcal{B}$ if any element in $\mathcal{A}$ that is invertible in $\mathcal{B}$ is also invertible in $\mathcal{A}$. Inverse-closedness has numerous applications in time–frequency analysis, sampling theory, numerical analysis and optimization. It has been established for infinite matrices, integral operators, and pseudo-differential operators satisfying various off-diagonal decay conditions, see the survey papers [18, 26, 40] for historical remarks and [15, 23, 29, 35, 39] and references therein for recent advances. For an inverse-closed subalgebra $\mathcal{A}$ of a Banach algebra $\mathcal{B}$, it is known that for any element $A \in \mathcal{A}$, its spectra in the algebras $\mathcal{A}$ and $\mathcal{B}$ are the same. Therefore by the Sylvester-Rosenblum theorem, we have the following result on solving the operator Sylvester equation (1.6) in an inverse-closed subalgebra $\mathcal{A}$.

Theorem 2.1. *Let $\mathcal{B}$ be a Banach algebra and $\mathcal{A}$ be its inverse-closed Banach subalgebra of $\mathcal{B}$. If $A, B, Q \in \mathcal{A}$ and the spectra of A and B in $\mathcal{B}$ are disjoint, then there is a unique solution X to the operator Sylvester equation* (1.6) *in $\mathcal{A}$.*

A quantitative version of inverse-closedness is norm-controlled inversion [4, 20, 30, 44]. Here we say that an inverse-closed subalgebra $\mathcal{A}$ of a Banach algebra $\mathcal{B}$ admits a *norm-controlled inversion* if there exists a nonnegative function $h : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ that is bounded on any compact subset of $\mathbb{R}_+ \times \mathbb{R}_+$ such that

$$\|A^{-1}\|_{\mathcal{A}} \le h(\|A^{-1}\|_{\mathcal{B}}, \|A\|_{\mathcal{A}}) \tag{2.1}$$

for all $A \in \mathcal{A}$ that are invertible in $\mathcal{B}$, where $\mathbb{R}_+$ is the set of all nonnegative real numbers, and $\|\cdot\|_{\mathcal{A}}$ and $\|\cdot\|_{\mathcal{B}}$ are norms on Banach algebras $\mathcal{A}$ and $\mathcal{B}$ respectively. We remark that not every inverse-closed Banach subalgebra has norm-controlled inversion. In particular, the classical Wiener algebra of periodic functions with summable Fourier coefficients does not admit a norm-controlled inversion in the Banach algebra of all bounded periodic functions. We also observe that a good estimate for the norm-control function h in (2.1) to the inversion is important for some mathematical and engineering applications [10, 45], and the norm-control function associated with some norm-controlled inversion subalgebras may have polynomial growth [14, 19, 21, 39, 41].

Given a complex Hilbert space H, we say that a linear operator $A \in \mathcal{B}(H)$ on H is normal if $A^\star A = AA^\star$ [37]. For a normal operator $A \in \mathcal{B}(H)$, we have

$$\|A\|_{\mathcal{B}(H)} = \sup\{|z|,\ z \in \sigma_{\mathcal{B}(H)}(A)\}. \tag{2.2}$$

In this work, we consider solving the operator Sylvester equation (1.6) in an inverse-closed subalgebra $\mathcal{A}$ that admits a norm-controlled inversion in the $C^\star$-algebra $\mathcal{B}(H)$.

Theorem 2.2. *Let H be a complex Hilbert space, and $\mathcal{A}$ be a Banach subalgebra of the operator algebra $\mathcal{B}(H)$ that admits norm-controlled inversion in $\mathcal{B}(H)$. If $A, B, Q \in \mathcal{A}$, and A, B are normal operators in $\mathcal{B}(H)$ with their spectra in $\mathcal{B}(H)$ being disjoint, then there is a bivariate function g on $\mathbb{R}_+ \times \mathbb{R}_+$ that is bounded on any compact subset of $\mathbb{R}_+ \times \mathbb{R}_+$ such that*

$$\|X\|_{\mathcal{A}} \le g((d(A,B))^{-1}, \|A\|_{\mathcal{A}} + \|B\|_{\mathcal{A}}) \tag{2.3}$$

holds for the unique solution X of the operator Sylvester equation (1.6)*, where $d(A,B)$ is the distance of the spectra of A and B in $\mathcal{B}(H)$.*

