Elementary Approximation Theory

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Important Notice

This note is written for the one-semester lectures for Honor Students at National University of Singapore.

Part of this note is based on the following materials: "An Introduction to the Approximation of Functions" by Theodore J. Rivlin (Dover Publications, Inc. 1981), the lecture notes for the module Approximation Theory by Professor Toh Kim Chuan, "Approximation of Functions" by G. G. Lorentz (Chelsea Publishing Co. 1986.), "Approximation by Spline Functions" by Gunther Nurnberger (Springer Verlag 1989), and "Theory of Approximation" by N. I. Achieser (Dover Publications, Inc. 1992).

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Chapter 1

Introduction

The main problem in the theory of approximation can be stated as follows: Suppose that $f(x)$ and $F(x, A_1, \ldots, A_r)$ are two functions on certain point set, where $A_1, \ldots, A_r$ are parameters. It is required to determine the paramaters that the deviation of the function $F(x, A_1, \ldots, A_r)$ from the function $f(x)$ shall be minimum.

Usually the function f to be approximated may have complicated structure and hard to handle, and the approximating function $F(x, A_1, \ldots, A_r)$ should have simple structure and be easily implemented, such as trigonometric polynomials, polynomials, splines, or finite linear combinations of some "simple" functions.

Set

$$\Pi_n := \left\{ \frac{a_0}{2} + \sum_{k=1}^{n} (a_k \cos kx + b_k \sin kx) : \ a_0 \in \mathbf{R}, a_k, b_k \in \mathbf{R} \quad \text{for } 1 \le k \le n \right\},$$

and

$$P_n := \left\{ \sum_{k=0}^{n} a_k x^k : \ a_k \in \mathbf{R} \text{ for } 1 \le k \le n \right\}, \quad n \ge 0.$$

We may take a trigonometric polynomial in Π_{r-1} as the function $F(x, A_1, \ldots, A_r)$ to approximate a periodic function, a polynomial in P_{r-1} to approximate a function on an interval.

For the set X of points $x_0, x_1, \ldots, x_n$ in the interval $[a, b]$ labeled as $a = x_0 < x_1 < \cdots < x_n = b$ and any positive integer m, we denote the set of all C^{m-1} functions on $[a, b]$ which agree with polynomials of degree at most m on each subinterval $[x_{i-1}, x_i], 1 \le i \le n$, by $S_m(X)$. The functions in $S_m(X)$ is known as spline of degree m at the knots $x_0, \ldots, x_n$. Due to the flexibility of knots and higher approximation order by splines, we may take a spline as the function $F(x, A_1, \ldots, A_r)$ if we require that the approximating function concoides with the original function $f(x)$.

We may use different distance to measure the difference between the function $f(x)$ and the approximating function $F(x, A_1, \ldots, A_r)$. For example, if we consider the bounded contiuous functions f and g on a finite interval $[a, b]$, we can take the least upper bound of the absolute value of their difference, i.e.,

$$\|f - g\|_\infty = \sup_{x \in [a,b]} |f(x) - g(x)|,$$

as the distance between two functions. If you consider the approximation of a vector in Euclidean space $\mathbf{R}^n$,

$$\mathbf{R}^n = \{(x_1, \ldots, x_n) : \; x_i \in \mathbf{R}, \; 1 \leq i \leq n\},$$

by vectors in a linear subspace in $\mathbf{R}^n$, we often use the standard distance $|x - y|$ between two points $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$

$$|x - y| = \Big(\sum_{i=1}^{n} |x_i - y_i|^2 \Big)^{1/2},$$

to measure their distance. If the function $f(x)$ represents an image and the approximating function $F(x, A_1, \ldots, A_r)$ represents the compressed image, due to the behaviour of our human vision, we may use

$$\|f - F(\cdot, A_1, \ldots, A_r)\|_1 = \int_a^b |f(x) - F(x, A_1, \ldots, A_r)| dx$$

to measure the difference between the original image and the compressed image and hence to justify the effectiveness of the compression.

For many applications, it is quite important to the explicit construction of the best approximation for a given function f and the parameters $A_1, \ldots, A_r$. For the approximation problem by elements in a finite dimensional subspace M of a Hilbert space H, for any given function in H, we can construct the least approximating element in M explicitly, in fact, the least approximating elment is the orthogonal project on M. It is proved that if H is the space of all 2π-periodic square integrable functions and if M is the space of all trigonomotric polynomials of degree n, i.e., $M = \Pi_n$, then the least square approximating function Π_n is jsut the partial sum of the corresponding Fourier series of the given function in H. The situation to find the best approximation becomes difficult when we consider the approximation in a normed linear space instead of a Hilbert space. The process to find the best approximation is nonlinear in general. For instance, it is known that the best uniform approximating constant to a continuous function on the interval $[a, b]$ is $(M + m)/2$, where M and m are the maximam and minimum of the function on $[a, b]$ (see Section 2.2.1 and 5.2 for details). In general, we do not have any linear/nonlinear algorithm to find the best uniform approximating function in a normed linear space, even for the case $F(x, A_1, \ldots, A_r)$ is a polynomial. So in some situations, we use certain good linear approximations with explicit expression, such as Bernstein polynomials, instead of best approximation.

In some applications, we restrict ourselves on the accurancy to approximate the original function, hence we have certain flexibility on parameters. To this end, we need study the approximation order of certain type of approximations and determine how many paramaters is enough to meet our requirement. For instance, the uniform approximation error of a Hölder continuous function of order $\alpha \in (0, 1]$ by polynomials of degree at most n is dominated by $cn^{-\alpha}$ for some positive constant c. Hence if we want to find a polynomial p to approximate the Hölder continuous function of order α with accurancy $\epsilon > 0$, then from the above observation we see that we can find the polynomial p of degree n_ϵ to meet the requirement, where n_0 is the minimal integer larger than $(c/\epsilon)^{1/\alpha}$.

The inverse problem in the theory of approximation is quite intersting. In that situation, the problem is to study certain properties of the function f through the approximating functions $F(x, A_1, \ldots, A_r)$. It is known that given a 2π-periodic continuous function f if the approximating error by trigonometric polynomials of degree at most n is dominated by $cn^{-\alpha}$, then f is Hölder continuous of order α, where $\alpha \in (0, 1)$ and where $c > 0$ is independent of $n \geq 1$.

Chapter 2

Approximation by Trigonometric Polynomials

In this chapter, we study the uniform approximation problem to a continuous 2π-periodic function by trigonometric polynomials.

2.1 Trigonometric Polynomials and Modulus of Continuity

2.1.1 Fourier Series

???????

2.1.2 Bernstein Theorem

In this section, we prove the Bernstein's theorem and consider the inverse problem of approximation.

Theorem 2.1.1 *Let p be trigonometric polynomial of degree n. Then*

$$\|p'\|_\infty \leq n\|p\|_\infty. \tag{2.1.1}$$

Proof. Suppose on the contrary that the theorem is not true. Then there exists $p \in \Pi_n$ so that

$$\|p\|_\infty = 1 \quad \text{and} \quad \|p'\|_\infty > n.$$

Set $L = \|p'\|_\infty/n$. Without loss of generality we assume that $p'(x_0) = nL$ for some $x_0 \in \mathbf{R}$, otherwise replacing p by $-p$ instead. Define

$$S(x) = L \sin n(x - x_0) - p(x).$$

Then the trigonometric polynomial S takes values of alternating signs at $x_k = x_0 + (2k-1)\pi/(2n), k = 1, \ldots, 2n$. Therefore between two of those points, S has a zero, and hence S has $2n$ different zeros. By Rolle's theorem,

$$S'(x) = nL \cos n(x - x_0) - p'(x)$$

also has $2n$ different zeros. One of those zeros is x_0, since $S'(x_0) = nL - p'(x_0) = 0$. Also,

$$S''(x) = -n^2 L \sin n(x - x_0) - p''(x)$$

vanishes at x_0 since $p'(x)$ takes the maximam. Moreover, S'' has, again by Rolle's theorem, $2n$ zeros between zeros of S'. Therefore S' has at least $2n+1$ zeros, which yields that S'' is identically zero. Hence S is a constant, which a contradiction since S changes sign on the line. □

2.1.3 Modulus of Continuity

In this subsection, we introduce the concept of modulus of continuity, which is a quantity to measure the smoothness of a continuous function. Also we introduce the class of Hölder continuous functions.

Given a continuous function f on a set K, we define the modulus of continuity of f on a set K, to be denoted by $\omega(f, K, \delta)$, or $\omega(f, \delta), \delta > 0$ for short, by

$$\omega(f, \delta) = \sup_{x_1, x_2 \in K, |x_1 - x_2| \leq \delta} |f(x_1) - f(x_2)|.$$

Theorem 2.1.2 *Let f be a continuous function on the interval $[a, b]$, and $\omega(f, \delta), \delta > 0$ be its modulus of continuity. Then*

(i) $\omega(f, \delta_1) \leq \omega(f, \delta_2)$ for any $0 < \delta_1 \leq \delta_2$;

(ii) $\lim_{\delta \to 0} \omega(f, \delta) = 0$ if f is uniform continuous;

(iii) $\omega(f, \lambda\delta) \leq (1 + \lambda)\omega(f, \delta)$ for any $\lambda, \delta > 0$;

(iv) If f has bounded derivative f' on $[a, b]$, then

$$\omega(f, \delta) \leq \|f'\|_\infty \delta. \tag{2.1.2}$$

Proof. The assertion (i) is obvious, the assertion (ii) follows easily from the definition of uniform continuity, and the assertion (iv) is true by mean value theorem.

Now we prove the third assertion. Let the integer n be so chosen that $n \leq \lambda < n+1$. Using the first assertion, we obtain

$$\omega(f, \lambda\delta) \leq \omega(f, (n+1)\delta).$$

Therefore it suffices to prove

$$\omega(f, n\delta) \leq n\omega(f, \delta) \tag{2.1.3}$$

for any positive integer n and any $\delta > 0$. For any $x_1, x_2 \in [a, b]$ with $|x_1 - x_2| < n\delta$, we insert $n-1$ equally spaced knots $z_j = x_1 + (x_2 - x_1)j/n$, $0 \leq j \leq n$, in the interval between x_1 and x_2. Then $z_0 = x_1, z_n = x_2$, $|z_{j+1} - z_j| = |x_2 - x_1|/n < \delta$ for all $0 \leq j \leq n$, and

$$\begin{aligned} |f(x_2) - f(x_1)| &= |f(z_n) - f(z_0)| \leq \sum_{j=0}^{n-1} |f(z_{j+1}) - f(z_j)| \\ &\leq \sum_{j=0}^{n-1} \omega(f, \delta) = n\omega(f, \delta). \end{aligned}$$

Taking supremum on all points $x_1, x_2 \in [a, b]$ with $|x_1 - x_2| \leq n\delta$ in the above estimate yields (2.1.3) and hence the third assertion follows. □

We say that a continuous function f satisfies the *Lipschitz condition* of order $\alpha \in (0, 1]$ with constant K if

$$\omega(f, [a, b], \delta) \leq K\delta^\alpha \quad \forall\, \delta > 0. \tag{2.1.4}$$

A continuous function satisfying Lipschitz condition is also called Hölder continuous function. The index α in (2.1.4) is known as Hölder exponent of f. The class of all continuous functions with Hölder exponent α is denoted by Lip_α or C^α in some literatures. By Theorem 2.1.2, a continuous function with bounded derivative belongs to Lip_1.

Example 2.1.3 *Show that the function f defined by $f(x) := |x|^\alpha, x \in [-1, 1]$, belongs to Lip_α, where $0 < \alpha \leq 1$.*

Solution Let $x_1, x_2 \in [-1,1]$ satisfies $|x_1 - x_2| \le \delta$. Then either $x_1, x_2 \in [-2\delta, 2\delta]$ or $x_1, x_2 \in [-1,1]\backslash[-\delta,\delta]$. For the first case that $x_1, x_2 \in [-2\delta, 2\delta]$,

$$|f(x_1) - f(x_2)| \le |f(x_1)| + |f(x_2)| \le 2(2\delta)^\alpha = 2^{1+\alpha}\delta^\alpha. \tag{2.1.5}$$

For the second case that $x_1, x_2 \in [-1,1]\backslash[-\delta,\delta]$, by mean value theorem, there exists ξ between x_1 and x_2 such that

$$|f(x_1) - f(x_2)| = |f'(\xi)||x_1 - x_2| \le \alpha\delta^{\alpha-1}\delta = \alpha\delta^\alpha. \tag{2.1.6}$$

Combining (2.1.5) and (2.1.6) proves $f \in \mathrm{Lip}_\alpha$. □

2.2 Least Square Approximation and Planchel Formula

Denote the space of all square integrable 2π-periodic functions by $L^2_{2\pi}$. One may verify that $L^2_{2\pi}$ is an inner product space with the inner product defined by

$$\langle f, g\rangle = \int_0^{2\pi} f(x)g(x)dx \quad \forall\ f, g \in L^2_{2\pi}.$$

Let Π_n be the space of all trigonometric polynomials of degree n,

$$\Pi_n = \left\{\frac{a_0}{2} + \sum_{k=0}^{n}(a_k\cos kx + b_k \sin kx) :\ a_0 \in \mathbf{R} \text{ and } a_k, b_k \in \mathbf{R}, 1 \le k \le n\right\}.$$

One may easily verify that Π_n is a linear subspace of $L^2_{2\pi}$, and

$$\left\{\frac{1}{\sqrt{2\pi}}, \frac{1}{\sqrt{\pi}}\cos x, \frac{1}{\sqrt{\pi}}\sin x, \ldots, \frac{1}{\sqrt{\pi}}\cos nx, \frac{1}{\sqrt{\pi}}\sin nx\right\}$$

is an orthonormal basis of Π_n. Therefore by Theorems 6.1.3 and 6.2.1, we have the following result about the least square approximation to a function in $L^2_{2\pi}$ by trigonometric polynomials of degree at most n.

Theorem 2.2.1 *Let $f \in L^2_{2\pi}$. Then the least square approximation to f from Π_n is*

$$p_n = \frac{a_0}{2} + \sum_{k=1}^{n}(a_k \cos kx + b_k \sin kx),$$

where

$$a_k = \frac{1}{\pi}\int_0^{\pi} f(x)\cos kx dx,\ 0 \le k \le n \tag{2.2.1}$$

and

$$b_k = \frac{1}{\pi}\int_0^{\pi} f(x)\sin kx dx,\ 1 \le k \le n. \tag{2.2.2}$$

For any function $f \in L^2_{2\pi}$, we associate f with a Fourier series,

$$f \sim \frac{a_0}{2} + \sum_{k=1}^{\infty}(a_k \cos kx + b_k \sin kx), \tag{2.2.3}$$

where a_0 and $a_k, b_k, k \ge 1$, are defined as in (2.2.1) and (2.2.2). Define the partial sum $S_n f$ of degree n of the Fourier series (2.2.3) by

$$S_n f = \frac{a_0}{2} + \sum_{k=1}^{n}(a_k \cos kx + b_k \sin kx).$$

Therefore as a consequence of Theorem 2.2.1, we have

Theorem 2.2.2 *Let $f \in L^2_{2\pi}$. Then the partial sum $S_n f$ of degree n of its corresponding Fourier series is the least square approximation to f out of Π_n.*

Let p_n be as in Theorem 2.2.1. By direct computation, we have

$$\begin{aligned}
\|p_n\|_2^2 &= \pi \left\| \frac{a_0}{\sqrt{2}} \times \frac{1}{\sqrt{2\pi}} + \sum_{k=1}^{n}\left(a_k \frac{\cos kx}{\sqrt{\pi}} + b_k \frac{\sin kx}{\sqrt{\pi}}\right)\right\|_2^2 \\
&= \pi\left(\frac{a_0^2}{2} + \sum_{k=1}^{n}(a_k^2 + b_k^2)\right).
\end{aligned} \tag{2.2.4}$$

This together with the fact that p_n is orthogonal projection onto Π_n (see Theorem 6.2.1) leads to

$$\begin{aligned}
0 &\le \|f - p_n\|_2^2 = \|f\|_2^2 - \langle p_n, p_n\rangle \\
&= \|f\|_2^2 - \pi\left(\frac{a_0^2}{2} + \sum_{k=1}^{n}(a_k^2 + b_k^2)\right).
\end{aligned} \tag{2.2.5}$$

Letting $n \to \infty$ in (2.2.5) yields

$$\|f\|_2^2 \ge \frac{\pi a_0^2}{2} + \sum_{k=1}^{\infty}\pi(a_k^2 + b_k^2). \tag{2.2.6}$$

For the case that f is a trigonometric polynomial, the inequality (2.2.6) becomes an equality since $f = p_n$ for sufficiently large n. Moreover, by the density of trigonometric polynomials in $L^2_{2\pi}$, the inequality (2.2.6) is an identity for all $f \in L^2_{2\pi}$, which is known as Planchel formula.

