Graph Fourier Transform on Directed Graphs

Qiyu Sun

Department of Mathematics
University of Central Florida

IEEE Signal Processing Society Webinar, March 28, 2025

Overview

- Why Graph Fourier transform (GFT)?
  It is one of the fundamental tools in graph signal processing to **decompose** graph signals into different frequency components and **effectively represent** graph signals with regularity using **various modes of variation**
- What we have to describe networks and understand data on networks?
  - Adjacent matrix, Laplacian, graph shifts and their variation
  - Finite Fourier transform on circulant graphs
- How to define GFT?
  - GFT on undirected graphs: a conventional approach is based on the eigendecomposition of the graph Laplacian
  - This eigendecomposition method does not apply to directed graph settings. Several approaches have been proposed to define GFTs on directed graphs, including Jordan decomposition of Laplacian, eigendecomposition of the magnetic Laplacian, SVD-decomposition of Laplacian, and their variants

UCF

- Why Graph Fourier transform (GFT)?
  It is one of the fundamental tools in graph signal processing to **decompose** graph signals into different frequency components and **effectively represent** graph signals with regularity using **various modes of variation**
- What we have to describe networks and understand data on networks?
  - Adjacent matrix, Laplacian, graph shifts and their variation
  - Finite Fourier transform on circulant graphs
- How to define GFT?
  - GFT on undirected graphs: a conventional approach is based on the eigendecomposition of the graph Laplacian
  - This eigendecomposition method does not apply to directed graph settings. Several approaches have been proposed to define GFTs on directed graphs, including Jordan decomposition of Laplacian, eigendecomposition of the magnetic Laplacian, SVD-decomposition of Laplacian, and their variants

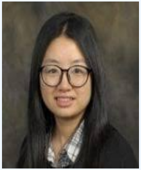
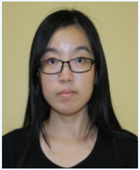
UCF

Overview

- Why Graph Fourier transform (GFT)?
  It is one of the fundamental tools in graph signal processing to **decompose** graph signals into different frequency components and **effectively represent** graph signals with regularity using **various modes of variation**
- What we have to describe networks and understand data on networks?
  - Adjacent matrix, Laplacian, graph shifts and their variation
  - Finite Fourier transform on circulant graphs
- How to define GFT?
  - GFT on undirected graphs: a conventional approach is based on the eigendecomposition of the graph Laplacian
  - This eigendecomposition method does not apply to directed graph settings. Several approaches have been proposed to define GFTs on directed graphs, including Jordan decomposition of Laplacian, eigendecomposition of the magnetic Laplacian, SVD-decomposition of Laplacian, and their variants

- Graph Fourier transform (GFT) is one of the fundamental tools in graph signal processing.
- What we have for networks: Adjacent matrix, Laplacian and their variations (graph shifts).
- What we have known: The GFT on undirected graphs has been well-studied, and several approaches have been proposed to define GFTs on directed graphs.
- The plan of this webinar?
  - graph Laplacian and graph shifts
  - GFT on undirected graphs
  - GFT on directed graphs (Jordan Decomposition and Magnetic Laplacian)
  - GFT based on the singular value decompositions of graph Laplacian.
  - GFT based on the singular value decompositions of graph Laplacian on product graphs
  - SVD-based GFTs: Robustness and good approximation of signals with regularity by band limiting procedure.

This presentation is based on joint works with Yang Chen, Cheng Cheng, Nazar Emirov, Junzheng Jiang, YeoJu Lee and Cong Zheng. [1]

[1] C. Cheng, Y. Chen, Y. J. Lee and Q. Sun, SVD-based graph Fourier transforms on directed product graphs, *IEEE Transactions on Signal and Information Processing over Networks*, 9(2023), 531-541; Y. Chen, C. Cheng and Q. Sun, Graph Fourier transform based on singular value decomposition of directed Laplacian, *Sampling Theory, Signal Processing, and Data Analysis*, 12(2023), article no. 24; N. Emirov, C. Cheng, J. Jiang and Q. Sun, Polynomial graph filter of multiple shifts and distributed implementation of inverse filtering *Sampling Theory, Signal Processing, and Data Analysis*, 20(2022), Article No. 2; C. Zheng, C. Cheng and Q. Sun, Wiener Filters on Graphs and Distributed Implementations, *Digital Signal Processing*, 162(2025), Paper No. 105156.

Graph Fourier transform

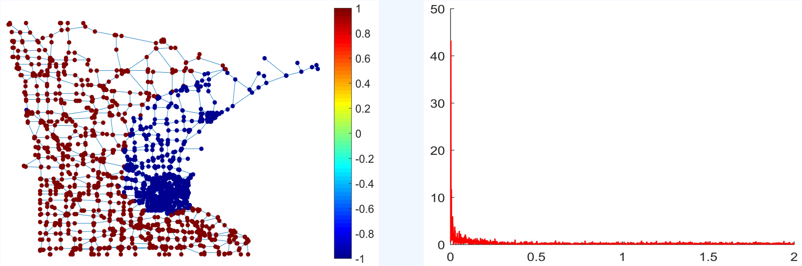


Figure: Left: Piecewise signal $\mathbf{x}$ on the undirected Minnesota traffic graph; Right: magnitude of GFT $|\hat{\mathbf{x}}| = |\mathbf{U}^T\mathbf{x}| = [|\langle \mathbf{u}_1, \mathbf{x}\rangle|, \ldots, |\langle \mathbf{u}_N, \mathbf{x}\rangle|]^T$, where $\mathbf{L} = \mathbf{U}\Lambda\mathbf{U}^T = \sum_{n=1}^N \lambda_n \mathbf{u}_n \mathbf{u}_n^T$ (Eigendecomposition of graph Laplacian $\mathbf{L}$)

Observation: The graph signal $\mathbf{x}$ is decomposed into different frequency components (mode of variation) effectively $\mathbf{x} = \sum_{n=1}^N \langle \mathbf{u}_n, \mathbf{x}\rangle \mathbf{u}_n$ (frequency components $\langle \mathbf{u}_n, \mathbf{x}\rangle \mathbf{u}_n$; modes of variation $\mathbf{u}_n$).

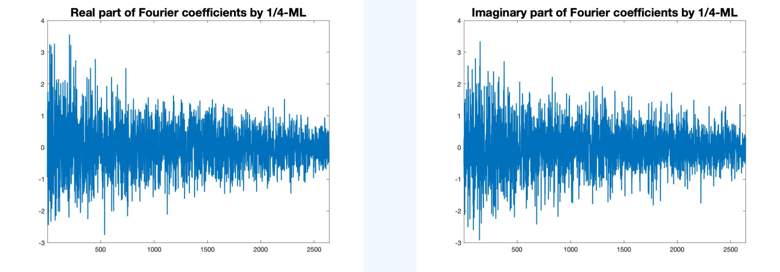


Figure: Plot on the left and right are the real part $\Re(\mathbf{V}_q^H \mathbf{x}_0)$ and imaginary part $\Im(\mathbf{V}_q^H \mathbf{x}_0)$ of the GFT by q-ML $\mathbf{L}^{(q)} = \mathbf{V}_q \Lambda_q \mathbf{V}_q^H$ with $q = 1/4$, where $\mathbf{x}_0$ is the piecewise constant signal on a weighted Minnesota traffic graph. The relative percentage of signal energy $\left(\sum_{k=0}^{M-1} |\mathbf{v}_{k;q}^H \mathbf{x}_0|^2\right)^{1/2} / \|\mathbf{x}_0\|_2$ for the first $M = 20, 50$ and 100 frequencies (about 0.76%, 1.89% and 3.79% of the total 2640 frequencies) are $0.1289, 0.2385, 0.3289$ respectively.