Taking $A = 0$ (resp., $B = 0$), the corresponding operator Sylvester equation (1.6) becomes a trivial inverse problem $BX = Q$ (resp., $-XA = Q$), and it has a unique solution X in the subalgebra $\mathcal{A}$. Therefore the estimate in (2.3) for the solution of the operator Sylvester equation (1.6) could be considered as the correspondence of the norm estimate (2.1) for the inversion in the Sylvester setting. We remark that an appropriate norm estimate for the solution of Sylvester equations could be essential for some mathematical and engineering applications, such as stability of dynamical systems and optimal control.

The bivariate function h in (2.1) is known as a norm-control function of the norm-controlled inversion subalgebra $\mathcal{A}$. We remark that the norm-control function h can be chosen so that it is monotonic about every variable, i.e.,

$$0 \le h(s_1, t_1) \le h(s_2, t_2) \ \text{ if } 0 \le s_1 \le s_2 \text{ and } 0 \le t_1 \le t_2. \tag{2.4}$$

Otherwise, we may replace the original norm-control function h by the following bivariate function

$$\tilde{h}(s,t) = \sup_{0\le u\le s, 0\le v\le t} h(u,v) \text{ for } \ s,t \ge 0,$$

which is well-defined by the boundedness assumption for the original norm-control function on any bounded set. Applying a similar argument, we see that the function g in (2.3) could be selected to be monotonic about every variable.

Let $m \ge 2$ be an integer. Given a Banach algebra $\mathcal{B}$, we say that it is a symmetric *-algebra if the spectrum $\sigma_{\mathcal{B}}(A^\star A)$ of $A^\star A$ is contained in $[0,\infty)$ for any $A \in \mathcal{B}$, and that its Banach subalgebra $\mathcal{A}$ is *differential* if there exist $\theta \in (0, m-1]$ and an absolute constant D satisfying

$$\|A^m\|_{\mathcal{A}} \le D\|A\|_{\mathcal{A}}^{m-\theta}\|A\|_{\mathcal{B}}^{\theta} \quad \text{for all} \quad A \in \mathcal{A} \tag{2.5}$$

[8, 11, 20, 25, 32, 39]. In [38, Theorem 4.1], it is shown that a differential *-subalgebra admits a norm-controlled inversion with the norm-control function having subexponential growth.

Combining [38, Theorem 4.1] and Theorem 2.2, we have the following corollary on solving the operator Sylvester equation (1.6) in differential *-subalgebra.

Corollary 2.3. *Let H be a complex Hilbert space, and $\mathcal{A}$ be a *-subalgebra of $\mathcal{B}(H)$ with common identity and involution *. If $\mathcal{A}$ is a differential subalgebra of $\mathcal{B}(H)$, and $A, B \in \mathcal{A}$ are normal operators in $\mathcal{B}(H)$ with their spectra in $\mathcal{B}(H)$ being disjoint, then the operator Sylvester equation* (1.6) *has a norm-controlled solution in the differential *-subalgebra $\mathcal{A}$.*

3. Sylvester equations for infinite matrices and integral operators

In this section, we apply the conclusion in Corollary 2.3 to solve Sylvester equations in Banach algebras of localized infinite matrices and integral operators.