Theorem 2.2.3 *Let $f \in L^2_{2\pi}$, and $\frac{a_0}{2} + \sum_{k=1}^{\infty}(a_k \cos kx + b_k \sin kx)$ be its corresponding Fourier series. Then*

$$\|f\|_2^2 = \frac{\pi a_0^2}{2} + \pi \sum_{k=1}^{\infty}(a_k^2 + b_k^2).$$

2.3 Approximation by Convolution Operators with Nonnegative Kernel

Let $C_{2\pi}$ be the space of all 2π-periodic continuous functions. We say that an operator T on the space of $C_{2\pi}$ is a *convolution operator* if there exists a continuous function $k \in C_{2\pi}$ such that

$$Tg(\theta) = \int_{-\pi}^{\pi} g(\phi)k(\theta - \phi)d\phi.$$

The function k is said to be the *kernel* of the convolution operator T. If

$$\int_{-\pi}^{\pi} k(\phi)d\phi = 1,$$

then we say that T is a convolution operator with *normalized kernel* k. If $k \geq 0$, then we say that the convolution operator T has *nonnegative kernel.*

For a continuous function g, we associate g with its Fourier series

$$g \sim \frac{a_0}{2} + \sum_{k=1}^{\infty}(a_k \cos k\theta + b_k \sin k\theta),$$

where

$$\begin{aligned} a_k &= \frac{1}{\pi}\int_{-\pi}^{\pi} g(\theta)\cos k\theta d\theta, \quad k = 0, 1, \ldots, \\ b_k &= \frac{1}{\pi}\int_{-\pi}^{\pi} g(\theta)\sin k\theta d\theta, \quad k = 1, \ldots. \end{aligned}$$

Denote the partial sum of the Fourier series of g by $S_n g$,

$$S_n g(\theta) = \frac{a_0}{2} + \sum_{k=1}^{n}(a_k \cos k\theta + b_k \sin k\theta).$$

It is known that $S_n g$ is the least square approximation to g out of Π_n (see Theorem 2.2.2 for details).

To study the uniform approximation to g out of Π_n, we need a weighted version of partial sum of Fourier series. For an infinite triangulation $\rho_{k,n} \in \mathbf{R}, 1 \le k \le n, n = 1, 2, \ldots$, we define weighted partial sum of Fourier series $Q_n g$ by

$$Q_n g(\theta) = \frac{a_0}{2} + \sum_{k=1}^{n} \rho_{k,n}(a_k \cos k\theta + b_k \sin k\theta),\ \ n \ge 1.$$

Obviously $Q_n = S_n$ when all entries in the infinite triangulation are identically one, and the weighted partial sum Q_n of Fourier series is a convolution operator with kernel $u_n(\theta)/\pi$, where

$$u_n(\theta) = \frac{1}{2} + \sum_{k=1}^{n} \rho_{k,n} \cos k\theta. \tag{2.3.1}$$

For the usual partial sum S_n, the corresponding kernel $k_n(x)$ is

$$k_n(\theta) = \frac{\sin(n + 1/2)\theta}{2\pi \sin \theta/2}. \tag{2.3.2}$$

For the uniform approximation by convolution operator with nonnegative kernel, we have the following result.

Theorem 2.3.1 *Let T be a convolution operator with normalized nonnegative kernel k. Then for any $g \in C_{2\pi}$ and $n \ge 1$,*

$$|g(\theta) - Tg(\theta)| \le \left[1 + \frac{n\pi}{\sqrt{2}}(1 - \rho_1)^{1/2}\right] \omega(g, n^{-1}), \tag{2.3.3}$$

where

$$\rho_1 = \int_{-\pi}^{\pi} k(\phi) \cos \phi d\phi, \tag{2.3.4}$$

and $\omega(g, \theta)$ is the modulus of continuity of the continuous function g on $\mathbf{R}$.

Proof. By the normalization condition of the kernel k, we have

$$\begin{aligned} g(\theta) - Tg(\theta) &= \int_{-\pi}^{\pi} (g(\theta) - g(\phi))k(\theta - \phi)d\phi \\ &= \int_{-\pi}^{\pi} (g(\theta) - g(\theta - \phi))k(\phi)d\phi. \end{aligned}$$

Therefore,

$$
\begin{aligned}
|g(\theta) - Tg(\theta)| &\le \int_{-\pi}^{\pi} |g(\theta) - g(\theta - \phi)| k(\phi) d\phi \\
\le & \int_{-\pi}^{\pi} \omega(g, |\phi|) k(\phi) d\phi \le \omega(g, n^{-1}) \int_{-\pi}^{\pi} (1 + n|\phi|) k(\phi) d\phi \\
\le & \ \omega(g, n^{-1}) + n\omega(g, n^{-1}) \int_{-\pi}^{\pi} |\phi| k(\phi) d\phi,
\end{aligned}
$$

where we have also used Theorem 2.1.2 and the assumption that k is a nonnegative kernel. Hence it suffices to justify

$$\int_{-\pi}^{\pi} |\phi| k(\phi) d\phi \le \frac{\pi}{\sqrt{2}} (1 - \rho_1)^{1/2}. \tag{2.3.5}$$

One may show that $x^{-1} \sin x$ is monotonously decreasing on $[0, \pi/2]$, and hence

$$|x| \le \frac{\pi}{2} |\sin x|, \ \ x \in \Big[-\frac{\pi}{2}, \ \frac{\pi}{2} \Big]. \tag{2.3.6}$$

By the Cauchy inequality and by the nonnegativeness and normalization condition of the kernel k, we obtain

$$
\begin{aligned}
\int_{-\pi}^{\pi} |\sin \frac{\phi}{2}| k(\phi) d\phi \ \ &\le \ \ \left(\int_{-\pi}^{\pi} \Big|\sin \frac{\phi}{2}\Big|^2 k(\phi) d\phi \right)^{1/2} \times \left(\int_{-\pi}^{\pi} k(\phi) d\phi \right)^{1/2} \\
&= \ \ \left(\int_{-\pi}^{\pi} \frac{1 - \cos \phi}{2} k(\phi) d\phi \right)^{1/2} = \left(\frac{1 - \rho_1}{2} \right)^{1/2}.
\end{aligned} \tag{2.3.7}
$$

Then the estimate in (2.3.5) follows from (2.3.6) and (2.3.7). □

2.4 Best Uniform Approximation by Trigonometric Polynomials

In this section, we estimate the approximation error $E_n(g)$ between a function $g \in C_{2\pi}$ and its best uniform approximating polynomial in Π_n,

$$E_n(g) = \inf_{p \in \Pi_n} \|g - p\|.$$

The main result of this section is stated as follows, which is known as Jackson's theorem.

Theorem 2.4.1 *Let $g \in C_{2\pi}$ and $\omega(g, \delta)$ denote the modulus of continuity of the function g on* $\mathbf{R}$. *Then*

$$E_n(g) \le 6\omega(g, n^{-1}), \ n \ge 1. \tag{2.4.1}$$

As a consequence of Theorem 2.4.1, we have the following result about the uniform approximation of a Hölder continuous function by trigonometric polynomials.

Corollary 2.4.2 *Let g be continuous and satisfy a Lipschitz condition of order α with constant K, where $0 < \alpha \le 1$. Then*

$$E_n(g) \le 6Kn^{-\alpha}, \ n \ge 1.$$

By Theorems 2.1.2 and 2.4.1, we have

Corollary 2.4.3 *The space of all trigonometric polynomials is dense in $C_{2\pi}$.*

We shall use a convolution operator with nonnegative kernel to approximation the identity and then establish the estimate (2.4.1). To this end, we construct a nonnegative kernel.

Lemma 2.4.4 *For $n \ge 1$, set*

$$c_{k,n} = (n+2)^{-1/2} \sin \frac{(k+1)\pi}{n+2}, \ k = 0, 1 \ldots, n,$$

define

$$u_n(\theta) = \left| \sum_{k=0}^{n} c_{k,n} e^{-ik\theta} \right|^2,$$

and write

$$u_n(\theta) = \rho_{0,n} + \sum_{k=1}^{n} \rho_{k,n} \cos k\theta$$

for some $\rho_{k,n}, 0 \le k \le n, n \ge 1$. Then

$$\rho_{0,n} = \frac{1}{2}, \tag{2.4.2}$$

and

$$n\pi \sqrt{\frac{1-\rho_{1,n}}{2}} \le 5. \tag{2.4.3}$$

Proof. By direct computation, we have

$$\begin{aligned} u_n(\phi) &= \sum_{k=0}^{n} c_{k,n}^2 + 2\sum_{k=0}^{n-1} c_{k,n}c_{k+1,n}\cos\phi \\ &\quad +2\sum_{k=0}^{n-2} c_{k,n}c_{k+2,n}\cos 2\phi + \cdots + 2c_{0,n}c_{n,n}\cos n\phi. \end{aligned}$$

Therefore

$$\rho_{0,n} = \frac{1}{n+2}\sum_{k=0}^{n}\sin^2\frac{(k+1)\pi}{n+2} \tag{2.4.4}$$

and

$$\rho_{1,n} = \frac{2}{n+2}\sum_{k=0}^{n-1}\sin\frac{(k+1)\pi}{n+2}\sin\frac{(k+2)\pi}{n+2}. \tag{2.4.5}$$

By direct computation, we get

$$\begin{aligned} &\sum_{k=0}^{n}\sin^2\frac{(k+1)\pi}{n+2} = \frac{1}{2}\sum_{k=0}^{n}\left(1-\cos\frac{2(k+1)\pi}{n+2}\right) \\ &= \frac{n+1}{2} - \frac{1}{2}\mathrm{Re}\left(e^{2\pi i/(n+2)}\frac{1-e^{2(n+1)\pi i/(n+2)}}{1-e^{2\pi i/(n+1)}}\right) \\ &= \frac{n+1}{2} + \frac{1}{2} = \frac{n+2}{2}, \end{aligned} \tag{2.4.6}$$

and

$$\begin{aligned} &\sum_{k=0}^{n-1}\sin\frac{(k+1)\pi}{n+2}\sin\frac{(k+2)\pi}{n+2} \\ &= \frac{1}{2}\sum_{k=0}^{n}\left(\sin\frac{k\pi}{n+2}+\sin\frac{(k+2)\pi}{n+2}\right)\sin\frac{(k+1)\pi}{n+2} \\ &= \sum_{k=0}^{n}\cos\frac{\pi}{n+2}\sin^2\frac{(k+1)\pi}{n+2} = \frac{n+2}{2}\cos\frac{\pi}{n+2}. \end{aligned} \tag{2.4.7}$$

Hence (2.4.2) follows from (2.4.4) and (2.4.6). By (2.4.5) and (2.4.7), we obtain

$$\begin{aligned} &n\pi\sqrt{\frac{1-\rho_{1,n}}{2}} = n\pi\sqrt{\frac{1-\cos\frac{\pi}{n+2}}{2}} \\ &= n\pi\sin\frac{\pi}{2n+4} \le \frac{n\pi^2}{2n+4} \le \frac{\pi^2}{2} \le 5. \end{aligned}$$

This proves (2.4.3). □

Now we start to prove Theorem 2.4.1.

Proof of Theorem 2.4.1. Let u_n be defined as in Lemma 2.4.4, and Q_n be the convolution operator with kernel u_n/π. By Lemma 2.4.4, the kernel of the convolution operator Q_n is nonnegative and satisfies the normalization condition. Therefore by Theorem 2.3.1,

$$\|g - Q_n g\| \leq \left[1 + \frac{n\pi}{\sqrt{2}}(1-\rho_1)^{1/2}\right]\omega(g, n^{-1}), \tag{2.4.8}$$

where

$$\rho_1 = \frac{1}{\pi}\int_{-\pi}^{\pi} u_n(\theta)\cos\theta d\theta.$$

By Lemma 2.4.4, $\rho_1 = \rho_{1,n}$. This together with (2.4.3) and (2.4.8) yields the estimate (2.4.1). □

2.5 Comparison of Best Uniform Approximation and Least Square Approximation

In this section, we compare the L^∞ error between a 2π-periodic function g and its least square approximating polynomial in Π_n, and between the function g and its best uniform approximating polynomial in Π_n.

Theorem 2.5.1 *Suppose that $g \in C_{2\pi}$ and let $q_n \in \Pi_n$ be the least square approximation to g out of Π_n. Then*

$$\|g - q_n\|_\infty \leq \Big(4 + \frac{4\ln n}{\pi^2}\Big)E_n(g), \tag{2.5.1}$$

where $E_n(g) = \min_{q\in\Pi_n}\|g-q\|_\infty$.

By Theorems 2.4.1 and 2.5.1, the least square approximating polynomial to a Hölder continuous function converges uniformly.

Corollary 2.5.2 *Let f be 2π-periodic and belong to Lip_α for some $\alpha > 0$. Then the partial sum $S_n f$ of its corresponding Fourier series converges to f uniformly as n tends to infinity.*

To prove Theorem 2.5.1, we need the following lemma.

Lemma 2.5.3

$$\frac{1}{\pi}\int_0^{\pi} \frac{|\sin(n+1/2)\theta|}{\sin\theta/2}d\theta < 3 + \frac{4\ln n}{\pi^2}, \quad n \geq 1. \tag{2.5.2}$$

Proof. For $n = 1$,

$$\frac{1}{\pi}\int_0^{\pi} \frac{|\sin 3\theta/2|}{\sin\theta/2}d\theta \leq \frac{1}{\pi}\int_0^{\pi} \left|2\cos^2\frac{\theta}{2} + \cos\theta\right| d\theta \leq 3.$$

Hence the estimate (2.5.1) holds for $n = 1$.

Now we suppose that $n \geq 2$. Set

$$I_n = \int_0^{\pi} |\cos n\theta| d\theta$$

and

$$J_n = \int_0^{\pi} \frac{|\sin n\theta|}{\tan\theta/2}d\theta.$$

For I_n, we have

$$I_n = \int_0^{\pi} |\cos n\theta| d\theta = \frac{1}{n}\int_0^{n\pi} |\cos\theta| d\theta = 2. \tag{2.5.3}$$

Note that $\tan x \geq x$ for all $0 \leq x \geq \pi/2$ since the function h defined by $h(x) = \tan x - x$ is monotonously increasing on $[0, \pi/2]$. Therefore

$$\begin{aligned} J_n &\leq 2\int_0^{\pi} \frac{|\sin n\theta|}{\theta}d\theta = 2\int_0^{n\pi} \frac{|\sin\theta|}{\theta}d\theta \\ &= 2\int_0^{\pi} \frac{|\sin\theta|}{\theta}d\theta + 2\int_0^{\pi} \sin\theta \times \sum_{k=1}^{n-1} \frac{1}{\theta + k\pi}d\theta \\ &\leq 2\int_0^{\pi} \frac{|\sin\theta|}{\theta}d\theta + 2\int_0^{\pi} \sin\theta \times \sum_{k=1}^{n-1} \frac{1}{k\pi}d\theta \\ &= 2\int_0^{\pi} \frac{|\sin\theta|}{\theta}d\theta + \frac{4 + 4\ln(n-1)}{\pi}. \end{aligned} \tag{2.5.4}$$

By numerical computation,

$$\int_0^{\pi} \frac{\sin\theta}{\theta}d\theta \approx 1.8524. \tag{2.5.5}$$

Combining (2.5.3), (2.5.4) and (2.5.5), we obtain

$$\frac{1}{\pi}\int_0^\pi \frac{|\sin(n+1/2)\theta|}{\sin\theta/2}d\theta \le \frac{1}{\pi}\int_0^\pi \frac{|\sin n\theta|}{\tan\theta/2}d\theta + \frac{1}{\pi}\int_0^\pi |\cos n\theta| d\theta$$
$$\le \Big(\frac{2}{\pi}+\frac{2}{\pi}\int_0^\pi \frac{\sin\theta}{\theta}d\theta + \frac{4}{\pi^2}\Big) + \frac{4\ln n}{\pi^2} < 3 + \frac{4\ln n}{\pi}.$$

Hence the estimate (2.5.2) follows for $n \ge 2$. □

Now we start to prove Theorem 2.5.1.