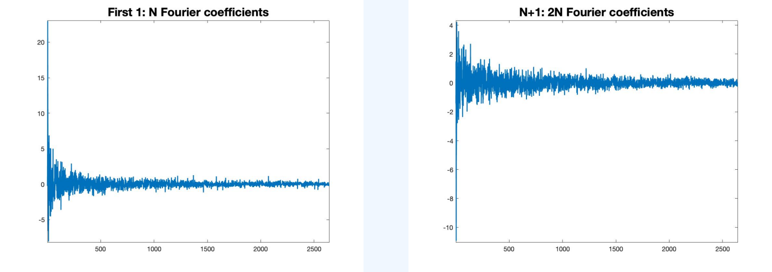


Figure: The main component $(\mathbf{U}^T + \mathbf{V}^T)\mathbf{x}_0/2$ (left) and companion component $(\mathbf{U}^T - \mathbf{V}^T)\mathbf{x}_0/2$ of the SVD-based GFT $\mathcal{F}\mathbf{x}_0$ of the signal $\mathbf{x}_0$, where $\mathbf{x}_0$ is a piecewise constant signal on the weighted directed Minnesota traffic graph with the weights a_{ij} on adjacent edges (j, i) being randomly chosen in the interval $[0, 2]$, and $\mathbf{L} = \mathbf{U}\Sigma\mathbf{V}^T = \sum_{n=1}^{N} \sigma_n \mathbf{u}_n \mathbf{v}_n^T$. The relative percentage of signal energy $(\sum_{k=0}^{M-1} |(\mathbf{u}_k + \mathbf{v}_k)^T \mathbf{x}_0/2|^2 + |(\mathbf{u}_k - \mathbf{v}_k)^T \mathbf{x}_0/2|^2)^{1/2}/\|\mathbf{x}_0\|_2$ for the first $M = 20, 50$ and 100 frequencies (about 0.76%, 1.89% and 3.79% of the total 2640 frequencies) are $0.7005, 0.7655, 0.8166$, where $\|\mathbf{x}\|_2 = \sqrt{\mathbf{x}^T\mathbf{x}}$.

Graph Signal Processing, Graph Laplacian and Graph Shifts

Graph Signal Processing

- Graph signal processing provides an innovative framework to handle data residing on spatially distributed sensor networks, neural networks, social networks and many other.
- A graph can be represented by $\mathcal{G} := (V, E)$, where $V = \{1, 2, \cdots, N\}$ denotes the node set and $E = \{e_{ij}\}_{i,j \in V}$ contains the edges linking the nodes. The graph topology offers a flexible tool to model the interrelationship between data on networks [2]

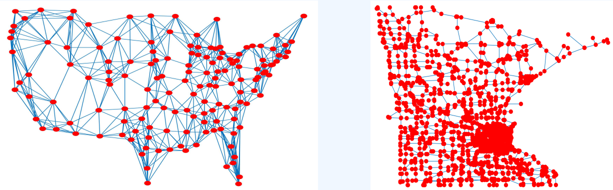
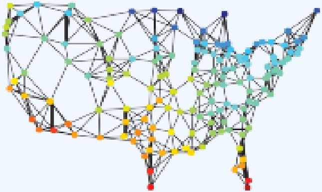
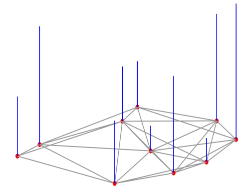


Figure: US temperature graph (left) and Minnesota traffic graph (right).

[2]Antonio Ortega, *Introduction to Graph Signal Processing*, Cambridge
ess, 2021

- A graph signal is defined as a vector residing on the graph nodes, denoted by $\mathbf{x} = [x_1, x_2, \cdots, x_N]^T$.





- Adjacency matrix $\mathbf{A} = [a_{ij}]_{i,j \in V}$ of a (un)directed graph $\mathcal{G} = (V, E)$, where $a_{ij} = w_{ij}$ is (i, j) is an edge, and $a_{ij} = 0$ if (i, j) is not an edge.
- Degree matrix $\mathbf{D} = \mathrm{diag}(d_i)_{i \in V}$, where $d_i = \sum_{j \in V} a_{ij}$ is the number of edges connecting to the vertex i.
- Laplacian $\mathbf{L} = \mathbf{D} - \mathbf{A}$ and symmetrically normalized Laplacian $\mathbf{L}^{\mathrm{sym}} = \mathbf{D}^{-1/2}\mathbf{L}\mathbf{D}^{-1/2}$ (all eigenvalues of $\mathbf{L}$ is nonnegative, and all eigenvalues of $\mathbf{L}^{\mathrm{sym}}$ are contained in $[0, 2]$ for undirected graphs.)

- **Graph shifts S** $= (s(i,j))_{i,j\in V}$ if $s(i,j) \neq 0$ only if either $j = i$ or $(i,j) \in E$.
- Examples: Adjacent matrix **A**, Laplacian **L**, symmetric (random walk) normalized Laplacian matrix, and their variations.
- **Commutative graph shifts** $\mathbf{S}_1, \ldots, \mathbf{S}_d$ if $\mathbf{S}_i\mathbf{S}_j = \mathbf{S}_j\mathbf{S}_i$, $1 \leq i, j \leq d$.
- The concept of commutative graph shifts may play a similar role in graph signal processing as the one-order delay $z_1^{-1}, \ldots, z_d^{-1}$ in classical multi-dimensional signal processing, and in practice graph shifts are selected to have specific features and physical interpretation [3]

[3] N. Emirov, C. Cheng, J. Jiang and Q. Sun, Polynomial graph filter of multiple shifts and distributed implementation of inverse filtering *Sampling Theory, Signal Processing, and Data Analysis*, 20(2022), Article No. 2

UCF

From Laplacian to graph shifts

- **Graph shifts S** $= (s(i,j))_{i,j\in V}$ if $s(i,j) \neq 0$ only if either $j = i$ or $(i,j) \in E$.
- Examples: Adjacent matrix **A**, Laplacian **L**, symmetric (random walk) normalized Laplacian matrix, and their variations.
- **Commutative graph shifts** $\mathbf{S}_1, \ldots, \mathbf{S}_d$ if $\mathbf{S}_i\mathbf{S}_j = \mathbf{S}_j\mathbf{S}_i,\ 1 \leq i,j \leq d$.
- The concept of commutative graph shifts may play a similar role in graph signal processing as the one-order delay $z_1^{-1}, \ldots, z_d^{-1}$ in classical multi-dimensional signal processing, and in practice graph shifts are selected to have specific features and physical interpretation [3]

[3] N. Emirov, C. Cheng, J. Jiang and Q. Sun, Polynomial graph filter of multiple shifts and distributed implementation of inverse filtering *Sampling Theory, Signal Processing, and Data Analysis*, 20(2022), Article No. 2