Let $\ell^p := \ell^p(\mathbb{Z}^d), 1 \le p \le \infty$, be the Banach space of all p-summable sequences on $\mathbb{Z}^d$ with the norm denoted by $\|\cdot\|_p$. Given $1 \le p \le \infty$ and $\alpha \ge 0$, we define the Gröchenig-Schur algebra of infinite matrices by

$$\mathcal{A}_{p,\alpha} = \Big\{ A = (a(i,j))_{i,j\in\mathbb{Z}^d}, \ \|A\|_{\mathcal{A}_{p,\alpha}} < \infty \Big\}, \tag{3.1}$$

the Baskakov-Gohberg-Sjöstrand algebra of infinite matrices by

$$\mathcal{C}_{p,\alpha} = \Big\{ A = (a(i,j))_{i,j\in\mathbb{Z}^d}, \ \|A\|_{\mathcal{C}_{p,\alpha}} < \infty \Big\}, \tag{3.2}$$

and the Beurling algebra of infinite matrices by

$$\mathcal{B}_{p,\alpha} = \Big\{ A = (a(i,j))_{i,j\in\mathbb{Z}^d}, \ \|B\|_{\mathcal{B}_{p,\alpha}} < \infty \Big\} \tag{3.3}$$

respectively, where $u_\alpha(i,j)=(1+|i-j|)^\alpha, \alpha\ge 0$, are polynomial weights on $\mathbb{Z}^{2d}$,

$$\|A\|_{\mathcal{A}_{p,\alpha}}=\max\Big\{\sup_{i\in\mathbb{Z}^d}\big\|\big(a(i,j)u_\alpha(i,j)\big)_{j\in\mathbb{Z}^d}\big\|_p,\ \ \sup_{j\in\mathbb{Z}^d}\big\|\big(a(i,j)u_\alpha(i,j)\big)_{i\in\mathbb{Z}^d}\big\|_p\Big\}, \tag{3.4}$$

$$\|A\|_{\mathcal{C}_{p,\alpha}}=\Big\|\Big(\sup_{i-j=k}|a(i,j)|u_\alpha(i,j)\Big)_{k\in\mathbb{Z}^d}\Big\|_p, \tag{3.5}$$

and

$$\|A\|_{\mathcal{B}_{p,\alpha}}=\Big\|\Big(\sup_{|i-j|\ge|k|}|a(i,j)|u_\alpha(i,j)\Big)_{k\in\mathbb{Z}^d}\Big\|_p \tag{3.6}$$

[1, 3, 6, 14, 15, 17, 22, 29, 39, 42, 46, 48, 49]. Clearly, we have

$$\mathcal{B}_{p,\alpha}\subset\mathcal{C}_{p,\alpha}\subset\mathcal{A}_{p,\alpha}\ \text{ for all } 1\le p\le\infty \text{ and } \alpha\ge 0. \tag{3.7}$$

The above inclusion become an equality for $p=\infty$, which is also known as the Jaffard algebra [24].

For $1\le p\le\infty$ and $\alpha>d-d/p$, it is known that $\mathcal{A}_{p,\alpha}$, $\mathcal{C}_{p,\alpha}$ and $\mathcal{B}_{p,\alpha}$ are differential *-subalgebras of $\mathcal{B}(\ell^2)$, the algebra of all bounded linear operators on ℓ^2. This together with Corollary 2.3 yields the following conclusion on solving the Sylvester equation (1.6) in the above three algebras of infinite matrices.

Theorem 3.1. *Let $d\ge 1, 1\le p\le\infty, \alpha>d-d/p$, and $\mathcal{A}$ be either the Gröchenig-Schur algebra $\mathcal{A}_{p,\alpha}$, or the Baskakov-Gohberg-Sjöstrand algebra $\mathcal{C}_{p,\alpha}$, or the Beurling algebra $\mathcal{B}_{p,\alpha}$. If $A,B\in\mathcal{A}$ have their spectra $\sigma_{\mathcal{B}(\ell^2)}(A)$ and $\sigma_{\mathcal{B}(\ell^2)}(B)$ in $\mathcal{B}(\ell^2)$ being disjoint, then for every $Q\in\mathcal{A}$, the operator Sylvester equation* (1.6) *has a unique solution in $\mathcal{A}$. Furthermore, if A,B are normal in $\mathcal{B}(\ell^2)$, then there is a bivariate function g on $\mathbb{R}_+\times\mathbb{R}_+$ such that* (2.3) *holds.*