Proof of Theorem 2.5.1. Let $p_n^* \in \Pi_n$ be so chosen that $E_n(g) = \|g - p_n^*\|_\infty$. By Theorem 2.2.2, $q_n - p_n^*$ is the partial sum $S_n(g-p_n^*)$ of degree n of the corresponding Fourier series of $g - p_n^*$. By (2.3.1) and (2.3.2), we have

$$(q_n - p_n^*)(\theta) = \int_{-\pi}^{\pi}(g-p_n^*)(\theta-\phi)\frac{\sin(n+1/2)\phi}{2\pi\sin\phi/2}d\phi.$$

Therefore

$$\|q_n - p_n^*\|_\infty \le \|g-p_n^*\|_\infty \times \int_{-\pi}^{\pi}\frac{|\sin(n+1/2)\phi|}{2\pi|\sin\phi/2|}d\phi \le \Big(3+\frac{4\ln n}{\pi^2}\Big)E_n(g). \tag{2.5.6}$$

Hence

$$\|g-q_n^*\|_\infty \le \|q_n^* - p_n^*\|_\infty + \|g-p_n^*\|_\infty \le \Big(4+\frac{4\ln n}{\pi^2}\Big)E_n(g).$$

This completes the proof of Theorem 2.5.1. □

2.6 Bernstein's Theorem

In this section, we prove the Bernstein's theorem and consider the inverse problem of approximation.

Theorem 2.6.1 *Let p be trigonometric polynomial of degree n. Then*

$$\|p'\|_\infty \le n\|p\|_\infty. \tag{2.6.1}$$

Proof. Suppose on the contrary that the theorem is not true. Then there exists $p \in \Pi_n$ so that

$$\|p\|_\infty = 1 \quad \text{and} \quad \|p'\|_\infty > n.$$

Set $L = \|p'\|_\infty/n$. Without loss of generality we assume that $p'(x_0) = nL$ for some $x_0 \in \mathbf{R}$, otherwise replacing p by $-p$ instead. Define

$$S(x) = L \sin n(x - x_0) - p(x).$$

Then the trigonometric polynomial S takes values of alternating signs at $x_k = x_0 + (2k-1)\pi/(2n), k = 1, \dots, 2n$. Therefore between two of those points, S has a zero, and hence S has $2n$ different zeros. By Rolle's theorem,

$$S'(x) = nL \cos n(x - x_0) - p'(x)$$

also has $2n$ different zeros. One of those zeros is x_0, since $S'(x_0) = nL - p'(x_0) = 0$. Also,

$$S''(x) = -n^2 L \sin n(x - x_0) - p''(x)$$

vanishes at x_0 since $p'(x)$ takes the maximam. Moreover, S'' has, again by Rolle's theorem, $2n$ zeros between zeros of S'. Therefore S' has at least $2n+1$ zeros, which yields that S'' is identically zero. Hence S is a constant, which a contradiction since S changes sign on the line. $\square$

Theorem 2.6.2 *Let f be a 2π-periodic continuous function and $E_n(f), n \geq 0$, be the best approximation error to f from Π_n. Then there exist a positive constant C independent of $\delta > 0$ so that*

$$\omega(f, \delta) \leq C\delta \sum_{0 \leq n \leq \delta^{-1}} E_n(f) \quad \forall\, \delta > 0. \tag{2.6.2}$$

By Corollary 2.4.2 and Theorem 2.6.2, we have the following result due to Bernstein.

Corollary 2.6.3 *A 2π-periodic continuous function f satisfies the Lipschitz condition of order α if and only if the corresponding approximation error $E_n(f)$ from Π_n satisfies*

$$E_n(f) \leq Cn^{-\alpha},$$

where $0 < \alpha < 1$.

Now we prove Theorem 2.6.2.

Proof of Theorem 2.6.2. Let $p_n \in \Pi_n$ be the best approximating trigonometric polynomial to f in Π_n. Then

$$\|f - p_n\|_\infty = E_n(f). \tag{2.6.3}$$

For $\delta > 1$, we have

$$\omega(f, \delta) = \omega(f - p_0, \delta) \leq \|f - p_0\|_\infty.$$

Hence the estimate (2.6.2) follows from $\delta > 1$.

Recall that $E_n(f)$ is monotonuously decreasing. Then it suffices to prove

$$\omega(f, 2^{-k}) \leq C 2^{-k} \sum_{l=0}^{k-1} 2^l E_{2^l}(f), \tag{2.6.4}$$

where we have also used the first and third assertions of Theorem 2.1.2. Write $f = f - p_{2^k} + p_{2^k}$. By (2.6.3) and the definition of modulus of continuity, we have

$$\omega(f, 2^{-k}) = \omega(f - p_0, 2^{-k}) \leq 2\|f - p_{2^k}\|_\infty + 2^{-k}\|p'_{2^k} - p'_0\|_\infty. \tag{2.6.5}$$

Write

$$p_{2^k} - p_0 = (p_{2^k} - p_{2^{k-1}}) + (p_{2^{k-1}} - p_{2^{k-2}}) + \cdots + (p_2 - p_1) + (p_1 - p_0).$$

By Bernstein's theorem (Theorem 2.6.1) and the monotonicity of $E_n(f)$, it follows that

$$\begin{aligned}
\|p'_{2^k} - p'_0\|_\infty &\leq \sum_{l=1}^{k} \|(p_{2^l} - p_{2^{l-1}})'\|_\infty + \|(p_1 - p_0)'\|_\infty \\
&\leq \sum_{l=1}^{k} 2^l \|p_{2^l} - p_{2^{l-1}}\|_\infty + \|p_1 - p_0\|_\infty \\
&\leq \sum_{l=1}^{k} 2^{l+1} E_{2^{l-1}}(f) + 2E_0(f) \qquad (2.6.6)
\end{aligned}$$

Combining (2.6.5) and (2.6.6) leads to the estimate (2.6.2) for $\delta < 1$. $\square$

Exercises

1. Let h be the hat function defined by $h(x) = \max(0, 1 - |x|), x \in [-1, 1]$. Compute $\omega(h, [-1, 1], \delta)$ for all $0 < \delta < 2$.

2. Let f be a continuous function on $[a,b]$, and $[c,d]$ be a subinterval of $[a,b]$. Show that

$$\omega(f,[c,d],\delta) \le \omega(f,[a,b],\delta).$$

3. Let f be a 2π-periodic continuous function. Show that

$$\omega(f,[a,a+2\pi],\delta) \le 2\omega(f,[b,b+2\pi],\delta),\ \ \delta > 0,$$

where $a, b \in \mathbf{R}$.

4. Let f be a continuous function on $[a,b]$, and let g be a continuous funftion from $[c,d]$ to $[a,b]$ and satisfy the Lipschitz condition of order one with constant K. Show that the composition of f and g, to be denoted by h, satisfies

$$\omega(h,[c,d],\delta) \le \omega(f,[a,b],K\delta),\ \ \delta > 0,$$

5. Show that $|x|^\alpha \ln|x|, x \in [0,1]$ belongs to Lip_β for all β with $0 < \beta < \alpha$, but does not belong to Lip_α, where $0 < \alpha \le 1$.

6. Show that

$$\frac{1}{2} + \sum_{k=1}^{n} \cos k\theta = \frac{\sin(n+\frac{1}{2})\theta}{2\sin\frac{1}{2}\theta},\ \ 1 \le n \in \mathbf{Z}.$$

7. Show that $E_n(f,[a,b]) = E_n(f - p_n,[a,b])$ for any $p_n \in P_n$, where P_n is the space of all polynomial of degree at most n, and where $E_n(f) = \inf_{p\in P_n} \|f - p\|$.

8. Let $n \ge 1$ and $\rho_{k,n} \in \mathbf{R}, 1 \le k \le n$. For 2π-periodic function g, define $Q_n g$ by

$$Q_n g(\theta) = \frac{a_0}{2} + \sum_{k=1}^{n} \rho_{k,n}(a_k \cos k\theta + b_k \sin k\theta),$$

where a_0 and $a_k, b_k, 1 \le k \le n$ are the Fouries coefficents of g. Show that Q_n is a convolution operator with kernel $u_n(\theta)/\pi$, where

$$u_n(\theta) = \frac{1}{2} + \sum_{k=1}^{n} \rho_{k,n} \cos k\theta.$$

9. Let

$$k_n(x) = (2n\pi)^{-1}\left(\frac{\sin nx/2}{\sin x/2}\right)^2.$$

Denote the convolution operator with kernel k_n by σ_n. Show that if f is 2π-periodic and satisfies the Lipschitz condition of order α for some $0 < \alpha < 1$, then there exists a positive constant C such that

$$\|\sigma_n f - f\|_\infty \le C n^{-\alpha},\ \ n \ge 1.$$

10. Let the Fourier series of a 2π-periodic continuous function f be $a_0/2 + \sum_{k=1}^{\infty} a_k \cos kx$, and let $E_n(f)$ be the best approximation error to f from Π_n. Prove that

$$\frac{1}{2}|a_{n+1}| \leq E_n(f) \leq \sum_{k=n+1}^{\infty} |a_k|.$$

7. Find all least square approximations to $|\cos x|$ out of $\Pi_n, n = 1, 2, 3$.

8. Compute the L^2 norm of the least square approximation to $|\cos x|$ out of $\Pi_n, n \geq 1$, and justify whether it converges to the L^2 norm of $|\cos x|$ or not.

Chapter 3

Approximation by Polynomials

In this chapter, we consider the uniform polynomial approximation problem.

3.1 Lagrange Interpolation

In the approximation to a given continuous function by functions with simple structure, it is natural to require that the approximating function coincides with the given function at certain points of the interval.

Theorem 3.1.1 *Let $f \in C([a,b])$, $a \leq x_1 < x_2 < \ldots < x_n \leq b$ be given, and G be an n-dimensional subspace of $C([a,b])$ with $g_1, \ldots, g_n$ being a basis. If*

$$det \begin{pmatrix} g_1(x_1) & \cdots & g_n(x_1) \\ \vdots & \ddots & \vdots \\ g_1(x_n) & \cdots & g_n(x_n) \end{pmatrix} \neq 0,$$

then there exists unique $g \in G$ so that

$$g(x_i) = f(x_i), \ 1 \leq i \leq n. \tag{3.1.1}$$

Moreover the coefficients in the solution $g = \sum_{i=1}^{n} a_i g_i$ satisfies the linear system

$$\begin{pmatrix} g_1(x_1) & \cdots & g_n(x_1) \\ \vdots & \ddots & \vdots \\ g_1(x_n) & \cdots & g_n(x_n) \end{pmatrix} \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} = \begin{pmatrix} f(x_1) \\ \vdots \\ f(x_n) \end{pmatrix}. \tag{3.1.2}$$

Proof. Write $g = \sum_{i=1}^n a_i g_i$. Substituting the above expression into the condition (3.1.1) leads to

$$\sum_{i=1}^n a_i g_i(x_j) = f(x_j),\ 1 \le j \le n,$$

which can be written as the matrix form (3.1.2). Hence the result follows from standard result of linear algebra. $\square$

To problem to find a function $g \in G$ to satisfy (3.1.1) is known as *Lagrange interpolation problem*, and the points $x_1, \ldots, x_n$ are known as the *knots* of the interpolation. For the simplicity, we denote the following determinant

$$det \begin{pmatrix} g_1(x_1) & \cdots & g_n(x_1) \\ \vdots & \ddots & \vdots \\ g_1(x_n) & \cdots & g_n(x_n) \end{pmatrix}$$

by

$$D\begin{pmatrix} g_1 & \cdots & g_n \\ x_1 & \cdots & x_n \end{pmatrix},$$

where $g_1, \ldots, g_n$ are continuous functions and $x_1, \ldots, x_n$ are points on the real line.

From the proof of Theorem 3.1.1, we see that the Lagrange interpolation problem is solvable if and only if the linear system (3.1.2) is solvable. For instance, for the space spanned by $g_1(x) = 1$ and $g_2(x) = \sin x, x \in [0, \pi]$, one may verify that

$$D\begin{pmatrix} g_1 & g_2 \\ x_1 & x_2 \end{pmatrix} \neq 0$$

for all $x_1, x_2 \in [0, \pi]$ with $x_1 + x_2 \neq \pi$, and hence the corresponding Lagrange interpolation is solvable and the solution is unique for any given continuous function f on $[0, \pi]$, in fact,

$$g(x) = \frac{\sin x_2 f(x_1) - \sin x_1 f(x_2)}{\sin x_2 - \sin x_1} + \frac{f(x_1) - f(x_2)}{\sin x_2 - \sin x_1} \sin x.$$

For the case that $x_1 = \pi - x_2$, the corresponding Lagrange interpolation problem is solvable for a given continuous function f if and only if $f(x_1) = f(x_2)$.

3.2 Polynomial Interpolation

In this section, we consider the Lagrange interpolation problem with the interpolating functions being polynomials of degree at most $n-1$.

3.2.1 Explicit Formula

It is known that $\{1, x, \ldots, x^{n-1}\}$ is a basis of P_{n-1}, and

$$D\begin{pmatrix} 1 & \cdots & x^{n-1} \\ x_1 & \cdots & x_n \end{pmatrix} = \prod_{1\le i<j\le n} (x_j - x_i) \neq 0$$

for any interpolating knots $a \le x_1 < \ldots < x_n \le b$, since the determinant is a Vandermonde determinant. Therefore by Theorem 3.1.1, we have

Theorem 3.2.1 *Let $f \in C([a,b])$, $a \le x_1 < x_2 < \ldots < x_n \le b$ be given. Then there exists unique $g \in P_{n-1}$ so that*

$$g(x_i) = f(x_i),\ 1 \le i \le n. \tag{3.2.1}$$

Moreover the coefficients in the solution $g = \sum_{i=0}^{n-1} a_i x^i$ satisfies the linear system

$$\begin{pmatrix} 1 & \cdots & x_1^{n-1} \\ \vdots & \ddots & \vdots \\ 1 & \cdots & x_n^{n-1} \end{pmatrix} \begin{pmatrix} a_0 \\ \vdots \\ a_{n-1} \end{pmatrix} = \begin{pmatrix} f(x_1) \\ \vdots \\ f(x_n) \end{pmatrix}. \tag{3.2.2}$$

By Theorem 3.2.1, the construction of the interpolating polynomial reduces to solving a linear system (3.2.2). Unlike the linear system (3.1.2) not having explicit solution in general, the linear system (3.2.2) can be solved explicitly.

Given distinct points $x_1, \ldots, x_n \in [a,b]$, one may easily verify that the polynomials $l_i, 1 \le i \le n$,

$$l_i(x) = \frac{(x-x_1)\cdots(x-x_{i-1})(x-x_{i+1})\cdots(x-x_n)}{(x_i-x_1)\cdots(x_i-x_{i-1})(x_i-x_{i+1})\cdots(x_i-x_n)}, \tag{3.2.3}$$

are of degree at most $n-1$, and satisfy

$$l_i(x_j) = \begin{cases} 1 & j \neq i \\ 0 & j = i \end{cases} \quad \text{for } i,j = 1, \ldots, n. \tag{3.2.4}$$

In the case $n = 1$, we set $l_1(x) = 1$. Therefore for any given continuous function f on $[a, b]$, the polynomial $L_{n-1} \in P_{n-1}$ defined by

$$L_{n-1}(x) = \sum_{i=1}^{n} f(x_i) l_i(x) \tag{3.2.5}$$

satisfies

$$L_{n-1}(x_i) = f(x_i), \ i = 1, \ldots, n.$$

Hence $L_{n-1}(x)$ is the interpolating polynomial of degree at most $n-1$ to f at the knots $x_1, \ldots, x_n$ by Theorem 3.2.1. The polynomial $L_{n-1}(x)$ is called the *Lagrange interpolating polynomials* to f at $x_1, \ldots, x_n$, and the polynomials $l_i(x), 1 \leq i \leq n$, in (3.2.3) are called the *fundamental polynomials* for the interpolation at $x_1, \ldots, x_n$.