UCF

- Given two finite graphs $\mathcal{G}_1 = (V_1, E_1)$ and $\mathcal{G}_2 = (V_2, E_2)$ with adjacency matrices $\mathbf{A}_1$ and $\mathbf{A}_2$, define their **Cartesian product graph** $\mathcal{G}_1 \times \mathcal{G}_2$ has vertex set $V_1 \times V_2$ and adjacency matrix given by $\mathbf{A} = \mathbf{A}_1 \otimes \mathbf{I}_{\#V_2} + \mathbf{I}_{\#V_1} \otimes \mathbf{A}_2$. (Kronecker product)
- $\mathbf{L}_1^{\rm sym} \otimes \mathbf{I}_{\#V_2}$ and $\mathbf{I}_{\#V_1} \otimes \mathbf{L}_2^{\rm sym}$ are graph filters of the Cartesian product graph $\mathcal{G}_1 \times \mathcal{G}_2$, where $\mathbf{L}_i^{\rm sym}$ are symmetric normalized Laplacian matrices of the graph $\mathcal{G}_i, i = 1, 2$.
- Property: Let $\mathbf{S}_1, \ldots, \mathbf{S}_d$ be commutative graph shifts. Then they can be **upper-triangularized simultaneously** over $\mathbb{C}$

$$\widehat{\mathbf{S}}_k = \mathbf{U}\mathbf{S}_k\mathbf{U}^H, 1 \leq k \leq d$$

are upper triangular matrices for some unitary matrix $\mathbf{U}$. Furthermore, if $\mathbf{S}_k$ are symmetric and real-valued, then $\mathbf{U}$ can be chosen to be orthogonal and $\widehat{\mathbf{S}}_k, 1 \leq k \leq d$ be diagonal. [4]

[4] Theorem 2.3.3 of the book Matrix Analysis by Horn and Johnson, Cambridge University Press, (2012); N. Emirov, C. Cheng, J. Jiang and Q. Sun, Polynomial graph filter of multiple shifts and distributed implementation of inverse filtering *Sampling Theory, Signal Processing, and Data Analysis*, 20(2022), Article No. 2

- Given two finite graphs $\mathcal{G}_1 = (V_1, E_1)$ and $\mathcal{G}_2 = (V_2, E_2)$ with adjacency matrices $\mathbf{A}_1$ and $\mathbf{A}_2$, define their **Cartesian product graph** $\mathcal{G}_1 \times \mathcal{G}_2$ has vertex set $V_1 \times V_2$ and adjacency matrix given by $\mathbf{A} = \mathbf{A}_1 \otimes \mathbf{I}_{\#V_2} + \mathbf{I}_{\#V_1} \otimes \mathbf{A}_2$. (Kronecker product)
- $\mathbf{L}_1^{\mathrm{sym}} \otimes \mathbf{I}_{\#V_2}$ and $\mathbf{I}_{\#V_1} \otimes \mathbf{L}_2^{\mathrm{sym}}$ are graph filters of the Cartesian product graph $\mathcal{G}_1 \times \mathcal{G}_2$, where $\mathbf{L}_i^{\mathrm{sym}}$ are symmetric normalized Laplacian matrices of the graph $\mathcal{G}_i, i = 1, 2$.
- Property: Let $\mathbf{S}_1, \ldots, \mathbf{S}_d$ be commutative graph shifts. Then they can be **upper-triangularized simultaneously** over $\mathbb{C}$

$$\widehat{\mathbf{S}}_k = \mathbf{U}\mathbf{S}_k\mathbf{U}^H, 1 \le k \le d$$

are upper triangular matrices for some unitary matrix $\mathbf{U}$. Furthermore, if $\mathbf{S}_k$ are symmetric and real-valued, then $\mathbf{U}$ can be chosen to be orthogonal and $\widehat{\mathbf{S}}_k, 1 \le k \le d$ be diagonal. [4]

[4]Theorem 2.3.3 of the book Matrix Analysis by Horn and Johnson, Cambridge University Press, (2012); N. Emirov, C. Cheng, J. Jiang and Q. Sun, Polynomial graph filter of multiple shifts and distributed implementation of inverse filtering *Sampling Theory, Signal Processing, and Data Analysis*, 20(2022), Article No. 2

UCF

Graph signal processing, graph Laplacian and graph shifts: Summary

- Graph signal processing provides an innovative framework to handle data on networks.
- Graph shifts are **building blocks** for graph signal processing and they are designed and selected to have specific features and physical interpretation.
- Laplacian $\mathbf{L} = \mathbf{D} - \mathbf{A}$ and symmetrically normalized Laplacian $\mathbf{L}^{\mathrm{sym}} = \mathbf{D}^{-1/2}\mathbf{L}\mathbf{D}^{-1/2}$ and their variants are illustrative examples of graph shifts.
- Symmetric commutative graph shifts could be **diagonalized simultaneously** by some orthogonal matrix.

Graph Fourier transform on undirected graphs

- **Commutative graph shifts $\mathbf{S}_1, \ldots, \mathbf{S}_d$** if $\mathbf{S}_i\mathbf{S}_j = \mathbf{S}_j\mathbf{S}_i,\ 1 \leq i, j \leq d$.
- Starting from $\mathbf{S}_k, 1 \leq k \leq d$ being symmetric, real-valued and commutative, there exists an orthgonal matrix $\mathbf{U} = [\mathbf{u}_1, \ldots, \mathbf{u}_N]^T$ such that

$$\mathbf{S}_k = \sum_{n=1}^{N} \lambda_k(n)\mathbf{u}_k\mathbf{u}_k^T. \qquad (1)$$

(simultaneous eigendecomposition)

- Define graph Fourier transform (GFT) of a graph signal $\mathbf{x}$ by

$$\widehat{\mathbf{x}} = \mathbf{U}^T\mathbf{x} = [\langle \mathbf{u}_1, \mathbf{x}\rangle, \ldots, \langle \mathbf{u}_N, \mathbf{x}\rangle]^T.$$

- Model of variations (GFT): $\mathbf{u}_1, \ldots, \mathbf{u}_N$.
- Frequencies of GFT: $\boldsymbol{\lambda}_1 = [\lambda_1(1), \ldots, \lambda_k(1)], \ldots, \boldsymbol{\lambda}_N = [\lambda_1(N), \ldots, \lambda_k(N)]$.

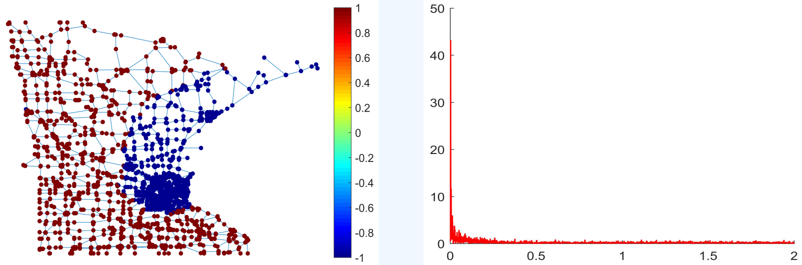


Figure: Left: Piecewise signal $\mathbf{x}$ on the undirected Minnesota traffic graph; Right (coordinate $\lambda_n \in [0, 2]$): magnitude of GFT $|\hat{\mathbf{x}}| = [|\langle \mathbf{u}_1, \mathbf{x}\rangle|, \ldots, |\langle \mathbf{u}_N, \mathbf{x}\rangle|]^T$, where $\mathbf{L} = \mathbf{U}\Lambda\mathbf{U}^T = \sum_{n=1}^{N} \lambda_n \mathbf{u}_n \mathbf{u}_n^T$ (Eigendecomposition of the Laplacian $\mathbf{L}$ (the graph shift))

Observation: The graph signal $\mathbf{x}$ is decomposed into different frequency components (mode of variation) effectively $\mathbf{x} = \sum_{n=1}^{N} \langle \mathbf{u}_n, \mathbf{x}\rangle \mathbf{u}_n$ (frequency components $\langle \mathbf{u}_n, \mathbf{x}\rangle \mathbf{u}_n$; modes of variation $\mathbf{u}_n$).