Let $\mathbb{Z}_+^d$ be the set of all d-tuples of nonnegative integers, and $L^p:=L^p(\mathbb{R}^d), 1\le p\le\infty$, be the space of all p-integrable functions on $\mathbb{R}^d$ with its norm denoted by $\|\cdot\|_p$. Take $1\le p\le\infty$, $\alpha>0$ and a positive integer $m\ge 1$, and consider Banach algebra $\mathcal{W}^m_{p,\alpha}$ of localized integral operators

$$Tf(x)=\int_{\mathbb{R}^d}K(x,y)f(y)dy \tag{3.8}$$

on the space L^2 with the norm defined by

$$\|T\|_{\mathcal{W}^m_{p,\alpha}}:=\max_{k,l\in\mathbb{Z}^d_+,|k|+|l|\le m-1,0<\delta\le 1}\|\partial_x^k\partial_y^l K(x,y)\|_{p,\alpha}+\delta^{-1}\|\omega_\delta(\partial_x^k\partial_y^l K(x,y))\|_{p,\alpha}, \tag{3.9}$$

where $u_\alpha(x,y)=(1+|x-y|)^\alpha$ is a polynomial weight on $\mathbb{R}^{2d}$, and for a kernel function $K(x,y), x,y\in\mathbb{R}^d$, we define its modulus of continuity $\omega_\delta(K)$ by

$$\omega_\delta(K)(x,y):=\sup_{|x'|\le\delta,\ |y'|\le\delta}|K(x+x',y+y')-K(x,y)|,\ \ x,y\in\mathbb{R}^d, \tag{3.10}$$

and set

$$\|K\|_{p,\alpha} := \max\Big(\sup_{x\in\mathbb{R}^d} \big\|K(x,\cdot)u_\alpha(x,\cdot)\big\|_p,\ \sup_{y\in\mathbb{R}^d} \big\|K(\cdot,y)u_\alpha(\cdot,y)\big\|_p\Big).$$

For $\alpha > d - d/p$, $\mathcal{W}^m_{p,\alpha}$ is a non-unital Banach algebra [13, 47]. Define the unital Banach algebra $\mathcal{IW}^m_{p,\alpha}$ induced from $\mathcal{W}^m_{p,\alpha}$ by

$$\mathcal{IW}^m_{p,\alpha} := \big\{\lambda I + T :\ \lambda \in \mathbb{C} \text{ and } T \in \mathcal{W}^m_{p,\alpha}\big\} \tag{3.11}$$

with

$$\|\lambda I + T\|_{\mathcal{IW}^m_{p,\alpha}} := |\lambda| + C_0\|T\|_{\mathcal{W}^m_{p,\alpha}} \tag{3.12}$$

for some positive constant C_0. With appropriate selection of the constant C_0 in (3.12), one may verify that $\mathcal{I}W^m_{p,\alpha}, 1 \le p \le \infty, m \ge 1, \alpha > d - d/p$, are differential *-subalgebra of $\mathcal{B}(L^2)$ (the *-algebra of bounded linear operators on L^2) [13, 47]. Therefore as a consequence of Corollary 2.3, we have the following result on the Sylvester equation (1.6) in the above algebra of localized integral operators.

Theorem 3.2. *Let $1 \le p \le \infty, m \ge 1, \alpha > d - d/p$, and let $\mathcal{I}W^m_{p,\alpha}$ be as in* (3.11)*. If $A, B \in \mathcal{IW}^m_{p,\alpha}$ have their spectra in $\mathcal{B}(L^2)$ being disjoint, then there exists a unique solution to the Sylvester equation* (1.6) *for every $Q \in \mathcal{I}W^m_{p,\alpha}$. Furthermore, if A, B are normal in $\mathcal{B}(L^2)$, then there is a bivariate function g on $\mathbb{R}_+ \times \mathbb{R}_+$ such that* (2.3) *holds.*

The reader may refer to [13, 16, 23] for additional Banach algebras of localized integral operators and pseudo-differential operators.

4. Proof of Theorem 2.2

We say that a bounded and open set D in the complex plane is a *Cauchy domain* if it contains only a finite number of components with the closures of any two of them being disjoint, and its boundary ∂D is composed of a finite positive number of closed positive oriented rectifiable Jordan curves with no two of those curves intersecting. To prove the main theorem, we need a technical lemma in [7, 33] for the unique solution of the operator Sylvester equation (1.6).