3.2.2 Divided Difference

Let $f \in C([a, b])$ and points $x_1 < x_2 < \ldots < x_{n+1}$ be given. We define *divided difference* of order n of f with respect to the points $x_1, \ldots, x_{n+1}$ by

$$f[x_1, \ldots, x_{n+1}] = a_n,$$

where $p(t) = \sum_{i=0}^{n} a_i x^i$ is the unique interpolating polynomial in P_{n-1} to the corresponding Lagrange problem from P_{n-1}. By (3.2.2), we have

$$f[x_1, \ldots, x_{n+1}] = \frac{D\begin{pmatrix} 1 & \cdots & x^{n-1} & f \\ x_1 & \cdots & x_n & x_{n+1} \end{pmatrix}}{D\begin{pmatrix} 1 & \cdots & x^n \\ x_1 & \cdots & x_{n+1} \end{pmatrix}}. \tag{3.2.6}$$

From the above expression, we can compute the divided difference by computing two determinants. Having the following property about the divided difference, we can compute the divided difference recursively.

Theorem 3.2.2 *Let $f \in C([a, b])$ and points $x_1 < x_2 < \ldots < x_n$ be given. Then*

$$f[x_1, \ldots, x_n] = \frac{f[x_2, \ldots, x_n] - f[x_1, \ldots, x_{n-1}]}{x_n - x_1}. \tag{3.2.7}$$

Proof. For any $f \in C([a, b])$, define

$$L(f) = f[x_1, \ldots, x_n] - \frac{f[x_2, \ldots, x_n] - f[x_1, \ldots, x_{n-1}]}{x_n - x_1}.$$

By (3.2.6), there exist coefficients $b_j, 1 \leq j \leq n$, such that

$$L(f) = \sum_{j=1}^{n} b_j f(x_j).$$

Therefore it suffices to prove that

$$b_j = 0, \;\; j = 1, 2, \cdots, n. \tag{3.2.8}$$

For $f(t) = x^i, 0 \leq i \leq n-3$, it follows from (3.2.6) that

$$f[x_1, \ldots, x_n] = f[x_2, \ldots, x_n] = f[x_1, \ldots, x_{n-1}] = 0,$$

which leads to

$$L(x^i) = 0 \quad \forall\; 0 \leq i \leq n-3. \tag{3.2.9}$$

For $f(x) = x^{n-2}$,

$$f[x_1, \ldots, x_n] = 0$$

by (3.2.6), and

$$f[x_2, \ldots, x_n] = f[x_1, \ldots, x_{n-1}] = 1$$

since the Lagrange interpolating polynomials to x^{n-2} at the knots $x_1, \ldots, x_{n-1}$ and at the knots $x_2, \ldots, x_n$ are the same, x^{n-2} itself. This shows that

$$L(x^{n-2}) = 0. \tag{3.2.10}$$

For the polynomial $w(x) = (x-x_1)\ldots(x-x_n)$, the interpolating polynomial at the knots $x_1, \ldots, x_n$ is zero since the values of the function w on $x_1, \ldots, x_n$ are zero. Write

$$w(x) = x^n - \Big(\sum_{i=1}^{n} x_i\Big) x^{n-1} - r(x)$$

for some polynomial r of degree at most $n-2$. Note that the interpolating polynomial to $(\sum_{i=1}^{n} x_i) x^{n-1} + r(x)$ at the knots $x_1, \ldots, x_n$ is the polynomial itself. Therefore

$$0 = w[x_1, \ldots, x_n] = h_n[x_1, \ldots, x_n] - \sum_{i=1}^{n} x_i.$$

where $h_n(x) = x^n$, which yields

$$L(x^{n-1}) = 1 - \frac{\sum_{i=2}^{n} x_i - \sum_{i=1}^{n-1} x_i}{x_n - x_1} = 0. \tag{3.2.11}$$

Combining (3.2.9), (3.2.10) and (3.2.11), we obtain

$$\sum_{j=1}^{n} b_j x_j^i = 0, \ 0 \le i \le n-1.$$

Hence (3.2.8) follows. $\square$

The above theorem shows that the divided differences can be easily computed according to the following scheme:

$$\begin{array}{ccccc}
f[x_1] & & & & \\
 & f[x_1, x_2] & & & \\
f[x_2] & & & & \\
 & & & f[x_1, \ldots, x_{n-1}] & \\
\vdots & \vdots & \cdots & & f[x_1, \cdots, x_n] \\
 & & & f[x_2, \ldots, x_n] & \\
f[x_{n-1}] & & & & \\
 & f[x_{n-1}, x_n] & & & \\
f[x_n] & & & &
\end{array}$$

By Theorem 3.2.2, for a differentiable function f, we have

$$\lim_{x_2 \to x_1} f[x_1, x_2] = f'(x_1).$$

So the divided difference is widely used as a replacement of the derivative in numerical computation of solutions of differential equations.

3.2.3 Newton Form of Interpolating Polynomials

In this section, we give another explicit expression of Lagrange interpolating polynomials. The advantage for such a formula is at least that we need only add one term when we add one knot.

Theorem 3.2.3 *Let $f \in C([a, b])$ and $a \le x_1 < \ldots < x_n \le b$ be given. The unique polynomial p of degree at most $n-1$ which solves the corresponding Lagrange interpolation problem can be written as*

$$p(x) = f[x_1] + f[x_1, x_2](x - x_1) + \cdots + f[x_1, \ldots, x_n](x - x_1) \cdots (x - x_{n-1}).$$

Proof. For $1 \le j \le n$, let p_j be the unique polynomial of degree at most $j-1$ which solves the Lagrange interpolation problem for $x_1 < x_2 < \ldots < x_j$. It is obvious that $p_1(x) = f[x_1]$. Inductively, we assume that

$$p_j(x) = f[x_1] + f[x_1, x_2](x - x_1) + \cdots + f[x_1, \ldots, x_j](x - x_1) \cdots (x - x_{j-1}).$$

Therefore it suffices to prove that

$$p_{j+1}(x) = p_j(x) + f[x_1, \ldots, x_j](x - x_1) \cdots (x - x_j).$$

Denote the polynomial at the right hand side by $\tilde{p}_{j+1}$. Then $p_{j+1} - \tilde{p}_{j+1}$ is a polynomial of degree at most $j-1$ by the definition of the divided difference. On the other hand,

$$\tilde{p}_{j+1}(x_i) = p_j(x_i) = f(x_i) = p_{j+1}(x_i), \ 1 \le i \le j.$$

This shows that $p_{j+1}(x) - \tilde{p}_{j+1}(x)$ has at least j roots, which together with the degree property of the polynomial $p_{j+1} - \tilde{p}_{j+1}$ proves $p_{j+1} \equiv \tilde{p}_{j+1}$. $\square$

3.3 Least Square Approximation

In this section, we consider the least square polynomial approximation of a function in weighted L^2 space.

A *weight* w on the interval $[a, b]$ is a positive measurable function w on $[a, b]$. For instance, the functions w_1 and w_2 defined by $w_1(x) := 1$ and $w_2(x) := (1 - x^2)^{-1/2}, x \in [-1, 1]$, are weight functions on $[-1, 1]$, and the Gaussian function w_3 defined by $w_3(x) = (2\pi)^{-1/2} e^{-x^2/2}$ is a weight on $(-\infty, \infty)$.

For a weight w on $[a, b]$, we define the weighted L^2 space by

$$L^2_w([a, b]) := \{f : \ f \text{ is measurable on } [a, b] \text{ and } \|f\|_{2,w} < \infty\},$$

where

$$\|f\|_{2,w} := \left(\int_a^b |f(x)|^2 w(x) dx \right)^{1/2}.$$

One may verify that $L^2_w([a, b])$ is an inner product space with the inner product $\langle \cdot, \cdot \rangle_w$ on $L^2_w([a, b])$ defined by

$$\langle f, g \rangle_w = \int_a^b f(x) g(x) w(x) dx, \quad \forall \ f, g \in L^2_w([a, b]).$$

Let P_n be the space of all polynomials of degree at most n. If the interval $[a, b]$ is a finite interval and the weight w is integrable, then for any $p \in P_n$

$$\|p\|_{2,w}^2 = \int_a^b |p(x)|^2 w(x)dx \le \|p\|_\infty^2 \int_a^b w(x)dx < \infty$$

and hence $P_n \subset L_w^2([a, b])$, where we have used the fact that a polynomial is bounded on any finite interval. So in this section, we restrict ourselves to consider the weighted L^2 space on a finite interval and the weight being integrable.

By Theorems 6.1.2 and 6.2.1, we have the following result about the least square approximation to a weighted L^2 function by polynomials.

Theorem 3.3.1 *Let w be an integrable weight on the finite interval, and let $f \in L_w^2([a, b])$. Then $p_n^* \in \Pi_n$ is a least square approximation to f if and only if*

$$\langle f - p_n^*, p\rangle_w = 0 \quad \forall\, p \in \Pi_n.$$

Moreover, $p_n^*(x) = \sum_{k=0}^n \alpha_k^* x^k$, where

$$\begin{pmatrix} \langle 1, 1\rangle_w & \cdots & \langle x^n, 1\rangle_w \\ \vdots & \ddots & \vdots \\ \langle 1, x^n\rangle_w & \cdots & \langle x^n, x^n\rangle_w \end{pmatrix} \begin{pmatrix} \alpha_0^* \\ \vdots \\ \alpha_n^* \end{pmatrix} = \begin{pmatrix} \langle f, 1\rangle_w \\ \vdots \\ \langle f, x^n\rangle_w \end{pmatrix}. \tag{3.3.1}$$

Example 3.3.2 *Let the function f and the weight on $[-1, 1]$ be defined by $f(x) = e^x$ and $w(x) = 1, x \in [-1, 1]$. Find the least square approximation in P_2.*

Solution Let p_2^* be the least square approximation to f in P_2, and write $p_2^*(t) = \alpha_0^* + \alpha_1^* t + \alpha_2^* t^2$. Then by Theorem 3.3.1, the coefficients $\alpha_0^*, \alpha_1^*, \alpha_2^*$ satisfy the following linear system:

$$\begin{pmatrix} \langle 1, 1\rangle & \langle x, 1\rangle & \langle x^2, 1\rangle \\ \langle 1, x\rangle & \langle x, x\rangle & \langle x^2, x\rangle \\ \langle 1, x^2\rangle & \langle x, x^2\rangle & \langle x^2, x^2\rangle \end{pmatrix} \begin{pmatrix} \alpha_0^* \\ \alpha_1^* \\ \alpha_2^* \end{pmatrix} = \begin{pmatrix} \langle e^x, 1\rangle \\ \langle e^x, x\rangle \\ \langle e^x, x^2\rangle \end{pmatrix}.$$

Simplifying the above system leads to

$$2 \begin{pmatrix} 1 & 0 & 1/3 \\ 0 & 1/3 & 0 \\ 1/3 & 0 & 1/5 \end{pmatrix} \begin{pmatrix} \alpha_0^* \\ \alpha_1^* \\ \alpha_2^* \end{pmatrix} = \begin{pmatrix} e - 1/e \\ 2/e \\ e - 5/e \end{pmatrix}.$$

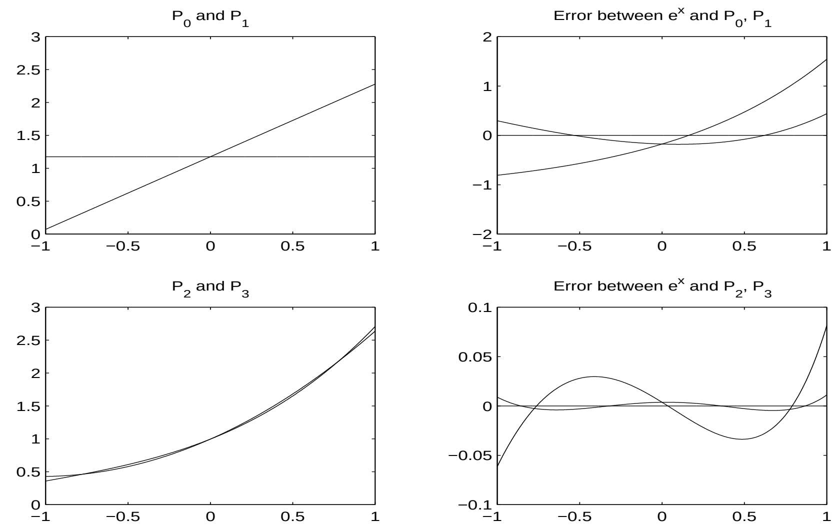

Figure 3.1: Figure 3.1: The least square approximating polynomials of degree $0, 1, 2, 3$ to the exponential function on $[-1, 1]$ and the errors.

Therefore

$$p_2^*(x) = -\Big(\frac{3e}{4} - \frac{33}{4e}\Big) + \frac{3}{e} + \Big(\frac{15e}{4} - \frac{105}{4e}\Big)x^2.$$

is the least square approximation to e^x on $[-1, 1]$ out of P_2.

From the figure 6.1.5, the approximation errors between e^x on $[-1, 1]$ and the least square approximating polynomials of degree $0, 1, 2, 3$ have the oscillatory properties. This inspires us to consider the oscillatory property of the approximation error between a function and its least square approximating polynomials.

To this end, we introduce a concept called *simple zero*. A point x_0 on the real line is said to be a simple zero of a continuous function f if f changes sign at x_0, that is, there exist intervals so that $f \not\equiv 0$ on $[a, x_0]$ and $[x_0, b]$, and that either (i) $f(x) \geq 0$ on $[x_0, b]$ and $f(x) \leq 0$ on $[a, x_0]$, or (ii) $f(x) \leq 0$ on $[x_0, b]$ and $f(x) \geq 0$ on $[a, x_0]$. For the later applications, for two simple zeros x_0 and x_1 of a continuous function f, we consider them as *distinct simple zeros* when $f(x) \not\equiv 0$ on the interval between x_0 and x_1.

For instance, $x_0 = 0$ is a simple zero of the function f defined by $f(x) =$

$x, x \in [-\pi, \pi]$, and of the piecewise linear function g defined by

$$g(x) = \begin{cases} x+1 & \text{if } x < -1 \\ 0 & \text{if } -1 \le x \le 1 \\ x-1 & \text{if } x > 1 \end{cases}$$

But $x_0 = 0$ is not a simple of the function x^2 on $[-1, 1]$ and of the function $\max(|x| - 1, 0)$ as well. For the function g defined above, $-1, 1, 0$ are thought as identical simple zeros.

Theorem 3.3.3 *Let f be a continuous function on the finite interval $[a, b]$, and let the weight function $w(x)$ is strictly positive and continuous. Suppose p_n^* is the least square approximation to f out of P_n, and that $e_n(x) = f(x) - p_n^*(x)$ is not identically zero. Then $e_n(x)$ has at least $n + 1$ distinct simple zeros in (a, b).*

Proof. Suppose on the contrary that $f(x) - p_n^*(x)$ has only k distinct simple zeros in $[a, b]$, where $k \le n$. Let these zeros be labeled as follows

$$a < x_1 < x_2 < \ldots < x_k < b.$$

First we claim that $k \ge 1$. By Theorem 3.3.1,

$$\int_a^b e_n(x)w(x)dx = 0,$$

which implies that $e_n(x)$ must take both positive and negative values in $[a, b]$, and then there is at least one simple zero by intermediate value theorem. This proves our claim $k \ge 1$.

Define

$$q(x) = (x - x_1) \cdots (x - x_k).$$

Then $q \in P_n$ by the assumption that $k \le n$. By the assumption on the simple zeros $f(x) - p_n^*(x)$ changes sign at intervals $[x_i, x_{i+1}], 0 \le i \le k$, where we set $x_0 = a$ and $x_{k+1} = b$, since otherwise there are more than k simple zeros for $f - p_n^*$. Also note that $q(x)$ changes sign at intervals $[x_i, x_{i+1}], 0 \le i \le k$ too. Therefore either $(f(x) - p_n^*(x))q(x) \ge 0$ for all $x \in [a, b]$ or $(f(x) - p_n^*(x))q(x) \le 0$ for all $x \in [a, b]$. On the other hand,

$$\int_a^b (f(x) - p_n^*(x))q(x)w(x)dx = 0$$

by Theorem 3.3.1. Therefore

$$(f(x) - p_n^*(x))q(x)w(x) = 0 \quad \forall\, x \in [a, b].$$

Thus $p_n^*(x) = f(x)$, which is a contradiction. $\square$

3.4 Best Approximation by Polynomials

For a continuous function f on $[a, b]$, denote the uniform approximation error between f and the best uniform approximating polynomial of degree at most n by $E_n(f, [a, b])$, or $E_n(f)$ for short,

$$E_n(f, [a, b]) = \inf_{p \in P_n} \|f - p\|_\infty.$$

In this section, we establish the following estimate to the uniform approximation error $E_n(f, [a, b])$.