- **Polynomial filter**: Let $\mathbf{S}_1, \ldots, \mathbf{S}_d$ be commutative filters. A graph filter $\mathbf{A}$ is a polynomial filter if

$$\mathbf{A} = h(\mathbf{S}_1, \ldots, \mathbf{S}_d) = h_0 \mathbf{I} + \sum_{l=1}^{L} \sum_{\alpha_1 + \ldots + \alpha_d = l} p_{\alpha_1, \ldots, \alpha_d} \mathbf{S}_1^{\alpha_1} \cdots \mathbf{S}_d^{\alpha_d}, \quad (2)$$

- The filtering procedure $\mathbf{x} \to \mathbf{Ax}$ can be implemented in a one-hop communication network. [5]
- Convolution associated with **b**: $\widehat{\mathbf{b} * \mathbf{x}} = \widehat{\mathbf{b}} \otimes \widehat{\mathbf{x}}$
- In the frequency domain, the filtering procedure can be described as $\widehat{\mathbf{Ax}} = h(\Lambda_1, \ldots \Lambda_d)\widehat{\mathbf{x}}$. Hence **a polynomial filter is a convolution**. The converse is true if frequencies $\lambda_1, \ldots, \lambda_N$ of the GFT are distinct.

[5] N. Emirov, C. Cheng, J. Jiang and Q. Sun, *Sampling Theory, Signal Processing, and Data Analysis*, 20(2022), Article No. 2

- **Polynomial filter**: Let $\mathbf{S}_1, \ldots, \mathbf{S}_d$ be commutative filters. A graph filter $\mathbf{A}$ is a polynomial filter if

$$\mathbf{A} = h(\mathbf{S}_1, \ldots, \mathbf{S}_d) = h_0 \mathbf{I} + \sum_{l=1}^{L} \sum_{\alpha_1+\ldots+\alpha_d=l} p_{\alpha_1,\ldots,\alpha_d} \mathbf{S}_1^{\alpha_1} \cdots \mathbf{S}_d^{\alpha_d}, \quad (2)$$

- The filtering procedure $\mathbf{x} \to \mathbf{A}\mathbf{x}$ can be implemented in a one-hop communication network. [5]
- Convolution associated with $\mathbf{b}$: $\widehat{\mathbf{b} * \mathbf{x}} = \widehat{\mathbf{b}} \otimes \widehat{\mathbf{x}}$
- In the frequency domain, the filtering procedure can be described as $\widehat{\mathbf{A}\mathbf{x}} = h(\Lambda_1, \ldots \Lambda_d)\widehat{\mathbf{x}}$. Hence **a polynomial filter is a convolution**. The converse is true if frequencies $\lambda_1, \ldots, \lambda_N$ of the GFT are distinct.

[5] N. Emirov, C. Cheng, J. Jiang and Q. Sun, *Sampling Theory, Signal Processing, and Data Analysis*, 20(2022), Article No. 2

Graph shifts and graph Fourier transform on undirected graphs: Summary

- **Commutative graph shifts $\mathbf{S}_1, \ldots, \mathbf{S}_d$** on undirected graphs are selected to be symmetric, real-valued and commutative.
- Define **GFT** on undirected graph by $\hat{\mathbf{x}} = \mathbf{U}^T\mathbf{x}$ where $\mathbf{U}$ is the orthogonal matrix $\mathbf{U}$ to diagonalize $\mathbf{S}_1, \ldots, \mathbf{S}_d$.
- GFT provide a tool to decompose graph signals into different frequency components and **effectively represent** graph signals with regularity using **various modes of variation**.
- Polynomial filtering is widely used in graph signal processing. It can be implemented distributedly in the one-hop communication network. **Polynomial filtering is a convolution** and the converse holds if all frequencies are distinct.

Graph Fourier Transform on Directed Graphs I

The conventional definition of **GFT** on undirected graph $\hat{\mathbf{x}} = \mathbf{U}^T\mathbf{x}$ is based on simultaneous eigendecomposition of graph shifts $\mathbf{S}_1, \ldots, \mathbf{S}_d$,

The eigendecomposition approach does not apply for the directed setting directly. Various approaches to define GFT on directed graph have been proposed. The following are some of them:

- Jodan decomposition $\mathbf{L} = \mathbf{B}^{-1}\mathbf{J}\mathbf{B}$, where $\mathbf{J}$ is diagonal and define $\hat{\mathbf{x}} = \mathbf{B}\mathbf{x}$.
- Magnetic Laplacian $L^{(q)} = \mathbf{V}_q\Lambda_q\mathbf{V}_q^H, 0 \leq q < 1$ and define $\hat{\mathbf{x}} = \mathbf{V}_q^H\mathbf{x}$.
- Optimization-based: Use certain optimization to find model of variation $\mathbf{u}_1, \ldots, \mathbf{u}_N$, and define $\widehat{\mathbf{x}} = [\langle \mathbf{u}_1, \mathbf{x}\rangle, \ldots, \langle \mathbf{u}_N, \mathbf{x}\rangle]^T$.
- SVD-based: $\mathbf{L} = \mathbf{U}\Sigma\mathbf{V}$ (SVD), and define DFT by $\widehat{\mathbf{x}} = \begin{bmatrix} (\mathbf{U}+\mathbf{V})\mathbf{x}/2 \\ (\mathbf{U}-\mathbf{V})\mathbf{x}/2 \end{bmatrix}$.
- Polar decomposition of adjacent matrix $\mathbf{A} = \mathbf{P}\mathbf{Q} = \mathbf{Q}\mathbf{F}$, where $\mathbf{P}, \mathbf{F}$ are positive semi-definite (PSD) Hermitian matrix and $\mathbf{Q}$ is unitary. Write $\mathbf{P} = \mathbf{V}_p\Lambda_p\mathbf{V}_p$, $\mathbf{F} = \mathbf{V}_f\Lambda_f\mathbf{V}_f^T$ and $\mathbf{Q} = \mathbf{U}_q\Lambda_q\mathbf{U}_q^H$.
  ... as GFT

The conventional definition of **GFT** on undirected graph $\hat{\mathbf{x}} = \mathbf{U}^T\mathbf{x}$ is based on simultaneous eigendecomposition of graph shifts $\mathbf{S}_1, \ldots, \mathbf{S}_d$,

The eigendecomposition approach does not apply for the directed setting directly. Various approaches to define GFT on directed graph have been proposed. The following are some of them:

- Jodan decomposition $\mathbf{L} = \mathbf{B}^{-1}\mathbf{J}\mathbf{B}$, where $\mathbf{J}$ is diagonal and define $\hat{\mathbf{x}} = \mathbf{B}\mathbf{x}$.
- Magnetic Laplacian $L^{(q)} = \mathbf{V}_q \Lambda_q \mathbf{V}_q^H, 0 \leq q < 1$ and define $\hat{\mathbf{x}} = \mathbf{V}_q^H\mathbf{x}$.
- Optimization-based: Use certain optimization to find model of variation $\mathbf{u}_1, \ldots, \mathbf{u}_N$, and define $\widehat{\mathbf{x}} = [\langle \mathbf{u}_1, \mathbf{x}\rangle, \ldots, \langle \mathbf{u}_N, \mathbf{x}\rangle]^T$.
- SVD-based: $\mathbf{L} = \mathbf{U}\Sigma\mathbf{V}$ (SVD), and define DFT by $\widehat{\mathbf{x}} = \begin{bmatrix} (\mathbf{U}+\mathbf{V})\mathbf{x}/2 \\ (\mathbf{U}-\mathbf{V})\mathbf{x}/2 \end{bmatrix}$.
- Polar decomposition of adjacent matrix $\mathbf{A} = \mathbf{P}\mathbf{Q} = \mathbf{Q}\mathbf{F}$, where $\mathbf{P}, \mathbf{F}$ are positive semi-definite (PSD) Hermitian matrix and $\mathbf{Q}$ is unitary. Write $\mathbf{P} = \mathbf{V}_p\Lambda_p\mathbf{V}_p$, $\mathbf{F} = \mathbf{V}_f\Lambda_f\mathbf{V}_f^T$ and $\mathbf{Q} = \mathbf{U}_q\Lambda_q\mathbf{U}_q^H$.