Lemma 4.1. *Let $\mathcal{B}$ be a Banach algebra and $A, B, Q \in \mathcal{B}$. If the spectra $\sigma_{\mathcal{B}}(A)$ and $\sigma_{\mathcal{B}}(B)$ of A and B in the algebra $\mathcal{B}$ are disjoint, then the Sylvester operator $T_{A,B}$ is invertible. Furthermore, for a Cauchy domain D such that $\sigma_{\mathcal{B}}(A) \subset D$, $\sigma_{\mathcal{B}}(B) \subset \mathbb{C}\backslash(D \cup \partial D)$, and its oriented boundary ∂D has total winding numbers $m \ge 1$ around $\sigma_{\mathcal{B}}(A)$ and 0 around $\sigma_{\mathcal{B}}(B)$, we have*

$$T^{-1}_{A,B}(Q) = -\frac{1}{2m\pi i}\int_{\partial D}(B - zI)^{-1}Q(zI - A)^{-1}dz.$$

Now we are ready to start the detailed proof of Theorem 2.2.

Proof of Theorem 2.2. Set $\mathcal{B} = \mathcal{B}(H)$ and define

$$\delta(A, B) = \min\big\{ \max(|\Re z - \Re w|, |\Im z - \Im w|) : \ z \in \sigma_{\mathcal{B}}(A), \ w \in \sigma_{\mathcal{B}}(B)\big\}$$

Then $\delta(A, B) > 0$ by the disjoint assumption for the spectra of A and B in the Banach algebra $\mathcal{B}$. Set $\delta'(A, B) = \delta(A, B)/3 > 0$, let N_0 be the integer part of $(\|A\|_{\mathcal{B}} + \delta'(A, B))/\delta'(A, B)$, and for every $k, k' \in \mathbb{Z}$ denote the closed square in the complex plane $\mathbb{C}$ with center $(k + k'i)\delta'(A, B)$ and side length $\delta'(A, B)$ by $S_{k,k'}$. Then we have

$$\sigma_{\mathcal{B}}(A) \subset \cup_{-N_0 \le k,k' \le N_0} S_{k,k'}. \tag{4.1}$$

Define the union of squares $S_{k,k'}, k, k' \in \mathbb{Z}$, with nonempty intersection with $\sigma_{\mathcal{B}}(A)$ by

$$D_1 = \cup_{S_{k,k'} \cap \sigma_{\mathcal{B}}(A) \ne \emptyset} S_{k,k'}.$$

Similarly, we let D_2 be the union of squares $S_{k,k'}, k, k' \in \mathbb{Z}$, with nonempty intersection with the domain D_1, and D_3 be the union of squares $S_{k,k'}, k, k' \in \mathbb{Z}$ with nonempty intersection with the domain D_2. By the above construction of domains D_1, D_2, D_3 and the distance definition $\delta(A, B)$ between spectra of A and B, we obtain

$$\sigma_{\mathcal{B}}(A) \subset D_1 \subset D_2 \subset Q(0, (2N_0 + 3)\delta'(A, B)) \ \text{ and } \ \sigma_{\mathcal{B}}(B) \subset \mathcal{C}\backslash D_3 \subset \mathcal{C}\backslash D_2, \tag{4.2}$$

where $Q(0, r)$ is the square in the complex plane with center zero and size length $r > 0$.