Theorem 3.4.1 *Let f be continuous function on the finite interval $[a, b]$. Then*

$$E_n(f) \leq 6\omega\Big(f, \frac{b-a}{2n}\Big), \;\; n \geq 1. \tag{3.4.1}$$

As a consequence of Theorem 3.4.1, we have

Corollary 3.4.2 *If f is continuous on the finite interval $[a, b]$ and satisfies the Lipschitz condition of order α with constant K, then*

$$E_n(f) \leq 6K \left(\frac{b-a}{2n}\right)^\alpha, \;\; n \geq 1.$$

By Theorem 2.1.2 and 3.4.1, the density of the space of all polynomials in $C([a, b])$ follows.

Corollary 3.4.3 *The space of all polynomials is dense in $C([a, b])$.*

Proof of Theorem 3.4.1. Let $f \in C([a, b])$, and define a 2π-periodic function g by

$$g(\theta) = f\Big(\frac{b+a}{2} + \frac{b-a}{2}\cos\theta\Big).$$

Then g is continuous and even. Also one may verify that

$$\omega(g, \delta) \leq \omega(f, (b-a)\delta/2). \tag{3.4.2}$$

Let q_n^* be the best uniform approximation to g out of Π_n, the space of all trigonometric polynomials of degree at most n. Therefore $q_n^*(-\cdot)$ is a best uniform approximation to g as well since g is even. By the convexity of the set of all best approximations, $(q_n^* + q_n^*(-\cdot))/2$ is a best uniform approximation to g out of Π_n. So without loss of generality we may assume that q_n^* is an even function. Therefore q_n^* is a polynomial of $\cos\theta$, i.e.,

$$q_n^*(\theta) = d_0 + d_1 \cos\theta + \ldots + d_n(\cos\theta)^n$$

for some $d_k \in \mathbf{R}, 0 \le k \le n$, where we have used the fact that $\cos k\theta$ is a polynomial of $\cos\theta$ of degree k. Putting

$$p_n(x) = d_0 + d_1 \frac{2x-b-a}{b-a} + \ldots + d_n\left(\frac{2x-b-a}{b-a}\right)^n$$

and using Theorem 2.4.1 leads to

$$\sup_{x\in[a,b]} |f(x) - p_n(x)| = \sup_{\cos\theta\in[-1,1]} \left| g(\cos\theta) - p_n\Big(\frac{a+b}{2} + \frac{b-a}{2}\cos\theta\Big)\right|$$

$$= \sup_{\theta\in[-\pi,\pi]} |g(\theta) - q_n^*(\theta)| \le 6\omega\Big(g, \frac{1}{n}\Big) \le 6\omega\Big(f, \frac{b-a}{2n}\Big).$$

This completes the proof. $\Box$.

For the approximation problem of a differentiable function f, we have the following estimate about the approximation error $E_n(f)$.

Theorem 3.4.4 *If f has l-th continuous derivatives on $[a,b]$ for all $0 \le l \le k$, then*

$$E_n(f) \le cn^{-k}\omega\left(f^{(k)}, \frac{b-a}{2(n-k)}\right) \tag{3.4.3}$$

for all $n > k$, where c is a positive constant independent of n (but depends on k).

Proof. At first we claim that

$$E_n(f) \le 6n^{-1}E_{n-1}(f') \tag{3.4.4}$$

for a differentiable function f on $[a,b]$. Suppose p_{n-1}^* be a polynomial of degree at most $n-1$ so that

$$\|f' - p_{n-1}^*\|_\infty = E_{n-1}(f'). \tag{3.4.5}$$

Then setting

$$p_n(x) = \int_0^x p_{n-1}^*(t)dt$$

and using (2.1.2), (3.4.5) and Theorem 3.4.1, we obtain

$$\begin{aligned} & E_n(f) = E_n(f - p_n) \\ \leq \;& 6n^{-1}\|f' - p_n'\|_\infty = 6n^{-1}\|f' - p_{n-1}^*\|_\infty \leq 6n^{-1}E_{n-1}(f'). \end{aligned}$$

This proves our claim (3.4.4).

Repeatedly using the claim (3.4.4), we get

$$\begin{aligned} E_n(f) &\leq 6n^{-1}E_{n-1}(f') \leq 6^2(n(n-1))^{-1}E_{n-2}(f'') \\ &\leq \cdots \leq 6^k(n(n-1)\cdots(n-k))^{-1}E_{n-k}(f^{(k)}) \\ &\leq 6^{k+1}(n(n-1)\cdots(n-k+1))^{-1}\omega\left(f^{(k)}, \frac{1}{n-k}\right) \\ &\leq cn^{-k}\omega\left(f^{(k)}, \frac{b-a}{2(n-k)}\right), \end{aligned}$$

where c is a constant independent of n, but dependent on k. □

By the proof of Theorem 3.4.1, we see that the constant c in the estimate (3.4.3) is dominated by 12^{k+1} for all $k \leq n/2$.

3.5 Characterization of Best Approximation

In this section, we characterize the best uniform approximating polynomials to a continuous function f on a finite interval $[a, b]$ in P_n.

A set of distinct points $x_0, x_1, \ldots, x_k$ satisfying $a \leq x_0 < x_1 < \ldots < x_{k-1} < x_k \leq b$ is called an *alternating set* for a continuous function g on $[a, b]$ if

$$|g(x_j)| = \|g\|_\infty,\ j = 0, \ldots, k \tag{3.5.1}$$

and

$$g(x_j) = -g(x_{j+1}),\ j = 0, \ldots, k-1. \tag{3.5.2}$$

For example, the set consisting of $-\pi/2, \pi/2$ is an alternating set for the sine function on $[-\pi, \pi]$.

Theorem 3.5.1 *Suppose $f \in C([a, b])$. Then p_n^* is a best uniform approximation on $[a, b]$ to f out of P_n if and only if there exists an alternating set for $f - p_n^*$ consisting of $n + 2$ points.*

To prove Theorem 3.5.1, we need a lemma.

Lemma 3.5.2 *Suppose $f \in C([a,b])$ and p_n^* is a best uniform approximation on $[a,b]$ to f out of P_n. Then there exist (at least) two distinct points $x_1, x_2 \in [a,b]$ such that*

$$|f(x_1) - p_n^*(x_1)| = |f(x_2) - p_n^*(x_2)| = \|f - p_n^*\|_\infty$$

and

$$f(x_1) - p_n^*(x_1) = -f(x_2) + p_n^*(x_2).$$

Proof. Set $m_1 := \max_{x\in[a,b]} f(x) - p_n^*(x)$ and $m_2 = \min_{x\in[a,b]} f(x) - p_n^*(x)$. Define

$$q_n^*(x) = p_n^*(x) + \frac{m_1 + m_2}{2}.$$

Then $q_n^*(x) \in P_n$ and

$$\|f - q_n^*\|_\infty = \frac{m_1 - m_2}{2} \le \|f - p_n^*\|_\infty \tag{3.5.3}$$

where we have used $|m_1|, |m_2| \le \|f - p_n^*\|_\infty$ to obtain the last inequality. On the other hand, p_n^* is the best approximation to f out of P_n, which implies that

$$\|f - p_n^*\|_\infty \le \|f - q_n^*\|_\infty. \tag{3.5.4}$$

Combining (3.5.3) and (3.5.4) leads to $m_1 = \|f - p_n^*\|_\infty$ and $m_2 = -\|f - p_n^*\|_\infty$. Therefore the set consisting of points x_1 and $x_2 \in [a,b]$, which are so chosen that $f(x_1) - p_n^*(x_1) = \max_{x\in[a,b]} f(x) - p_n^*(x)$ and $f(x_2) - p_n^*(x_2) = \min_{x\in[a,b]} f(x) - p_n^*(x)$, satisfies the required properties. □

Now we start to prove Theorem 3.5.1.

Proof of Theorem 3.5.1. First the sufficiency. Suppose that $p_n \in P_n$ and that the set of distinct points $x_0, \ldots, x_{n+1}$ forms an alternating set for $f - p_n$. Now we show that p_n^* is a best approximation. Suppose on the contrary that there exists $q_n \in P_n$ such that

$$\|f - q_n\|_\infty < \|f - p_n\|_\infty. \tag{3.5.5}$$

Recall that

$$p_n^*(x_j) - q_n(x_j) = [f(x_j) - q_n(x_j)] - [f(x_j) - p_n^*(x_j)].$$

Then it follows from (3.5.5) and the conditions (3.5.1) and (3.5.2) of an alternating set that the function $p_n^* - q_n$ alternates in sign as j runs from

0 to $n+1$. Hence the polynomial $q_n(x) - p_n^*(x)$ has at least one zero in each interval $(x_j, x_{j+1}), j = 0, \ldots, n$ for a total of at least $n+1$ zeros, which contradicts to the assumption that $q_n - p_n^*$ is a nonzero polynomial of degree at most n.

The necessity. Suppose that p_n^* is a best approximation to f. We may assume that $f \notin P_n$ since otherwise $p_n^* = f$ and then the whole question is trivial. Denote $\rho = \|f - p_n^*\|_\infty$. Then $\rho > 0$ by $f \neq p_n^*$. Let $x_0, \ldots, x_s$ be points of $[a,b]$ chosen so that $a = x_0 < x_1 < \ldots < x_s = b$ and so that $e(x) = f(x) - p_n^*(x)$ satisfies $|e(y) - e(z)| \leq \rho/2$ for all $y, z \in [x_j, x_{j+1}], 0 \leq j \leq s-1$. The existence of such a partition follows from the uniform continuity of $f - p_n^*$. Now we label every interval with a sign: positive, negative and zero. If the interval $[x_j, x_{j+1}]$ contains z such that $e(z) = \rho$, we label the interval $[x_j, x_{j+1}]$ by positive sign; If the interval $[x_j, x_{j+1}]$ contains z such that $e(z) = -\rho$, we label the interval by negative sign, otherwise we define the sign of the interval as zero. From the above construction, we see that

$$-\rho < f(x) - p_n^*(x) < \rho \tag{3.5.6}$$

for all x in some intervals with zero sign. For the later applications, we need relabel the sign of every interval. By Lemma 3.5.2, there exists at least one interval with positive sign and one with negative sign. Now we relabel the sign of every interval as follows: If the sign of the interval $[x_0, x_1]$ is positive or negative, then the new sign is the same as the old one. If the sign of $[x_0, x_1]$ is zero, then the new sign is labeled as positive (negative) if the closest interval with nonzero sign is positive (negative). Inductively we assume that all intervals $[x_j, x_{j+1}], j \leq l-1$, have been relabeled. Before we start to label the sign of the interval $[x_l, x_{l+1}]$, we observe that the sign of $[x_l, x_{l+1}]$ is either positive or negative if the sign of $[x_{l-1}, x_l]$ is zero. Now we label the sign of the interval of the interval $[x_l, x_{l+1}]$ as positive (negative) if either the sign of the interval is positive (negative), or the sign of the interval $[x_{l-1}, x_l]$ is positive (negative) and the signs of the intervals $[x_l, x_{l+1}]$ and $[x_{l+1}, x_{l+2}]$ (if there exists) are zero, and label the sign as zero otherwise. Continue this procedure until all intervals are labeled. Putting those connected intervals with same sign together leads to the following partition of the interval $[a,b]$, $a = y_0 < y_1 < \cdots < y_k = b$ so that either (i) the sign of $[y_l, y_{l+1}]$ is positive if $l = 0$ mod 3, zero if $l = 1$ mod 3, and negative if $l = 2$ mod 3, or (ii) the sign of $[y_l, y_{l+1}]$ is negative if $l = 0$ mod 3, zero if $l = 1$ mod 3, and positive if $l = 2$ mod 3.

Denote the maximum and minimum of the function $f(x) - p_n^*(x)$ for all x in the intervals with positive, zero and negative sign by m_1^+ and m_1^-, m_0^+

and m_0^-, and m_{-1}^+ and m_{-1}^- respectively. Then by (3.5.6) and the above construction of partition of the interval $[a, b]$, we have

$$\begin{cases} -\rho < m_1^- \le m_1^+ = \rho, \\ -\rho < m_0^- \le m_0^+ < \rho, \\ -\rho = m_{-1}^- \le m_{-1}^+ < \rho. \end{cases} \tag{3.5.7}$$

From the construction of the above partition of the interval $[a, b]$, we see that the proof is done if the number of the intervals $[y_l, y_{l+1}]$ with positive or negative sign exceeds $n+2$. Therefore it suffices to prove that the number of the intervals with zero sign exceeds $n+1$. Suppose on the contrary that the number, say k_0, of the intervals with zero sign is less than n. Choose points $z_j, 0 \le j \le k_0$ in the intervals with zero sign such that only one point is selected in each interval with zero sign. We label those points as $a < z_1 < z_2 < \ldots < z_{k_0} < b$, and define

$$q_n^*(x) = p_n^*(x) + \gamma(z_1 - x)(z_2 - x) \cdots (z_{k_0} - x),$$

where $0 \neq \gamma \in \mathbf{R}$ is chosen later. Obviously $q_n^* \in P_n$ by $k_0 \le n$ and

$$f(x) - q_n^*(x) = (f(x) - p_n^*(x)) - \gamma(z_1 - x)(z_2 - x) \cdots (z_{k_0} - x).$$

Choose the sign of γ be the same of the sign of the interval $[y_0, y_1]$. Then one may verify that $\gamma(z_1 - z) \cdots (z_{k_0} - z)$ are positive (negative) on the interval with positive (negative) sign. So any γ with sufficiently small magnitude,

$$\begin{aligned} -\rho &< m_1^- - \gamma(z_1 - x)(z_2 - x) \cdots (z_{k_0} - x) \le f(x) - q_n^*(x) \\ &\le \rho - \gamma(z_1 - x)(z_2 - x) \cdots (z_{k_0} - x) < \rho \end{aligned} \tag{3.5.8}$$

for any x in the interval with positive sign, where we have used (3.5.7). Similarly it follows (3.5.7) and the construction of $z_j, 1 \le j \le k_0$, that for γ with the same sign as the one of $[y_0, y_1]$ and sufficiently small magnitude,

$$\begin{aligned} -\rho &< -\rho - \gamma(z_1 - x)(z_2 - x) \cdots (z_{k_0} - x) \le f(x) - q_n^*(x) \\ &\le m_{-1}^+ - \gamma(z_1 - x)(z_2 - x) \cdots (z_{k_0} - x) < \rho \end{aligned} \tag{3.5.9}$$

for all x in the intervals with negative sign, and

$$\begin{aligned} -\rho &< m_0^- - \gamma(z_1 - x)(z_2 - x) \cdots (z_{k_0} - x) \le f(x) - q_n^*(x) \\ &\le m_0^+ - \gamma(z_1 - x)(z_2 - x) \cdots (z_{k_0} - x) < \rho \end{aligned} \tag{3.5.10}$$

for all x in the intervals with zero sign. Combining (3.5.8), (3.5.9) and (3.5.10) leads to

$$\|f - q_n^*\|_\infty < \|f - p_n^*\|_\infty$$

for γ with the same sign as the one of $[y_0, y_1]$ and sufficiently small magnitude, which is a contradiction since p_n^* is a best uniform approximation to f out of P_n. $\square$

An easy application of Theorem 3.5.1 leads to the explicit expression of the best uniform approximating constant to a continuous function.

Corollary 3.5.3 *Let f be a continuous on the finite interval $[a, b]$. Then the best approximating constant p_0^* to f is*

$$p_0^* = \frac{1}{2}\left[\max f(x) + \min f(x)\right],$$

and the approximation error $E_n(f)$ by constant functions can be computed by

$$E_n(f) = \frac{1}{2}\left[\max f(x) - \min f(x)\right].$$

Another application of Theorem 3.5.1 is the uniqueness of the best approximation to a continuous function by polynomials. Here we emphasize the L^∞ norm is not a strictly convex norm (see Section 2.1.4), and hence the uniqueness of the best approximation to a continuous function by a linear subspace does not hold in general.