as GFT

The conventional definition of **GFT** on undirected graph $\hat{\mathbf{x}} = \mathbf{U}^T\mathbf{x}$ is based on simultaneous eigendecomposition of graph shifts $\mathbf{S}_1, \ldots, \mathbf{S}_d$,
The eigendecomposition approach does not apply for the directed setting directly. Various approaches to define GFT on directed graph have been proposed. The following are some of them:

- Jodan decomposition $\mathbf{L} = \mathbf{B}^{-1}\mathbf{J}\mathbf{B}$, where $\mathbf{J}$ is diagonal and define $\hat{\mathbf{x}} = \mathbf{B}\mathbf{x}$.
- Magnetic Laplacian $L^{(q)} = \mathbf{V}_q\Lambda_q\mathbf{V}_q^H, 0 \leq q < 1$ and define $\hat{\mathbf{x}} = \mathbf{V}_q^H\mathbf{x}$.
- Optimization-based: Use certain optimization to find model of variation $\mathbf{u}_1, \ldots, \mathbf{u}_N$, and define $\widehat{\mathbf{x}} = [\langle \mathbf{u}_1, \mathbf{x}\rangle, \ldots, \langle \mathbf{u}_N, \mathbf{x}\rangle]^T$.
- SVD-based: $\mathbf{L} = \mathbf{U}\Sigma\mathbf{V}$ (SVD), and define DFT by $\widehat{\mathbf{x}} = \begin{bmatrix} (\mathbf{U}+\mathbf{V})\mathbf{x}/2 \\ (\mathbf{U}-\mathbf{V})\mathbf{x}/2 \end{bmatrix}$.
- Polar decomposition of adjacent matrix $\mathbf{A} = \mathbf{P}\mathbf{Q} = \mathbf{Q}\mathbf{F}$, where $\mathbf{P}, \mathbf{F}$ are positive semi-definite (PSD) Hermitian matrix and $\mathbf{Q}$ is unitary. Write $\mathbf{P} = \mathbf{V}_p\Lambda_p\mathbf{V}_p$, $\mathbf{F} = \mathbf{V}_f\Lambda_f\mathbf{V}_f^T$ and $\mathbf{Q} = \mathbf{U}_q\Lambda_q\mathbf{U}_q^H$.

as GFT

- Jordan decomposition for Laplacian: $\mathbf{L} = \mathbf{BJB}^{-1}$, and define $\widehat{\mathbf{x}} = \mathbf{B}^{-1}\mathbf{x}$. [6].
- Convenience for graph signal processing: Polynomial filtering is widely used in graph signal processing. It can be implemented distributedly in the one-hop communication network. **Polynomial filtering is a convolution** and the converse holds if all frequencies are distinct.

[6] J. A. Deri and J. M. F. Moura, IEEE J. Sel. Top. Signal Process., vol. 11, no. 6, pp. 785-795, Sept. 2017; J. Domingos and J. M. F. Moura, IEEE Trans. Signal Process., vol. 68, pp. 4422-4437, July 2020; A. Sandryhaila and J. M. F. Moura, quency analysis," IEEE Trans. Signal Process., vol. 62, no. 12, pp. 3042-3054, June 2014; A. Sandryhaila and J. M. F. Moura, IEEE Signal Process. Mag., vol. 31, no. 5, pp. 80-90, Sept. 2014; R. Singh, A. Chakraborty, and B. Manoj, Proc. IEEE Int. Conf. Signal Process. Commun., 2016, pp.1-5.

GFT based on Jordan decomposition

- Jordan decomposition for Laplacian: $\mathbf{L} = \mathbf{B}\mathbf{J}\mathbf{B}^{-1}$, and define $\widehat{\mathbf{x}} = \mathbf{B}^{-1}\mathbf{x}$.
- Convenience for graph signal processing: **Polynomial filtering is a convolution** and the converse holds if all frequencies are distinct.
- The GFT could have **complex** frequencies (diagonal entries of $\mathbf{J}$) and the Parseval identity $\|\hat{\mathbf{x}}\| = \|\mathbf{x}\|$ does **not** hold in general.
- Jordan decomposition of the Laplacian on directed graphs could be **numerically unstable and computationally expensive**, and hence it could be difficult to be applied for graph spectral analysis and decomposition on a large network[7]
- Not all matrices can be diagonalized.

[7] J. Domingos and J. M. F. Moura, IEEE Trans. Signal Process., vol. 68, pp. 4422-4437, July 2020.

- Connectivity matrix $A = (W + W^T)/2$ and directionality matrix $\Gamma^{(q)}$ with entries $\gamma_{v,v'} = \exp(2\pi i q(w_{v,v'} - w_{v',n}))$ where $0 \leq q < 1$ is the rotation matrix and W is the adjacent matrix of the directed graph.
- Define magnetic Laplacian $L^{(q)} = D - \Gamma^{(q)} \otimes \mathbf{A}$ where D is a diagonal matrix with diagonal entries $\sum_{v' \in V} w_{v,v'}, v \in V$.
- $L^{(q)} = \mathbf{V}_q \Lambda_q \mathbf{V}_q^H$, where $\mathbf{V}_q$ is unitary and Λ_q is a diagonal matrix with diagonal entries $\lambda_{q,k}, 0 \leq k \leq N-1$, in a nondecreasing order.
- DFT $\hat{\mathbf{x}} = \mathbf{V}_q^H \mathbf{x}$ [8]

[8] S. Furutani, T. Shibahara, M. Akiyama, K. Hato, and M. Aida, "Graph signal processing for directed graphs based on the Hermitian Laplacian," in Proc. Joint Eur. Conf. Mach. Learn. Knowl. Discov. Databases, 2020, pp. 447–463; X. Zhang, Y. He, N. Brugnone, M. Perlmutter, and M. J. Hirn, "MagNet: A neural network for directed graphs," in Proc. Adv. Neural Inf. Process Syst., 2021, pp. 27003–27015.

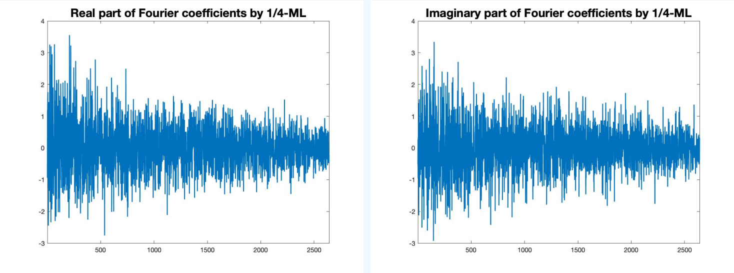


Figure: Plot on the left and right are the real part $\Re(\mathbf{V}_q^H \mathbf{x}_0)$ and imaginary part $\Im(\mathbf{V}_q^H \mathbf{x}_0)$ of the GFT by q-ML $\mathbf{L}^{(q)}$ with $q = 1/4$, where $\mathbf{x}_0$ is the piecewise constant signal on a weighted Minnesota traffic graph. The relative percentage of signal energy $\left(\sum_{k=0}^{M-1} |\mathbf{v}_{k;q}^H \mathbf{x}_0|^2\right)^{1/2} / \|\mathbf{x}_0\|_2$ for the first $M = 20, 50$ and 100 frequencies (about $0.76\%, 1.89\%$ and 3.79% of the total 2640 frequencies) are $0.1289, 0.2385, 0.3289$ respectively.