Let D be the interior of the domain D_2 with its boundary denoted by ∂D. With the positive direction on the boundary ∂D selected, we see that ∂D has total winding numbers $m \ge 1$ around $\sigma_{\mathcal{B}}(A)$ and 0 around $\sigma_{\mathcal{B}}(B)$. Furthermore, D is a Cauchy domain with the boundary ∂D being made of finitely many line segments and the length $\ell(\partial D)$ of its boundary ∂D is bounded, i.e.,

$$|z| \le \|A\|_{\mathcal{B}} + 2\sqrt{2}\delta'(A, B) \le \|A\|_{\mathcal{B}} + \delta(A, B) \text{ for all } z \in \partial D, \tag{4.3}$$

and

$$\ell(\partial D) \le 4(2N_0 + 3)^2\delta'(A, B) \le 48(\|A\|_{\mathcal{B}} + \delta(A, B))^2(\delta(A, B))^{-1}. \tag{4.4}$$

By (4.2), the spectra of $zI - A$ and $zI - B, z \in \partial D$, lie outside the square $Q(0, 2\delta'(A, B))$ in the complex plane with the origin as its center and $2\delta'(A, B)$ as its size length. This together with the normal property for operators A and B in $\mathcal{B}$ implies that

$$\begin{aligned} &\max(\|(zI - A)^{-1}\|_{\mathcal{B}}, \|(zI - B)^{-1}\|_{\mathcal{B}}) \\ \le &\max_{w \in \partial Q(0, 2\delta'(A,B))} |w|^{-1} \le (\delta'(A, B))^{-1} \text{ for all } z \in \partial D. \end{aligned} \tag{4.5}$$

Let $h(s, t), s, t \ge 0$, be the norm-control function for the subalgebra $\mathcal{A}$ which satisfies (2.4). By (4.2) and Lemma 4.1, the unique solution of the operator Sylvester equation (1.6) in $\mathcal{A}$ is given by

$$T^{-1}_{A,B}(Q) = -\frac{1}{2m\pi i} \int_{\partial D} (B - zI)^{-1} Q (zI - A)^{-1} dz, \ Q \in \mathcal{B},$$

where m is the total winding numbers $m \ge 1$ of the boundary ∂D around $\sigma_{\mathcal{B}}(A)$.

Combining the above expression for the solution of the operator Sylvester equation (1.6) and the estimates in (4.3), (4.4) and (4.5), we obtain

$$\begin{aligned}\|T_{A,B}^{-1}(Q)\|_{\mathcal{A}} &\le (2m\pi)^{-1}\int_{\partial D}\|(B-zI)^{-1}\|_{\mathcal{A}}\|Q\|_{\mathcal{A}}\|(zI-A)^{-1}\|_{\mathcal{A}}|dz|\\ &\le 24\pi^{-1}\|Q\|_{\mathcal{A}}(\|A\|_{\mathcal{B}}+\delta(A,B))^2(\delta(A,B))^{-1}\\ &\qquad\times\Big(h\big(3(\delta(A,B))^{-1},\ \max(\|A\|_{\mathcal{A}},\|B\|_{\mathcal{A}})+(\|A\|_{\mathcal{B}}+\delta(A,B))\|I\|_{\mathcal{A}}\big)\Big)^2. \end{aligned} \tag{4.6}$$

This complete the proof. □

References

[1] A. Aldroubi, A. Baskakov and I. Krishtal, Slanted matrices, Banach frames, and sampling, *J. Funct. Anal.*, **255**(2008), 1667–1691.

[2] S. Barnett and C. Storey, *Matrix Methods in Stability Theory*, Nelson, London, 1970.

[3] A. G. Baskakov, Wiener's theorem and asymptotic estimates for elements of inverse matrices, *Funktsional. Anal. i Prilozhen*, **24**(1990), 64–65.

[4] E. S. Belinskii, E. R. Liflyand and R. M. Trigub, The Banach algebra and its properties, *J. Fourier Anal. Appl.*, **3**(1997), 103–129.

[5] C. Bertram and H. Fassbender, A quadrature framework for solving Lyapunov and Sylvester equations, *Linear Algebra Appl.*, **622**(2021), 66–103.

[6] A. Beurling, On the spectral synthesis of bounded functions, *Acta Math.*, **81**(1949), 225–238.

[7] R. Bhatia and P. Rosenthal, How and why to solve the operator equation $AX - XB = Y$, *Bull. Lond. Math. Soc.*, **29**(1997), 1–21.

[8] B. Blackadar and J. Cuntz, Differential Banach algebra norms and smooth subalgebras of $C^\star$-algebras, *J. Operator Theory*, **26**(1991), 255–282.