Theorem 3.5.4 *Let f be a continuous function on $[a, b]$. Then for any $n \geq 0$ there exists unique polynomial $p_n^* \in P_n$, which is a best uniform approximating polynomial to $f \in C([a, b])$ in P_n.*

Proof. Set $E_n(f) = \inf_{p \in P_n} \|f - p\|_\infty$ and let p_n^* and $q_n^* \in P_n$ satisfy

$$\|f - p_n^*\|_\infty = \|f - q_n^*\|_\infty = E_n(f). \tag{3.5.11}$$

Then it suffices to prove that $p_n = p_n^*$. By the convexity of the set of all best approximations (Theorem 5.3.5),

$$\|f - (p_n^* + q_n^*)/2\|_\infty = E_n(f). \tag{3.5.12}$$

Set $r_n^* = (p_n^* + q_n^*)/2$. By Theorem 3.5.1, there exists an alternating set $x_0, \ldots, x_{n+1}$ for $f - r_n^*$, which is labeled so that $x_0 < x_1 < \ldots < x_{n+1}$. So

$$f(x_j) - r_n^*(x_j) = \frac{f(x_j) - p_n^*(x_j)}{2} + \frac{f(x_j) - q_n^*(x_j)}{2} = (-1)^{l+j} E_n(f)$$

for $l = 0$ or 1. This together with (3.5.11) implies that

$$f(x_j) - p_n^*(x_j) = f(x_j) - q_n^*(x_j) = (-1)^{l+j} E_n(f) \quad \forall\ j = 0, \ldots, n+1.$$

Thus

$$p_n^*(x_j) = q_n^*(x_j) \quad \forall\ j = 0, \ldots, n+1.$$

Hence $p_n^* = q_n^*$ since both are polynomials of degree at most n. □

3.6 Approximation by Bernstein Polynomials

In Section 5.2, we characterize the best approximating polynomials of a continuous function. Until now, there are few situations that the best approximating polynomials can be constructed explicitly. For instance, the best approximating constant to a continuous function is determined by the maximum and minimum of that function (see Corollary 3.5.3), and the best approximation polynomial of degree at most n to the polynomial x^{n+1} on $[-1, 1]$ is the Chebyshev polynomial $T_n(x)$ defined in Section 3.6 (see Theorem 1.9 in the book "An Introduction to the Approximation of Functions" by T. J. Rivlin). All those inspire us to consider explicit construction of polynomials to approximate a continuous function. In this section, we introduce the Bernstein polynomials and discuss the approximation power to a continuous function on $[0, 1]$ by corresponding Bernstein polynomials. Here given any function $h \in C([0, 1])$, we define its Bernstein polynomial of degree m, to be denoted by $B_m(h, t)$, by

$$B_m(h, t) = \sum_{k=0}^{m} h\left(\frac{k}{m}\right) \begin{pmatrix} m \\ k \end{pmatrix} t^k (1-t)^{m-k}, \ m \geq 1.$$

We also write $B_m(h, t)$ as $B_m h(t)$.

Theorem 3.6.1 *Let h be a continuous function on $[0, 1]$, and $B_m h$ be the corresponding Bernstein polynomials of degree m. Then*

$$\|h - B_m h\|_\infty \leq \frac{3}{2}\omega\Big(h, \frac{1}{\sqrt{m}}\Big), \ m \geq 1. \tag{3.6.1}$$

By Theorem 3.6.1, we have

Corollary 3.6.2 *Let h satisfy the Lipschitz condition of order α with constant K. Then*

$$\|B_m h - h\|_\infty \le \frac{3}{2} K m^{-\alpha/2},\ m \ge 1.$$

By Theorems 2.1.2 and 3.6.1, we show that for any continuous function h, $B_m h$ tends to h uniformly as m tends infinity.

Corollary 3.6.3 *Let h be a continuous function on $[0,1]$. Then*

$$\lim_{m\to\infty} \|B_m h - h\|_\infty = 0.$$

To prove Theorem 3.6.1, we need some basic properties of Bernstein polynomials.

Theorem 3.6.4 *Set $h_0(t) = 1, h_1(t) = t$ and $h_2(t) = t^2$. Then*

$$B_m h_0 = h_0, \tag{3.6.2}$$

$$B_m h_1 = h_1, \tag{3.6.3}$$

and

$$B_m h_2 = \frac{m-1}{m} h_2 + \frac{1}{m} h_1, \quad m \ge 2. \tag{3.6.4}$$

Proof. By binomial formula,

$$B_m h_0(t) = \sum_{k=0}^{m} \binom{m}{k} t^k (1-t)^{m-k} = (t + (1-t))^m = 1,$$

which proves (3.6.2). Similarly,

$$\begin{aligned}
B_m(h_1, t) &= \sum_{k=0}^{m} h_1\left(\frac{k}{m}\right) \binom{m}{k} t^k (1-t)^{m-k} \\
&= \sum_{k=0}^{m} \frac{k}{m} \binom{m}{k} t^k (1-t)^{m-k} \\
&= \sum_{k=1}^{m} \binom{m-1}{k-1} t^k (1-t)^{m-k} \\
&= t(t + (1-t))^{m-1} = t,
\end{aligned}$$

which leads to (3.6.3). Note that

$$\begin{aligned}
B_m(h_2 - h_1/m, t) &= \sum_{k=0}^{m} \Big(\frac{k(k-1)}{m^2}\Big) \left(\begin{array}{c} m \\ k \end{array} \right) t^k (1-t)^{m-k} \\
&= \frac{m-1}{m} \sum_{k=2}^{m} \left(\begin{array}{c} m-2 \\ k-2 \end{array} \right) t^k (1-t)^{m-k} \\
&= \frac{m-1}{m} t^2 (t + (1-t))^{m-2} = \frac{m-1}{m} t^2.
\end{aligned}$$

This together with (3.6.3) implies

$$B_m h_2 = \frac{1}{m} B_m h_1 + \frac{m-1}{m} h_2 = \frac{1}{m} h_1 + \frac{m-1}{m} h_2.$$

Hence (3.6.4) follows. □

Now we start to prove Theorem 3.6.1.
Proof of Theorem 3.6.1. By (3.6.2),

$$B_m h(t) - h(t) = \sum_{k=0}^{m} \Big(h\Big(\frac{k}{m}\Big) - h(t) \Big) \left(\begin{array}{c} m \\ k \end{array} \right) t^k (1-t)^{m-k}.$$

Therefore

$$\begin{aligned}
& |B_m h(t) - h(t)| \\
\le\ & \sum_{k=0}^{m} \Big| h\Big(\frac{k}{m}\Big) - h(t) \Big| \left(\begin{array}{c} m \\ k \end{array} \right) t^k (1-t)^{m-k} \\
\le\ & \sum_{k=0}^{m} \omega\Big(h, \Big|\frac{k}{m} - t\Big|\Big) \left(\begin{array}{c} m \\ k \end{array} \right) t^k (1-t)^{m-k} \\
\le\ & \sum_{k=0}^{m} \Big(1 + m^{1/2}\Big|\frac{k}{m} - t\Big|\Big) \omega(h, m^{-1/2}) \left(\begin{array}{c} m \\ k \end{array} \right) t^k (1-t)^{m-k} \\
\le\ & \omega(h, m^{-1/2}) + \sqrt{m}\,\omega(h, m^{-1/2}) \sum_{k=0}^{m} \Big|\frac{k}{m} - t\Big| \left(\begin{array}{c} m \\ k \end{array} \right) t^k (1-t)^{m-k}
\end{aligned}$$

where we have used the properties of modulus of continuity (Theorem 2.1.2). Hence it suffices to show

$$\sum_{k=0}^{m} \Big|\frac{k}{m} - t\Big| \left(\begin{array}{c} m \\ k \end{array} \right) t^k (1-t)^{m-k} \le \frac{1}{2\sqrt{m}}. \qquad (3.6.5)$$

By (3.6.2), (3.6.3) and (3.6.4), we get

$$
\begin{aligned}
& \sum_{k=0}^{m} \Big|\frac{k}{m} - t\Big| \begin{pmatrix} m \\ k \end{pmatrix} t^k (1-t)^{m-k} \\
\le & \left(\sum_{k=0}^{m} \Big|\frac{k}{m} - t\Big|^2 \begin{pmatrix} m \\ k \end{pmatrix} t^k (1-t)^{m-k}\right)^{1/2} \left(\sum_{k=0}^{m} \begin{pmatrix} m \\ k \end{pmatrix} t^k (1-t)^{m-k}\right)^{1/2} \\
= & \Big(B_m h_2(t) - 2t B_m h_1(t) + t^2 B_m h_0(t)\Big)^{1/2} (B_m h_0(t))^{1/2} \\
= & (t^2 + t(1-t)/m - 2t^2 + t^2)^{1/2} = \sqrt{\frac{t(1-t)}{m}}.
\end{aligned}
$$

This together with

$$t(1-t) = \frac{1}{4} - \Big(t - \frac{1}{2}\Big)^2 \le \frac{1}{4} \quad \forall\, t \in [0,1].$$

proves (3.6.5) and hence completes the proof of Theorem 3.6.1. $\square$

The estimate in Theorem 3.6.1 can not be improved in general. For example, by Theorem 3.6.1, for the function h defined by $h(t) = |t - 1/2|$ on $t \in [0,1]$, we have

$$\|B_m h - h\| \le \frac{3}{2\sqrt{m}}. \tag{3.6.6}$$

But for even m,

$$
\begin{aligned}
B_m h\Big(\frac{1}{2}\Big) - h\Big(\frac{1}{2}\Big) &= \sum_{k=0}^{m} \left(\Big|\frac{k}{m} - \frac{1}{2}\Big| - 0\right) \begin{pmatrix} m \\ k \end{pmatrix} \Big(\frac{1}{2}\Big)^k \Big(1 - \frac{1}{2}\Big)^{m-k} \\
&= 2^{-m+1} \sum_{k=0}^{m/2} \Big(\frac{1}{2} - \frac{k}{m}\Big) \begin{pmatrix} m \\ k \end{pmatrix} \\
&= 2^{-m-1} \left(\sum_{k=0}^{m} \begin{pmatrix} m \\ k \end{pmatrix} + \begin{pmatrix} m \\ m/2 \end{pmatrix}\right) \\
&\quad -2^{-m+1} \sum_{k'=0}^{m/2-1} \begin{pmatrix} m-1 \\ k' \end{pmatrix} \\
&= 2^{-m-1} \left(2^m + \begin{pmatrix} m \\ m/2 \end{pmatrix}\right) - 2^{-m} 2^{m-1} \\
&= 2^{-m-1} \begin{pmatrix} m \\ m/2 \end{pmatrix} \ge \frac{1}{2\sqrt{m}},
\end{aligned}
$$

where we have used the following Stirling formula

$$\sqrt{2\pi k}\Big(\frac{k}{e}\Big)^k < k! < \sqrt{2\pi k}\Big(\frac{k}{e}\Big)^k(1+\frac{1}{4k}).$$

Thus

$$\|B_m h - h\| \geq B_m h\Big(\frac{1}{2}\Big) - h\Big(\frac{1}{2}\Big) \geq \frac{1}{2\sqrt{m}}. \tag{3.6.7}$$

Comparing (3.6.6) and (3.6.7), we see that the estimate of approximation error by Bernstein polynomials can not be improved in general.

By Theorem 3.4.1, the uniform approximation error to h by polynomials in P_m is dominated by $3m^{-1}$, which is much smaller than $m^{-1/2}$ for sufficiently large m. Due to the difficulty to give an explicit expression of best uniform approximating polynomial to a continuous function, it is still a good choice to use Bernstein polynomial to approximate a continuous function on $[0,1]$ since Bernstein polynomial can be computed explicitly and easily.

3.7 Approximation by Interpolating Polynomials

In this section, we study the approximating power of interpolating polynomials as the number of interpolation knots increases. For simplicity, we restrict ourselves to consider the approximating property of interpolating polynomials on the interval $[-1,1]$. To this end, we consider an infinite triangular array of knots

$$X: \quad \begin{array}{llll} x_1^{(1)} & & & \\ x_1^{(2)} & x_2^{(2)} & & \\ x_1^{(3)} & x_2^{(3)} & x_3^{(3)} & \\ \vdots & & & \\ x_1^{(n)} & x_2^{(n)} & \cdots & x_n^{(n)} \\ \vdots & & & \end{array} \tag{3.7.1}$$

where for $n \geq 1$, $-1 \leq x_1^{(n)} < x_2^{(n)} < \ldots < x_n^{(n)} \leq 1$. We denote $\ell_j^{(n)}, j = 1,\ldots,n$, the fundamental polynomials corresponding to the interpolation problem at knots $-1 \leq x_1^{(n)} < x_2^{(n)} < \ldots < x_n^{(n)} \leq 1$. We call the function

$$\lambda_n(X,x) = \sum_{j=1}^{n} |l_j^{(n)}(x)|, \; n = 1,\ldots, \tag{3.7.2}$$

the *Lebesgue function* of order n of X, and the quantity

$$\Lambda_n(X) = \sup_{-1\le x\le 1} \lambda_n(X,x) \tag{3.7.3}$$

the *Lebesgue constant* of order n of X. Let L_n be the unique polynomial of degrees at most $n-1$ solving the Lagrange interpolation problem of f at the knots $-1 \le x_1^{(n)} < x_2^{(n)} < \ldots < x_n^{(n)} \le 1$. Set

$$G_n = G_n(f,X) := \|f - L_n\|_\infty. \tag{3.7.4}$$

Then we have the following estimate to the approximating error by interpolating polynomials L_n.

Theorem 3.7.1 *Let X be the infinity triangular array of knots in (3.7.1). Then*

$$G_n(f,X) \le (1+\Lambda_n(X))E_{n-1}(f),\ \ n = 1,\ldots, \tag{3.7.5}$$

where G_n is the approximating error by interpolating polynomials L_n in (3.7.4), $E_{n-1}(f)$ is the approximation error to f by the best uniform approximation polynomial in P_{n-1}, and $\Lambda_n(X)$ is the Lebesgue constant of order n of X in (3.7.3).

Proof. Let $p^*_{n-1} \in P_{n-1}$ be the best uniform approximation to f on $[-1,1]$ out of P_{n-1}, i.e.,

$$\|f - p^*_{n-1}\|_\infty = E_{n-1}(f). \tag{3.7.6}$$

By the uniqueness of the Lagrange interpolation polynomial,

$$L_n(p^*_{n-1}, X^{(n)}, x) = p^*_{n-1}(x), \tag{3.7.7}$$

where $X^{(n)}$ is the set of knots $x_1^{(n)},\ldots,x_n^{(n)}$. Therefore by (3.2.5), (3.7.6) and (3.7.7), we obtain

$$\begin{aligned}
|f(x) - L_n(x)| &\le |f(x) - p^*_{n-1}(x)| + |p^*_{n-1}(x) - L_n(x)| \\
&\le |f(x) - p^*_{n-1}(x)| + |L_n(p^*_{n-1} - f, X^{(n)}, x)| \\
&\le E_{n-1}(f) + \|f - p^*_{n-1}\|_\infty \sum_{l=1}^{n} |l_j^{(n)}(x)| \\
&\le E_{n-1}(f)\Big(1 + \sum_{j=1}^{n} |l_j^{(n)}(x)|\Big).
\end{aligned}$$

Hence the estimate follows by taking supremum at both sides of the above inequality. □

From Theorem 3.7.1, we see that the Lebesgue constant plays important role for a good approximation by interpolating polynomials. For the case that the zeros of Chebyshev polynomials of order n are chosen to be the knots at level n, we have the following estimate to the corresponding Lebesgue constant.