- $L^{(q)} = \mathbf{V}_q \Lambda_q \mathbf{V}_q^H$, where $\mathbf{V}_q$ is unitary and Λ_q is a diagonal matrix with diagonal entries $\lambda_{q,k}, 0 \leq k \leq N-1$, in a nondecreasing order.
- DFT $\hat{\mathbf{x}} = \mathbf{V}_q^H \mathbf{x}$
- The GFT could have **real** frequencies and the Parseval identity $\|\hat{\mathbf{x}}\| = \|\mathbf{x}\|$ hold.
- How to understand the rotation parameter $0 \leq q < 1$? It is unclear whether graph signals with regularity can decomposed into different frequency components **very effectively**.

GFT based on SVD of graph Laplacian

- Singular value decomposition (SVD) of the Laplacian ,

$$\mathbf{L} = \mathbf{U}\mathbf{\Sigma}\mathbf{V}^T = \sum_{k=0}^{N-1} \sigma_k \mathbf{u}_k \mathbf{v}_k^T \tag{3}$$

  where $\sigma_k, 0 \leq k \leq N-1$ are its nonnegative singular values considered as frequencies of GFT and $\mathbf{u}_k, \mathbf{v}_k, 0 \leq k \leq N-1$, as the associated left/right frequency components, where

$$\mathbf{U} = [\mathbf{u}_0, \ldots, \mathbf{u}_{N-1}] \text{ and } \mathbf{V} = [\mathbf{v}_0, \ldots, \mathbf{v}_{N-1}] \tag{4}$$

  are orthogonal matrices, and the diagonal matrix $\mathbf{\Sigma} = \mathrm{diag}(\sigma_0, \ldots, \sigma_{N-1})$ has singular values deployed on the diagonal in a nondecreasing order, i.e., $0 = \sigma_0 \leq \sigma_1 \leq \ldots \leq \sigma_{N-1}$.
- GFT $\widehat{\mathbf{x}} = \begin{pmatrix} (\mathbf{U}^T + \mathbf{V}^T)\mathbf{x}/2 \\ (\mathbf{U}^T - \mathbf{V}^T)\mathbf{x}/2 \end{pmatrix}$

UCF

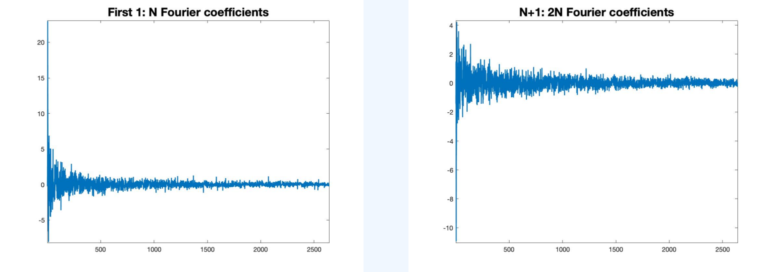


Figure: The main component $(\mathbf{U}^T + \mathbf{V}^T)\mathbf{x}_0/2$ (left) and companion component $(\mathbf{U}^T - \mathbf{V}^T)\mathbf{x}_0/2$ of the SVD-based GFT $\mathcal{F}\mathbf{x}_0$ of the signal $\mathbf{x}_0$, where $\mathbf{x}_0$ is a piecewise constant signal on the weighted directed Minnesota traffic graph with the weights a_{ij} on adjacent edges (j, i) being randomly chosen in the interval $[0, 2]$. The relative percentage of signal energy $(\sum_{k=0}^{M-1} |(\mathbf{u}_k + \mathbf{v}_k)^T\mathbf{x}_0/2|^2 + |(\mathbf{u}_k - \mathbf{v}_k)^T\mathbf{x}_0/2|^2)^{1/2}/\|\mathbf{x}_0\|_2$ for the first $M = 20, 50$ and 100 frequencies (about $0.76\%, 1.89\%$ and 3.79% of the total 2640 frequencies) are $0.7005, 0.7655, 0.8166$, where $\|\mathbf{x}\|_2 = \sqrt{\mathbf{x}^T\mathbf{x}}$.

UCF

- On the *directed circulant graph* $\mathcal{C}_d := \mathcal{C}_d(N)$, define the discrete Fourier transform matrix by

$$\mathrm{DFT}(\mathbf{x}) = \mathbf{W}^H \mathbf{x},$$

where $\mathbf{W} := \left(N^{-1/2}\omega_N^{ij}\right)_{0\le i,j\le N-1}$ and where $\omega_N = \exp(2\pi\sqrt{-1}/N)$ is the N-th root of the unit.

- The GFT based on SVD is given by

$$\mathcal{F}\mathbf{x} = \frac{1}{2}\begin{pmatrix}\mathbf{P}_1 & \mathbf{O}\\ \mathbf{O} & \mathbf{P}_1\end{pmatrix}^T \begin{pmatrix}\mathbf{R} & \mathbf{O}\\ \mathbf{O} & \mathbf{R}\end{pmatrix}^H \begin{pmatrix}\mathbf{P}_0 & \mathbf{O}\\ \mathbf{O} & \mathbf{P}_0\end{pmatrix} \begin{pmatrix}\mathbf{\Theta} & \mathbf{\Theta}\\ \mathbf{I} & -\mathbf{I}\end{pmatrix}^H \begin{pmatrix}\mathrm{DFT}(\mathbf{x})\\ \mathrm{DFT}(\mathbf{x})\end{pmatrix}, \tag{5}$$

where $\mathbf{R}$ is a rotation, $\mathbf{\Theta}$ is a phase adjustment matrix and $\mathbf{P}_0$ and $\mathbf{P}_1$ are permutations. [9]

[9] Y. Chen, C. Cheng and Q. Sun, Graph Fourier transform based on singular value decomposition of directed Laplacian, *Sampling Theory, Signal Processing, and Data Analysis*, 12(2023), article no. 24

Compared with the magnetic Laplacian, it decompose graph signals into different frequency components and represent them **more effectively**.

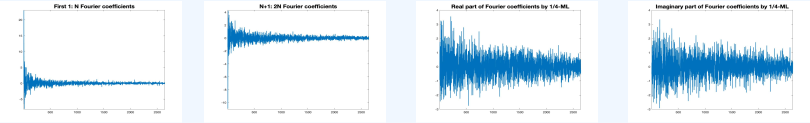


Figure: The relative percentage of signal energy for the first $M = 20, 50$ and 100 frequencies (about 0.76%, 1.89% and 3.79% of the total 2640 frequencies) are 0.7005, 0.7655, 0.8166 (left two, SVD-based DFT). The relative percentage of signal energy for the first $M = 20, 50$ and 100 frequencies (about 0.76%, 1.89% and 3.79% of the total 2640 frequencies) are 0.1289, 0.2385, 0.3289 respectively (right two, Magnetic-based DFT).

UCF

- Compared with the GFT based on a Jordan decomposition of Laplacian, a significant advantage of the proposed SVD-based GFT is the **numerical stability and low computational cost**.
- SVD-based GFT: real frequencies and Parseval identity holds, while Jordan-based GFT: complex frequencies and Parseval identity does not apply.
- However, SVD-based GFT can **not** be used to define convolution and hence establish the equivalence between convolution and polynomial filtering procedure.