[9] J. W. Bunce, Stabilizability of linear systems defined over $C^\star$-algebras, *Math. Syst. Theory*, **18**(1985), 237–250.

[10] C. Cheng, Y. Jiang and Q. Sun, Spatially distributed sampling and reconstruction, *Appl. Comput. Harmon. Anal.*, **47**(2019), 109–148.

[11] M. Christ, Inversion in some algebra of singular integral operators, *Rev. Mat. Iberoamericana*, **4**(1988), 219–225.

[12] B. D. Djordjevic, Singular Sylvester equation in Banach spaces and its applications: Fredholm theory approach, *Linear Algebra Appl.*, **622**(2021), 189–214.

[13] Q. Fang, Y. Shen, C. E. Shin and X. Tao, Norm-controlled inversion in Banach algebras of integral operators, *Banach J. Math. Anal.*, **17**(2023), Article No. 21, 29 pp.

[14] Q. Fang and C. E. Shin, Norm-controlled inversion of Banach algebras of infinite matrices, *Comptes Rendus. Math.*, **358**(2020), 407–414.

[15] Q. Fang, C. E. Shin and Q. Sun, Polynomial control on weighted stability bounds and inversion norms of localized matrices on simple graph, *J. Fourier Anal. Appl.*, **27** (2021), Article No. 83, 33 pp.

[16] Q. Fang, C. E. Shin and Q. Sun, Wiener's lemma for singular integral operators of Bessel potential type, *Monatsch. Math.*, **173** (2014), 35–54.

[17] I. Gohberg, M. A. Kaashoek and H. J. Woerdeman, The band method for positive and strictly contractive extension problems: an alternative version and new applications, *Integral Equations Operator Theory*, **12**(1989), 343–382.

[18] K. Gröchenig, Wiener's lemma: theme and variations, an introduction to spectral invariance and its applications, In *Four Short Courses on Harmonic Analysis: Wavelets, Frames, Time-Frequency Methods, and Applications to Signal and Image Analysis*, edited by P. Massopust and B. Forster, Birkhauser, Boston 2010.

[19] K. Gröchenig and A. Klotz, Noncommutative approximation: inverse-closed subalgebras and off-diagonal decay of matrices, *Constr. Approx.*, **32**(2010), 429–466.

[20] K. Gröchenig and A. Klotz, Norm-controlled inversion in smooth Banach algebra I, *J. London Math. Soc.*, **88**(2013), 49–64.

[21] K. Gröchenig and A. Klotz, Norm-controlled inversion in smooth Banach algebra II, *Math. Nachr.*, **287**(2014), 917–937.

[22] K. Gröchenig and M. Leinert, Symmetry of matrix algebras and symbolic calculus for infinite matrices, *Trans. Amer. Math. Soc.*, **358**(2006), 2695–2711.

[23] K. Gröchenig, C. Pfeuffer and J. Toft, Spectral invariance of quasi-Banach algebras of matrices and pseudodifferential operators, *Forum Math.*, 2024, https://doi.org/10.1515/forum-2023-0212

[24] S. Jaffard, Properiétés des matrices bien localisées prés de leur diagonale et quelques applications, *Ann. Inst. Henri Poincaré*, **7**(1990), 461–476.

[25] E. Kissin and V. S. Shulman, Differential properties of some dense subalgebras of $C^\star$-algebras, *Proc. Edinburgh Math. Soc.*, **37**(1994), 399–422.

[26] I. Krishtal, Wiener's lemma: pictures at exhibition, *Rev. Un. Mat. Argentina*, **52**(2011), 61–79.

[27] P. Lancaster and M. Tismenetsky, *The Theory of Matrices*, 2nd ed., Academic Press, 1985.

[28] A. Lyapunov, Problemes general de la stabilite du mouveement, 1892; reprinted as Ann. of Math. Stud. 17, Princeton University Press, 1947.

[29] N. Motee and Q. Sun, Sparsity and spatial localization measures for spatially distributed systems, *SIAM J. Control Optim.*, **55**(2017), 200–235.