Theorem 3.7.2 *Let T be the infinite triangular array of knots with $x_i^{(n)}$ being defined by*

$$x_i^{(n)} = -\cos\frac{(2i-1)\pi}{2n}, \ 1 \le i \le n, \ n \ge 1, \tag{3.7.8}$$

and denote the corresponding Lebesgue constant of order n by $\Lambda_n(T)$. Then we have

$$\Lambda_n(T) \le \frac{2}{\pi}\ln n + 4. \tag{3.7.9}$$

Proof. For the knots in (3.7.8), one may easily verify that

$$l_i(x) = \frac{\prod_{1\le j\le n, j\ne i}(\cos\theta + \cos\theta_j)}{\prod_{1\le j\le n, j\ne i}(-\cos\theta_i + \cos\theta_j)} = \frac{(-1)^{n+i-1}\sin\theta_i\cos n\theta}{\cos\theta + \cos\theta_i}, \tag{3.7.10}$$

where $x = \cos\theta$ and $\theta_i = (2i-1)\pi/(2n), 1 \le i \le n$. Denote the corresponding Lebesgue function of order n of T by $\lambda_n(x,T)$. Then it follows from (3.7.10) that

$$\begin{aligned} \lambda_n(x,T) &= \frac{|\cos n\theta|}{n}\sum_{i=1}^{n}\frac{|\sin\theta_i|}{|\cos\theta+\cos\theta_i|} \\ &= \frac{|\cos n\theta|}{2n}\sum_{i=1}^{n}\Big|\cot\frac{\theta+\theta_i}{2} - \cot\frac{\theta-\theta_i}{2}\Big|, \end{aligned} \tag{3.7.11}$$

where $x = \cos\theta$ and we denote $\cot x = \cos x/\sin x$ as usual. Clearly, $\lambda_n(\cos\theta, T)$ is an even function and has period π/n about θ. This implies that

$$\Lambda_n(T) = \max_{-1\le x\le 1}\lambda_n(x,T) = \max_{0\le\theta\le\pi/2n}\lambda_n(\cos\theta,T). \tag{3.7.12}$$

Recall that $|l_j(\cos\theta)|$ is monotonously decreasing about θ on $[0, \pi/(2n)]$ by (3.7.10) and the monotonicity of $\cos\theta$. This together with (3.7.12) implies that

$$\Lambda_n(T) = \lambda_n(1, T) = \frac{1}{n}\sum_{i=1}^{n} \cot\frac{\theta_i}{2}.$$

One may show that $\cot x \le 1/x$ for all $0 \le x \le \pi/2$ since $h(x) = \cot x - 1/x$ satisfies $h(0) = 0$ and $h'(x) \le 0$ for all $x \in [0, \pi/2]$. Thus

$$\begin{aligned}\Lambda_n(T) &= \frac{1}{n}\sum_{i=1}^{n} \cot\frac{\theta_i}{2} \le \frac{4}{\pi}\sum_{j=1}^{n} \frac{1}{2j-1} \\ &\le \frac{4}{\pi} + \frac{2}{\pi}\int_1^{2n-1} \frac{dx}{x} \le 4 + \frac{2\ln n}{\pi}\end{aligned}$$

This proves (3.7.9) and completes the proof. □

By Theorems 3.7.1 and 3.7.2, we see that

$$G_n(f, T) \le \Big(5 + \frac{2\ln n}{\pi}\Big) E_{n-1}(f).$$

Therefore if the modulus of continuity of f satisfies $\lim_{\delta\to 0} \omega(f, \delta)\ln\frac{1}{\delta} = 0$, then the Lagrange interpolating polynomial to f at the zeros of Chebyshev polynomials converges to f uniformly.

We remark that for any infinite triangular arrays X, the corresponding Lebesgue constant $\Lambda_n(X)$ satisfies

$$\Lambda_n(X) \ge \frac{2\ln n}{\pi^2} - 1.$$

For the infinite triangular arrays E from equally spaced points on $[-1, 1]$, the corresponding Lebesgue constant $\Lambda_n(E)$ as n tends to infinity increase exponentially and hence much worse that the asymptotic behavior of the Lebesgue constant corresponding to the zeros of Chebyshev polynomials (see Section 4.2 of Rivlin's book "An Introduction to the Approximation of Functions" for details).

Exercises

1. Show that $\cos k\theta$ is a polynomial of degree k about $\cos\theta$ with leading coefficients 2^{k-1} for $k \geq 1$.

2. Find all alternating sets for the function $\sin x$ on $[-2\pi, 2\pi]$.

3. Let $f(x) = x^2, x \in [0,1]$ and $p^*(x) = ax+b$ be the best uniform approximating affine function to f. Compute $\|f-p^*\|_\infty$ and find all alternating sets for $f-p^*$.

4. Suppose that $q \in P_n, f \in C([a,b])$, and

$$f(x_j) - q(x_j) = (-1)^j A_j$$

with $A_j > 0, j = 0, \ldots, n = 1$. Prove that

$$E_n(f) \geq \min(A_0, \ldots, A_{n+1}).$$

5. Let f be a continuous function on $[a,b]$. Then $p^* \in P_n$ is a best approximation to f if and only if for each polynomial q,

$$\max_{x\in A_0}(f(x) - p^*(x))q(x) \geq 0,$$

where A_0 denotes the set of all points $y \in [a,b]$ for which $|f(y) - p^*(y)| = \|f - p^*\|_\infty$.

6. Let $f(x) = |x - 1/2|, x \in [0,1]$. Compute the corresponding Berstein polynomials $B_m f$ of degree $0, 1, 2$.

7. Let $h_3(x) = x^3, x \in [0,1]$. Compute the corresponding Bernstein polynomials $B_m h_3$ of degree m, where $m \geq 3$.

8. For $k \geq 2$, show that the Bernstein polynomials $B_m h_k$ of the function $h_k(x) = x^k, x \in [0,1]$ are polynomials of degree k for all $m \geq k$, and also find the leading coefficients of $B_m h_k$.

9. Let h be bounded on $[0,1]$ and suppose that h has continuous second derivative h''. Show that

$$\lim_{n\to\infty} m(B_m h(x) - h(x)) = -\frac{x(1-x)}{2} f''(x).$$

Chapter 4

Spline Approximation

In this chapter, we study the Lagrange interpolation problem and its approximation power.

4.1 Piecewise Linear Interpolation

Polynomial interpolation has the drawback of producing approximations that may be excessively oscillatory between knots. If we abandon the requirement that the approximating functions are polynomials, a much more general family of approximating functions that suggests itself is the set of *piecewise polynomials*, i.e., functions that are polynomials, possibly different at different subdomains of the domain which we are approximating. In this section, we consider the interpolation problem by continuous piecewise linear functions and its approximating property.

Given a set $X = \{x_1, x_2, \ldots, x_n\}$ labeled so that $x_1 < x_2 < \cdots < x_n$. We let $S_1(X)$ be the space of all continuous function which agrees with an affine function on each subinterval $[x_i, x_{i+1}], i = 1, \ldots, n-1$. For any function $g \in S_1(X)$, it has "corner" where two linear pieces meets, and generally, have no derivative at a corner. For example, the hat function h define by $h(x) = \max(0, 1-|x|), x \in [-1, 1]$, is piecewise affine function and has corner at the points $-1, 0$ and 1.

Define

$$l_1(x) = \begin{cases} \frac{x_2-x}{x_2-x_1} & \text{if } x \in [x_1, x_2] \\ 0 & \text{otherwise,} \end{cases} \tag{4.1.1}$$

$$l_n(x) = \begin{cases} \frac{x-x_{n-1}}{x_2-x_1} & \text{if } x \in [x_{n-1}, x_n] \\ 0 & \text{otherwise,} \end{cases} \tag{4.1.2}$$

and

$$l_i(x) = \begin{cases} \frac{x - x_{i-1}}{x_i - x_{i-1}} & \text{if } x \in [x_{i-1}, x_i] \\ \frac{x_{i+1} - x}{x_{i+1} - x_i} & \text{if } x \in [x_i, x_{i+1}] \\ 0 & \text{otherwise} \end{cases} \tag{4.1.3}$$

for $2 \le i \le n-1$. One may easily verify that the functions $l_i(x), 1 \le i \le n$, satisfy

$$l_i(x_j) = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \ne j. \end{cases} \tag{4.1.4}$$

So the dimension of $S_1(X)$ is n.

To study the interpolation problem to a given continuous function f on a finite interval $[a, b]$, we need make some restrictions on the knots,

$$x_1 = a \quad \text{and} \quad x_n = b \tag{4.1.5}$$

By (4.1.1), (4.1.2) and (4.1.3), for any given continuous function f, the interpolating function g in $S_1(X)$ satisfying

$$f(x_i) = g(x_i),\ 1 \le i \le n,$$

is

$$g(x) = \sum_{i=1}^{n} f(x_i) l_i(x). \tag{4.1.6}$$

Because of (4.1.4), we call the functions $l_i, 1 \le i \le n$, the *fundamental piecewise linear functions* at the knots $x_1, \ldots, x_n$.

Let X be an infinite triangular array of knots in $[a, b]$,

$$X: \quad \begin{array}{llll} x_1^{(2)} & x_2^{(2)} & & \\ x_1^{(3)} & x_2^{(3)} & x_3^{(3)} & \\ \vdots & & & \\ x_1^{(n)} & x_2^{(n)} & \ldots & x_n^{(n)} \\ \vdots & & & \end{array} \tag{4.1.7}$$

and satisfy

$$a = x_1^{(n)} < x_2^{(n)} < \ldots < x_n^{(n)} = b, \quad \forall\, n \ge 2. \tag{4.1.8}$$

We define

$$\triangle_n = \max_{1 \le i \le n-1} |x_{i+1}^{(n)} - x_i^{(n)}|,\ n \ge 1. \tag{4.1.9}$$

For any continuous function f on $[a,b]$, we denote the interpolating piecewise linear function to f at the knots $x_1^{(n)}, \ldots, x_n^{(n)}$ by $g_n(x)$. Then we have the following estimate to the approximation error to f by interpolating piecewise linear function g_n.

Theorem 4.1.1 *Let X be an infinite triangular array of knots in (4.1.7) and satisfy (4.1.8), and let f, g_n and $\triangle_n, n \geq 1$, be defined by above. Then*

$$\|f - g_n\|_\infty \leq \omega(f, \triangle_n). \tag{4.1.10}$$

Proof. For $x \in [x_i^{(n)}, x_{i+1}^{(n)}],\ 1 \leq i \leq n-1$, we have

$$g_n(x) = \frac{x_{i+1}^{(n)} - x}{x_{i+1}^{(n)} - x_i^{(n)}} f(x_i^{(n)}) + \frac{x - x_i^{(n)}}{x_{i+1}^{(n)} - x_i^{(n)}} f(x_{i+1}^{(n)}),$$

which implies that

$$\begin{aligned} |f(x) - g_n(x)| &\leq \frac{x_{i+1}^{(n)} - x}{x_{i+1}^{(n)} - x_i^{(n)}} |f(x_i^{(n)}) - f(x)| + \frac{x - x_i^{(n)}}{x_{i+1}^{(n)} - x_i^{(n)}} |f(x_{i+1}^{(n)}) - f(x)| \\ &\leq \frac{x_{i+1}^{(n)} - x}{x_{i+1}^{(n)} - x_i^{(n)}} \omega(f, \triangle_n) + \frac{x - x_i^{(n)}}{x_{i+1}^{(n)} - x_i^{(n)}} \omega(f, \triangle_n) = \omega(f, \triangle_n). \end{aligned}$$

This completes the proof. □

4.2 Quadratic Spline Interpolation

In this section, we introduce the quadratic spline and study its corresponding interpolation and approximation problem.

4.2.1 Quadratic Spline

Suppose that X denotes the set of real numbers $x_0, \ldots, x_n$ labeled so that $a \leq x_0 < x_1 < \ldots < x_n \leq b$. Let $S_2(X)$ be the set of all functions $s(X, x) = s(x) \in C^1([a,b])$ so that in each interval $[x_i, x_{i+1}], 0 \leq i \leq n-1$, $s(x)$ agrees with a polynomial of degree at most 2. We call the function $s \in S_2(X)$ a quadratic spline, and the points $x_0, \ldots, x_n$ knots. Obviously $1, x, x^2$ are quadratic spline. One may easily verify that $(x + 2/3)_+^2$ is a quadratic spline with knots $-1, -2/3, 2/3, 1$, where we define $x_+ = \max(x, 0)$.

Theorem 4.2.1 *Let* $X = \{x_0, x_1, \ldots, x_n\}$ *be labeled so that* $a \le x_0 < x_1 < \ldots < x_n \le b$. *Then both*

$$\{1, x, x^2, (x - x_1)_+^2, \ldots, (x - x_{n-1})_+^2\}$$

and

$$\{1, x, x^2, (x - x_1)_-^2, \ldots, (x - x_{n-1})_-^2\}$$

are bases of $S_2(X)$, *where* $x_+ = \max(0, x)$ *and* $x_- = \min(0, x), x \in \mathbf{R}$.

Proof. First we prove that

$$\{1, x, x^2, (x - x_1)_+^2, \ldots, (x - x_{n-1})_+^2\}$$

is a basis of $S_2(X)$. Obviously, $1, x, x^2, (x - x_1)_+^2, \ldots, (x - x_{n-1})_+^2$ belong to $C^1([a, b])$ and agree with quadratic polynomials on every interval $[x_i, x_{i+1}], 0 \le i \le n - 1$. Hence they are quadratic splines on the knots X.

For any given $f \in S_2(X)$, denote the jump of the second derivative f'' from the left hand side of $x_i, 1 \le i \le n-1$, to the right hand side by a_i, and define

$$g(x) = \frac{1}{2} \sum_{i=1}^{n-1} a_i (x - x_i)_+^2.$$

Therefore $h := f - g \in S_2(X)$ and the jump of h'' from the left hand side of $x_i, 1 \le i \le n - 1$, to the right hand side is always zero. This together with the definition of a quadratic spline in $S_2(X)$ implies that h is a polynomial of degree at most two, say $h(x) = b_0 + b_1 x + b_2 x^2$. Hence f is linear combination of $1, x, x^2, (x - x_1)_+^2, \ldots, (x - x_{n-1})_+^2$. Moreover from the proof above, we see that $1, x, x^2, (x - x_1)_+^2, \ldots, (x - x_{n-1})_+^2$ are linear independent. Therefore they form a basis of $S_2(X)$.

We may similarly prove that $\{1, x, x^2, (x - x_1)_-^2, \ldots, (x - x_{n-1})_-^2\}$ is a basis. We omit the detail here. $\square$

By Theorem 4.2.1, we have

Corollary 4.2.2 *The dimension of the space* $S_2(X)$ *of quadratic splines is* $n + 2$, *and the dimension of* $S_2(X)$ *exceeds the number of knots by* 1.

4.2.2 Quadratic Spline Interpolation

In this section, we study the Lagrange interpolation problem by quadratic splines. Given points $a = x_0 < x_1 < \cdots < x_n = b$, we see that the dimension of quadratic spline is $n+2$, which exceeds the number $n+1$ of knots.