UCF

GFT on Directed Graphs I: Summary

- GFT based on Jordan decomposition of Laplacian: polynomial filtering procedure is a convolution. However, numerical instable and computationally expensive for large networks, complex frequencies, no Parseval identity
- GFT based on Magnetic Laplacian: real frequencies, Parseval identity. However it is unclear whether graph signals with regularity can decomposed into different frequency components effectively, and inconvenience to define convolution and establish the equivalence between convolution and polynomial filtering procedure
- GFT based on SVD of Laplacian: real frequencies, Parseval identity, numerical stability and low computational cost, effective representation of graph signals with regularity; however inconvenience to define convolution and establish the equivalence between convolution and polynomial filtering procedure

Graph Fourier Transform on Directed Graphs II: Product Graph

GFT: Product space

- Let $\mathcal{G}_1 = (V_1, E_1)$ and $\mathcal{G}_2 = (V_2, E_2)$ be two directed graphs of orders N_1 and N_2. Our illustrative example are the temporal line graphs and spatial graphs to describe time-varying data sets on directed networks.
- The *Cartesian product graph* $\mathcal{G} := \mathcal{G}_1 \boxtimes \mathcal{G}_2 = (V_1 \times V_2, E_1 \boxtimes E_2)$ has vertices $(v_1, v_2) \in V_1 \times V_2$ and edges between vertices (v_1, v_2) and $(\tilde{v}_1, \tilde{v}_2)$ if either $(v_1, \tilde{v}_1) \in E_1$ and $\tilde{v}_2 = v_2$, or $\tilde{v}_1 = v_1$ and $(v_2, \tilde{v}_2) \in E_2$
- Denote the adjacency, in-degree and (in-degree) Laplacian matrices of graphs $\mathcal{G}_l$ by $\mathbf{A}_l, \mathbf{D}_l$ and $\mathbf{L}_l = \mathbf{D}_l - \mathbf{A}_l, l = 1, 2$, respectively. Then

$$\mathbf{L}_{\boxtimes} = \mathbf{L}_1 \otimes \mathbf{I}_{N_2} + \mathbf{I}_{N_1} \otimes \mathbf{L}_2. \tag{6}$$

UCF

- $\mathbf{L}_\boxtimes = \mathbf{L}_1 \otimes \mathbf{I}_{N_2} + \mathbf{I}_{N_1} \otimes \mathbf{L}_2$.
- SVD decomposition $\mathbf{L}_\boxtimes = \mathbf{U}_\boxtimes \mathbf{\Sigma} \mathbf{V}_\boxtimes^T = \sum_{k=0}^{N-1} \sigma_k \mathbf{u}_k \mathbf{v}_k^T$, where $N = N_1 N_2$, $\mathbf{U}_\boxtimes = [\mathbf{u}_0, \ldots, \mathbf{u}_{N-1}]$ and $\mathbf{V}_\boxtimes = [\mathbf{v}_0, \ldots, \mathbf{v}_{N-1}]$ are orthogonal matrices, and the diagonal matrix $\mathbf{\Sigma} = \mathrm{diag}(\sigma_0, \ldots, \sigma_{N-1})$ has singular values of the Laplacian $\mathbf{L}_\boxtimes$ deployed on the diagonal in a nondecreasing order, i.e.,

$$0 = \sigma_0 \leq \sigma_1 \leq \ldots \leq \sigma_{N-1}.$$

- GFT $\mathcal{F}_\boxtimes \mathbf{x} = \begin{pmatrix} (\mathbf{U}_\boxtimes^T + \mathbf{V}_\boxtimes^T)\mathbf{x}/2 \\ (\mathbf{U}_\boxtimes^T - \mathbf{V}_\boxtimes^T)\mathbf{x}/2 \end{pmatrix}$
- The computational complexity to perform the SVD is $O(N_1^3 N_2^3)$.

Theorem

For a frequency bandwidth $M \in \{1, 2, \ldots, N\}$ of the GFT $\mathcal{F}_{\boxtimes}$, define the low frequency component of a graph signal $\mathbf{x}$ on $\mathcal{G}$ with bandwidth M by

$$\mathbf{x}_{M,\boxtimes} = \frac{1}{2}\sum_{k=0}^{M-1}(z_{1,k} + z_{2,k})\mathbf{u}_k + (z_{1,k} - z_{2,k})\mathbf{v}_k, \tag{7}$$

where $z_{1,k} = (\mathbf{u}_k + \mathbf{v}_k)^T\mathbf{x}/2$ and $z_{2,k} = (\mathbf{u}_k - \mathbf{v}_k)^T\mathbf{x}/2, 0 \le k \le M-1$. Then

$$\|\mathbf{x} - \mathbf{x}_{M,\boxtimes}\|_2 \le \frac{1}{2\sigma_{M-1}}\left(\|\mathbf{L}_{\boxtimes}\mathbf{x}\|_2 + \|\mathbf{L}_{\boxtimes}^T\mathbf{x}\|_2\right) \tag{8}$$

where σ_{M-1} is the cut-off frequency of the bandlimiting procedure (7).

- $\mathbf{L}_{\boxtimes} = \mathbf{L}_1 \otimes \mathbf{I}_{N_2} + \mathbf{I}_{N_1} \otimes \mathbf{L}_2$.
- Observation: $\mathbf{L}_1$ and $\mathbf{L}_2$ are graph Laplacian on $\mathcal{G}_1$ and $\mathcal{G}_2$ respectively, and $\mathbf{L}_1 \otimes \mathbf{I}_{N_2}$ and $\mathbf{I}_{N_1} \otimes \mathbf{L}_2$ are commutative graph shifts, representing diffusion on graph $\mathcal{G}_1$ and $\mathcal{G}_2$ respectively.
- SVD of the Laplacian matrices $\mathbf{L}_l, l = 1, 2$:

$$\mathbf{L}_l = \mathbf{U}_l \mathbf{\Sigma}_l \mathbf{V}_l^T = \sum_{i=0}^{N_l - 1} \sigma_{l,i} \mathbf{u}_{l,i} \mathbf{v}_{l,i}^T, \tag{9}$$

where $\sigma_{l,i}, 0 \le i \le N_l - 1$, are singular values of the Laplacian matrix $\mathbf{L}_l$ with a nondecreasing order, $\mathbf{U}_l = [\mathbf{u}_{l,0}, \ldots, \mathbf{u}_{l,N_l-1}]$ and $\mathbf{V}_l = [\mathbf{v}_{l,0}, \ldots, \mathbf{v}_{l,N_l-1}]$ are orthonormal matrices.

- Set $\mathbf{U}_\otimes = \mathbf{U}_1 \otimes \mathbf{U}_2$ and $\mathbf{V}_\otimes = \mathbf{V}_1 \otimes \mathbf{V}_2$.
- Define GFT $\mathcal{F}_\otimes : \mathbb{R}^N \longmapsto \mathbb{R}^{2N}$ on the product graph $\mathcal{G}$ by

$$\mathcal{F}_\otimes \mathbf{x} := \frac{1}{2} \begin{pmatrix} (\mathbf{U}_\otimes + \mathbf{V}_\otimes)^T \mathbf{x} \\ (\mathbf{U}_\otimes - \mathbf{V}_\otimes)^T \mathbf{x} \end{pmatrix} \tag{10}$$

  where $\mathbf{x} \in \mathbb{R}^N$ is a signal on the graph $\mathcal{G}$. [10]
- The computational complexity to perform the SVD is $O(N_1^3 + N_2^3)$ and find models of variation in the GFT $\mathcal{F}_\otimes \mathbf{x}$, comparing with the default one: $O((N_1 N_2)^3)$ and models of variation in the GFT $\mathcal{F}_\boxtimes \mathbf{x}$.