[30] N. Nikolski, In search of the invisible spectrum, Ann. Inst. Fourier (Grenoble), **49**(1999), 1925–1998.

[31] H. Radjavi and P. Rosenthal, Hyperinvariant subspaces for spectral and n-normal operators, *Acta Sci. Math. (Szeged)*, **32**(1971) 121–126.

[32] M. A. Rieffel, Leibniz seminorms for "matrix algebras converge to the sphere", In *Quanta of Maths, volume 11 of Clay Math. Proc.*, Amer. Math. Soc., Providence, RI, pp. 543–578, 2010.

[33] M. Rosenblum, On the operator eqaution $BX - XA = Q$, *Duke Math. J.*, **23**(1956), 263–269.

[34] W. E. Roth, The equations $AX - YB = C$ and $AX - XB = C$ in matrices, *Proc. Amer. Math. Soc.*, **3**(1952), 392–396.

[35] E. Samei and V. Shepelska, Norm-controlled inversion in weighted convolution algebra, *J. Fourier Anal. Appl.*, **25**(2019), 3018–3044.

[36] A. Sasane, The Sylvester equation in Banach algebras, *Linear Algebra Appl.*, **631**(2021), 1–9.

[37] I. H. Sheth, On normaloid operators, *Pacific J. Math.*, **28**(1969), 675–676.

[38] C. E. Shin and Q. Sun, Differential subalgebras and norm-controlled inversion, In *Operator Theory, Operator Algebras and Their Interactions with Geometry and Topology*, R. E. Curto, W. Helton, H. Lin, X. Tang, R. Yang and G. Yu eds., Birkhauser Basel, pp. 467–485, 2020.

[39] C. E. Shin and Q. Sun, Polynomial control on stability, inversion and powers of matrices on simple graphs, *J. Funct. Anal.*, **276**(2019), 148–182.

[40] C. E. Shin and Q. Sun, Wieners lemma: localization and various approaches, *Appl. Math. J. Chinese Univ.*, **28**(2013), 465–484.

[41] C. E. Shin and Q. Sun, Stability of localized operators, *J. Funct. Anal.*, **256**(2009), 2417–2439.

[42] J. Sjöstrand, Wiener type algebra of pseudodifferential operators, Centre de Mathematiques, Ecole Polytechnique, Palaiseau France, Seminaire 1994–1995, December 1994.

[43] D. C. Sorensen and A. C. Antoulas, The Sylvester equation and approximate balanced reduction, *Linear Algebra Appl.*, **351–352**(2002), 671–700.

[44] J. D. Stafney An unbounded inverse property in the algebra of absolutely convergent Fourier series, *Proc. Amer. Math. Soc.*, **18**(1967), 497–498.

[45] Q. Sun, Localized nonlinear functional equations and two sampling problems in signal processing, *Adv. Comput. Math.*, **40**(2014), 415–458.

[46] Q. Sun, Wiener's lemma for infinite matrices II, *Constr. Approx.*, **34**(2011), 209–235.

[47] Q. Sun, Wiener's lemma for localized integral operators, *Appl. Comput. Harmonic Anal.*, **25**(2008), 148–167.

[48] Q. Sun, Wiener's lemma for infinite matrices, *Trans. Amer. Math. Soc.*, **359**(2007), 3099–3123.

[49] Q. Sun, Wiener's lemma for infinite matrices with polynomial off-diagonal decay, *C. Acad. Sci. Paris Ser I*, **340**(2005), 567–570.

[50] H. L. Trentelman, A. A. Stoorvogel and M. Hautus, *Control Theory for Linear Systems*, Springer, 2001.

Fang: Department of Mathematics, Zhejiang University of Science and Technology, Hangzhou, Zhejiang 310023, China
Email address: qiquanfang@163.com

Shin: Department of Mathematics, Sogang University, Seoul, South Korea
Email address: shinc@sogang.ac.kr

Sun: Department of Mathematics, University of Central Florida, Orlando, FL 32816, USA.
Email address: qiyu.sun@ucf.edu