[10] C. Cheng, Y. Chen, Y. J. Lee and Q. Sun, SVD-based graph Fourier transforms on directed product graphs, *IEEE Transactions on Signal and Information Processing over Networks*, 9(2023), 531-541

Theorem

For a frequency bandwidth $1 \leq M \leq N$ of the GFT $\mathcal{F}_{\otimes}$ in (10), define the low frequency component of a graph signal $\mathbf{x}$ on $\mathcal{G}$ with bandwidth M by

$$\mathbf{x}_{M,\otimes} = \frac{1}{2} \sum_{(i,j)\in\mathcal{S}_M} (\mathbf{u}_{1,i} \otimes \mathbf{u}_{2,j})(\mathbf{u}_{1,i} \otimes \mathbf{u}_{2,j})^T \mathbf{x} + (\mathbf{v}_{1,i} \otimes \mathbf{v}_{2,j})(\mathbf{v}_{1,i} \otimes \mathbf{v}_{2,j})^T \mathbf{x}, \quad (11)$$

where $\mathcal{S}_M$ contains all pairs (i,j) with $\sigma_{1,i} + \sigma_{2,j}$ being some $\mu_k, 0 \leq k \leq M-1$. Then

$$\|\mathbf{x} - \mathbf{x}_{M,\otimes}\|_2 \leq \frac{1}{2\mu_{M-1}} \Big(\|(\mathbf{L}_1 \otimes \mathbf{I}_{N_2})\mathbf{x}\|_2 + \|(\mathbf{L}_1^T \otimes \mathbf{I}_{N_2})\mathbf{x}\|_2 + \|(\mathbf{I}_{N_1} \otimes \mathbf{L}_2)\mathbf{x}\|_2 + \|(\mathbf{I}_{N_1} \otimes \mathbf{L}_2^T)\mathbf{x}\|_2 \Big), \quad (12)$$

where μ_{M-1} is the cut-off frequency of the bandlimiting procedure (11).

- The hourly temperature data set measured in Celsius collected at 32 weather stations in the region of Brest (France) in January 2014.
- The temperature data set by matrices $\mathbf{X}_d = [\mathbf{x}_d(t_0) \ldots, \mathbf{x}_d(t_{23})], 1 \le d \le 31$, where the column vectors $\mathbf{x}_d(t_i), 0 \le i \le 23$, are the regional temperature at t_i-th hour of d-th day of January 2014.
- We model the matrices $\mathbf{X}_d, 1 \le d \le 31$, as signals on the Cartesian product graphs $\mathcal{T} \boxtimes \mathcal{S}$ of order $768 = 24 \times 32$, where $\mathcal{T}$ is the unweighted directed line graph with 24 vertices and $\mathcal{S}$ is the directed graph with 32 locations of weather observation stations as vertices and edges constructed by the 5 nearest neighboring stations in physical distances with weight $w(i,j) = 1 + r(i,j)$, where $r(i,j) \in [-0.2, 0.2]$ are randomly and independently selected with uniform distribution

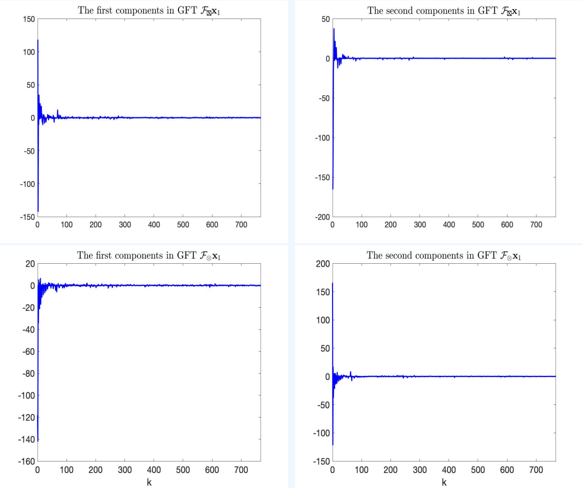


Figure: Plotted on the top left and right are the first component $(\mathbf{U}_{\boxtimes} + \mathbf{V}_{\boxtimes})^T \mathbf{x}_1/2$ and the second component $(\mathbf{U}_{\boxtimes} - \mathbf{V}_{\boxtimes})^T \mathbf{x}_1/2$ of the GFT $\mathcal{F}_{\boxtimes}\mathbf{x}_1$ of the signal $\mathbf{x}_1$ respectively, On the middle left and right are the first component $(\mathbf{U}_{\otimes} + \mathbf{V}_{\otimes})^T \mathbf{x}_1/2$ and the second component $(\mathbf{U}_{\otimes} - \mathbf{V}_{\otimes})^T \mathbf{x}_1/2$ of the GFT $\mathcal{F}_{\otimes}\mathbf{x}_1$ respectively.

- GFT on product space: GFT $\widehat{\mathbf{x}} = \begin{pmatrix} (\mathbf{U}_{\boxtimes}^T + \mathbf{V}_{\boxtimes}^T)\mathbf{x}/2 \\ (\mathbf{U}_{\boxtimes}^T - \mathbf{V}_{\boxtimes}^T)\mathbf{x}/2 \end{pmatrix}$ vs. $\mathcal{F}_{\otimes}\mathbf{x} := \frac{1}{2}\begin{pmatrix} (\mathbf{U}_{\otimes} + \mathbf{V}_{\otimes})^T\mathbf{x} \\ (\mathbf{U}_{\otimes} - \mathbf{V}_{\otimes})^T\mathbf{x} \end{pmatrix}$
- The time to find the left/right frequency components $\mathbf{u}_k, \mathbf{v}_k, 0 \le k \le 767$, of the GFT $\mathcal{F}_{\boxtimes}$ and the ones $\mathbf{u}_{1,i} \otimes \mathbf{u}_{2,j}, \mathbf{v}_{1,i} \otimes \mathbf{v}_{2,j}, 0 \le i \le 23, 0 \le j \le 31$, of the GFT $\mathcal{F}_{\otimes}$ are 0.1456 and 0.0255 seconds.
- It is observed that the hourly temperature data set $\mathbf{X}_1$ has about 99.56% and 99.60% energy concentrated on the first 32 out of total 768 (about 4.167%) frequencies of the GFTs $\mathcal{F}_{\boxtimes}$ and $\mathcal{F}_{\otimes}$, respectively. Hence weather data set has **similar energy concentration**.
- Frequencies σ_k of the GFT $\mathcal{F}_{\boxtimes}$, μ_k of the GFT $\mathcal{F}_{\otimes}$ and $\lambda_{q,k}$ of the GFT $\mathcal{F}_q$ satisfy $0 \le \sigma_k, \mu_k \le 13.3330$ and $0.1174 \le \lambda_{q,k} \le 13.5798, 0 \le k \le 767$. Of more interest, it is observed that $\mu_k - 0.4093 \le \sigma_k \le \mu_k$ and $\mu_k - 0.4738 \le \lambda_{q,k} \le \mu_k + 0.2469, 0 \le k \le 767$. Therefore the GFTs $\mathcal{F}_{\boxtimes}, \mathcal{F}_{\otimes}$ and $\mathcal{F}_q$ have **similar frequency information**

Conclusions and discussions

- Graph shifts are building blocks for graph signal processing and they are designed and selected to have specific features and physical interpretation.
- GFT provide a tool to decompose graph signals into different frequency components and **effectively represent** graph signals with regularity using **various modes of variation**. More mathematical tools should be explored for graph signal processing: convolution neural network, wavelet transform, Gabor analysis etc
- On undirected graph setting, a conventional definition of GFT is based on eigendecomposition of symmetric graph shifts.
- On directed graph setting, **various definitions of GFTs** have been proposed, including GFT based on Jordan decomposition of Laplacian, eigendecomposition of magnetic decomposition, singular value decomposition of Laplacian, polar decomposition of Laplacian, optimization of directed variation.

Personal website: https://sciences.ucf.edu/math/qsun/