Theorem 4.2.3 *Take points $a = x_0 < x_1 < \cdots < x_n = b$ and set $t_0 = x_0, t_{n+1} = x_n$ and $t_i = (x_{i-1} + x_i)/2, 1 \le i \le n$. Then for any given $f_i \in \mathbf{R}, 0 \le i \le n+1$, there exists unique quadratic spline $s \in S_2(X)$ so that*

$$s(t_i) = f_i,\ 0 \le i \le n+1. \tag{4.2.1}$$

Proof. Let $s \in S_2(X)$ satisfy (4.2.1) and denote $s(x_i)$ by $s_i, 0 \le i \le n$. From the values of s on x_{i-1}, t_i, x_i, the quadratic spline s agrees with

$$p_i(x) = 2s_{i-1}\frac{(x-t_i)(x-x_i)}{(x_i-x_{i-1})^2} - 4f_i\frac{(x-x_{i-1})(x-x_i)}{(x_i-x_{i-1})^2} + 2s_i\frac{(x-t_i)(x-x_{i-1})}{(x_i-x_{i-1})^2} \tag{4.2.2}$$

on $[x_{i-1}, x_i]$. By direct computation, we have

$$p_i'(x_{i-1}) = \frac{-3s_{i-1} + 4f_i - s_i}{x_i - x_{i-1}} \tag{4.2.3}$$

and

$$p_i'(x_i) = \frac{s_{i-1} - 4f_i + 3s_i}{x_i - x_{i-1}}. \tag{4.2.4}$$

Therefore it follows from $s \in C^1([a,b])$ that $p_i'(x_i) = p_{i+1}'(x_i)$, which together with (4.2.3) and (4.2.4) leads to

$$\delta_{i+1}s_{i-1} + 3(\delta_i + \delta_{i+1})s_i + \delta_i s_{i+1} = 4\delta_{i+1}f_i + 4\delta_i f_{i+1}, 1 \le i \le n-1, \tag{4.2.5}$$

where $\delta_i = x_i - x_{i-1}$. Set $s = (s_1, \ldots, s_{n-1})$,

$$v = 4(\delta_2 f_1 + \delta_1 f_2, \ldots, \delta_n f_{n-1} + \delta_{n-1} f_n)^T - (\delta_2 f_0, 0, \ldots, 0, \delta_{n-1} f_{n+1})^T,$$

and

$$A = \begin{pmatrix} 3(\delta_2+\delta_1) & \delta_1 & 0 & \cdots & 0 & 0 \\ \delta_3 & 3(\delta_3+\delta_2) & \delta_2 & \cdots & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \delta_{n-2} & 3(\delta_{n-2}+\delta_{n-3}) & \delta_{n-3} \\ 0 & 0 & \cdots & \cdots & \delta_{n-1} & 3(\delta_{n-1}+\delta_{n-2}) \end{pmatrix}.$$

Then we may write the linear system (4.2.5) as

$$As = v. \tag{4.2.6}$$

Note that A is a strictly diagonal dominated matrix. Here we say that a matrix $(a_{ij})_{1\le i,j\le m}$ is strictly diagonal dominated if $|a_{ii}| > \sum_{j\neq i}|a_{ij}|$ for all $1 \le i \le n$. Then A is nonsingular. Therefore the linear system (4.2.1) is solvable and the solution is unique by (4.2.6). $\square$

4.2.3 Quadratic Spline Approximation

Suppose that $X = \{x_0, x_1, \ldots, x_n\}$ satisfies $a = x_0 < x_1 < \cdots < x_n = b$. We define

$$\delta = \delta(X) = \max_{1\le i\le n} |x_i - x_{i-1}|.$$

Then we have the following estimate about the error between a continuous function and its approximating quadratic interpolation spline.

Theorem 4.2.4 *Suppose that f be a differentiable function on $[a, b]$ and $s \in S_2(X)$ satisfies*

$$s(a) = f(a),\ \ s(b) = f(b) \quad \text{and} \quad s\Big(\frac{x_i + x_{i-1}}{2}\Big) = f\Big(\frac{x_i + x_{i-1}}{2}\Big), 1 \le i \le n \tag{4.2.7}$$

Then

$$\|f - s\|_\infty \le C\delta\omega(f', \delta), \tag{4.2.8}$$

where C is a positive constant independent of X.

Proof. Set $a_i = (x_i - x_{i-1})^{-1}|s(x_i) - f(x_i)|, 0 \le i \le n$. Then it follows from (4.2.7) that

$$a_0 = a_n = 0. \tag{4.2.9}$$

By Taylor's expansion,

$$\begin{cases} |f(t_i) - f(x_i) + \frac{\delta_i}{2} f'(x_i)| \le \frac{\delta}{2}\omega(f', \delta) \\ |f(t_{i+1}) - f(x_i) - \frac{\delta_{i+1}}{2} f'(x_i)| \le \frac{\delta}{2}\omega(f', \delta). \end{cases} \tag{4.2.10}$$

Combining (4.2.5) and (4.2.10), we obtain

$$3(\delta_i + \delta_{i+1})\delta_i a_i \le \delta_i\delta_{i+1}a_{i+1} + \delta_{i+1}\delta_{i-1}a_{i-1} + 4\delta_i\delta_{i+1}\delta\omega(f', \delta),$$

which implies that

$$3\Big(1+\frac{\delta_i}{\delta_{i+1}}\Big)a_i \le a_{i+1} + 4\delta\omega(f',\delta) + \frac{\delta_{i-1}}{\delta_i}a_{i-1}. \tag{4.2.11}$$

By (4.2.9) and (4.2.11), we may prove

$$\left(\frac{8}{9}+\frac{\delta_i}{\delta_{i+1}}\right)a_i \le \frac{1}{3}a_{i+1} + 2\delta\omega(f',\delta) \tag{4.2.12}$$

for $1 \le i \le n-1$ by induction. Thus

$$a_i \le \frac{3}{8}a_{i+1} + \frac{9}{4}\delta\omega(f',\delta),\ 1 \le i \le n, \tag{4.2.13}$$

which together with (4.2.9) that

$$a_i \le \frac{18}{5}\delta\omega(f',\delta). \tag{4.2.14}$$

Therefore

$$|f(x_i) - s(x_i)| \le 4|x_i - x_{i-1}|\delta\omega(f',\delta). \tag{4.2.15}$$

Let $p_i^*(x)$ be the quadratic polynomial in (4.2.2) with s_{i-1}, f_i, s_i replaced by $f(x_{i-1}), f(t_i), f(x_i)$. Then

$$p_i^*(x_{i-1}) = f(x_{i-1}),\ p_i^*(t_i) = f(t_i),\ p_i^*(x_i) = f(x_i), \tag{4.2.16}$$

and

$$\|p_i^* - p_i\|_\infty \le 4\delta\omega(f',\delta) \tag{4.2.17}$$

by (4.2.15). By Taylor's formula, we obtain

$$\begin{aligned} & \left|(f-p_i^*)(x_{i-1}) - (f-p_i^*)(t_i) + \frac{1}{2}(f-p_i^*)'(t_i)\delta_i - \frac{f(x_{i-1}) - 2f(t_i) + f(x_i)}{2}\right| \\ + & \left|(f-p_i^*)(x_i) - (f-p_i^*)(t_i) - \frac{1}{2}(f-p_i^*)'(t_i)\delta_i - \frac{f(x_{i-1}) - 2f(t_i) + f(x_i)}{2}\right| \\ \le & \ \delta_i\omega(f',\delta), \end{aligned} \tag{4.2.18}$$

where we have used (4.2.2) to compute the second derivative of p_i^*. This together with (4.2.15) leads to

$$|(f-p_i^*)'(t_i)\delta_i| + |f(x_{i-1}) - 2f(t_i) + f(x_i)| \le 4\delta\omega(f',\delta). \tag{4.2.19}$$

Therefore the estimate (4.2.8) follows from (4.2.17) and (4.2.19), since for $x \in [x_{i-1}, x_i]$,

$$\begin{aligned} |f(x) - s(x)| &\leq |f(x) - p_i^*(x)| + |p_i(x) - p_i^*(x)| \\ &\leq |(f - p_i^*)'(t_i)||x - t_i| + \frac{2|f(x_i) - 2f(t_i) + f(x_i)|}{(x_i - x_{i-1})^2}(x - t_i)^2 \\ &\quad +\delta_i\omega(f', \delta) + 4\delta\omega(f', \delta) \\ &\leq 12\delta\omega(f', \delta). \end{aligned}$$

This proves (4.2.8). □

4.3 Cubic Spline Interpolation

In this section, we introduce the cubic spline and study the corresponding interpolation and approximation problem.

4.3.1 Cubic Spline Interpolation

Suppose that X denotes the set of real numbers $x_0, \ldots, x_n$, where

$$a \leq x_0 < x_1 < \ldots < x_n \leq b.$$

Let $S_3(X)$ be the set of all functions $s(X, x) = s(x) \in C^2([a, b])$ having the property that in each interval $[x_i, x_{i+1}], 0 \leq i \leq n - 1$, $s(x)$ agrees with a polynomial of degree at most 3. We call the function $s \in S_3(X)$ a cubic spline, and the points $x_0, \ldots, x_n$ knots. Obviously a polynomial of degree at most 3 is a cubic spline. Also one may verify that $\max(0, (x + 2/3)^3)$ is a cubic spline with knots $-1, -2/3, 2/3, 1$. By similar procedure in the proof of Theorem 4.2.1, we have

Theorem 4.3.1 *Let $X = \{x_0, x_1, \ldots, x_n\}$ be labeled so that $a \leq x_0 < x_1 < \ldots < x_n \leq b$. Then both*

$$\{1, x, x^2, x^3, (x - x_1)_+^3, \ldots, (x - x_{n-1})_+^3\}$$

and

$$\{1, x, x^2, x^3, (x - x_1)_-^3, \ldots, (x - x_{n-1})_-^3\}$$

are bases of $S_3(X)$, where $x_+ = \max(0, x)$ and $x_- = \min(0, x), x \in \mathbf{R}$. Also the dimension of the space $S_3(X)$ of cubic spline is $n + 3$.

By Theorem 4.3.1, the dimension of $S_3(X)$ is $n+3$, which exceeds the number of the knots X by 2. To this end, we need two additional restrictions before we start to consider the interpolating problem. We observe that the unique polynomial p of degrees at most 3 that satisfies

$$p(\alpha)=u_1,\ p(\beta)=u_2,\ p'(\alpha)=v_1,\ p'(\beta)=v_2,$$

is

$$\begin{aligned} p(x) &= u_1\left[\frac{(x-\beta)^2}{(\beta-\alpha)^2}+2\frac{(x-\alpha)(x-\beta)^2}{(\beta-\alpha)^3}\right] \\ &\quad +u_2\left[\frac{(x-\alpha)^2}{(\beta-\alpha)^2}-2\frac{(x-\beta)(x-\alpha)^2}{(\beta-\alpha)^3}\right] \\ &\quad +v_1\frac{(x-\alpha)(x-\beta)^2}{(\beta-\alpha)^3}+v_2\frac{(x-\alpha)^2(x-\beta)}{(\beta-\alpha)^2}, \end{aligned} \tag{4.3.1}$$

where $\alpha<\beta$. For the case $\alpha=0$ and $\beta=1$, the above polynomial is

$$p(x)=u_1(x-1)^2(1+2x)+u_2x^2(3-2x)+v_1x(x-1)^2+v_2x^2(x-1).$$

The above observation inspires us to consider the Lagrange interpolation by cubic spline with restriction on the first derivative on the boundary knots.

Theorem 4.3.2 *Let $X=\{x_0,\dots,x_n\}$ be points on $\mathbf{R}$ labelled so that $x_0<x_1<\cdots<x_n$. Fix numbers s_0' and s_n'. Then for any given $f_i, 0\le i\le n$, there exists a unique cubic spline $s\in S_3(X)$ satisfying*

$$s(f,X,x_i)=f_i,\ 0\le i\le n \tag{4.3.2}$$

and

$$s'(f,X,x_i)=s_i',\ i=0,n. \tag{4.3.3}$$

Proof. Let $s\in S_3(X)$ satisfies (4.3.2) and (4.3.3), and denote the $s'(x_i)=s_i', 1\le i\le n-1$. Then by (4.3.1), $s(x)$ agree with

$$\begin{aligned} p_i(x) &= f_{i-1}\left[\frac{(x-x_i)^2}{(x_i-x_{i-1})^2}+2\frac{(x-x_{i-1})(x-x_i)^2}{(x_i-x_{i-1})^3}\right] \\ &\quad +f_i\left[\frac{(x-x_{i-1})^2}{(x_i-x_{i-1})^2}-2\frac{(x-x_i)(x-x_{i-1})^2}{(x_i-x_{i-1})^3}\right] \\ &\quad +s_{i-1}'\frac{(x-x_{i-1})(x-x_i)^2}{(x_i-x_{i-1})^3}+s_i'\frac{(x-x_{i-1})^2(x-x_i)}{(x_i-x_{i-1})^2} \end{aligned} \tag{4.3.4}$$

on $[x_{i-1}, x_i], 1 \le i \le n$. By direct computation, we obtain

$$p_i''(x_{i-1}) = \delta_i^{-2}(-6f_{i-1} + 6f_i - 4\delta_i s_{i-1}' - 2\delta_i s_i') \tag{4.3.5}$$

and

$$p_i''(x_i) = \delta_i^{-2}(6f_{i-1} - 6f_i + 2\delta_i s_{i-1}' + 4\delta_i s_i') \tag{4.3.6}$$

where we set $\delta_i = x_i - x_{i-1}$ as usual. By $s \in S_3(X)$, we have

$$p_i''(x_i) = p_{i+1}''(x_i),\ 1 \le i \le n-1. \tag{4.3.7}$$

Combining (4.3.5), (4.3.6) and (4.3.7) leads to

$$\delta_{i+1} s_{i-1}' + 2(\delta_i + \delta_{i+1}) s_i' + \delta_i s_{i+1}' = 3\left[\frac{\delta_i}{\delta_{i+1}}(f_{i+1} - f_i) + \frac{\delta_{i+1}}{\delta_i}(f_i - f_{i-1})\right] \tag{4.3.8}$$

for $1 \le i \le n-1$. Set $s' = (s_1', \ldots, s_{n-1}')^T$,

$$\begin{aligned} v' &= 3\left(\frac{\delta_2}{\delta_1}(f_1 - f_0) + \frac{\delta_1}{\delta_2}(f_2 - f_1), \ldots, \frac{\delta_n}{\delta_{n-1}}(f_{n-1} - f_{n-2}) + \frac{\delta_{n-1}}{\delta_n}(f_n - f_{n-1})\right)^T \\ &\quad -(\delta_2 s_1', 0, \ldots, 0, \delta_{n-1} s_n') \end{aligned}$$

and

$$A = \begin{pmatrix} 2(\delta_2 + \delta_1) & \delta_1 & 0 & \cdots & 0 & 0 \\ \delta_3 & 2(\delta_3 + \delta_2) & \delta_2 & \cdots & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \delta_{n-2} & 2(\delta_{n-2} + \delta_{n-3}) & \delta_{n-3} \\ 0 & 0 & \cdots & \cdots & \delta_{n-1} & 2(\delta_{n-1} + \delta_{n-2}) \end{pmatrix}.$$

Then we may write the equation (4.3.8) as

$$As' = v'. \tag{4.3.9}$$

Clearly, A is strictly diagonal dominated. Therefore A is nonsingular and hence the linear system (4.3.9) is solvable and has a unique solution. □

From the proof of Theorem 4.3.2, we see that we can solve the Lagrange interpolation (4.3.2) and (4.3.3) with restriction on the derivative on the boundary knot by solving the linear system (4.3.9) to get the values of the first derivatives on all knots and then substituting them into (4.3.4) to find the final interpolating cubic spline. An alternative to find the interpolating cubic spline is to find a basis $e_1, \ldots, e_{n+3}$ of $S_3(X)$, then to establish the linear system to the solution $s(x) = \sum_{i=1}^{n+3} a_i e_i$ by

$$\begin{cases} p(x_i) = f(x_i), \ i = 0, \ldots, n \\ p'(x_0) = \tilde{s}_0 \\ p'(x_n) = \tilde{s}_n. \end{cases} \tag{4.3.10}$$

The dimension of the above linear system is $n+3$, and hence is larger than the dimension in the system (4.3.9). If the data $f_i, 0 \leq i \leq n$, and s'_0, s'_n, and the knots have certain symmetry, the dimension of the system (4.3.10) will be the minimal integer larger than $(n+3)/2$, which is quite smaller than the one of the system (4.3.9). For the cubic spline interpolating problem to the function $f(x) = x^5, x \in [-1, 1]$ at the knots $-1/2, 0, 1/2$, the interpolating cubic spline p satisfying (4.3.10) must be the linear combination of $1, x^2, |x|^3$. Then we need only solve a linear system of dimension 3. In fact for this particular problem, the corresponding cubic interpolating spline $p(x)$ is $-2x^2 + 3|x|^3$.

4.3.2 Extreme Property

In this section, we study the extreme property of interpolating cubic spline.

Theorem 4.3.3 *Suppose that $a = x_0 < x_1 < \ldots < x_n = b$ and $f \in C^2[a, b]$. If we take $f_i = f(x_i), 0 \leq i \leq n$ and consider the spline that satisfies*

$$\begin{cases} s(x_i) = f_i, \ 0 \leq i \leq n \\ s'(x_0) = f'(x_0), \ s'(x_n) = f'(x_n). \end{cases}$$

Then we have

$$\int_a^b [f''(x)]^2 dx - \int_a^b [s''(x)]^2 dx = \int_a^b [f''(x) - s''(x)]^2 dx.$$

Proof. Clearly, we have

$$\begin{aligned} & \int_a^b [f''(x) - s''(x)]^2 dx \\ = & \int_a^b [f''(x)]^2 - \int_a^b [s''(x)]^2 dx - 2 \int_a^b s''(x)[f''(x) - s''(x)] dx. \end{aligned}$$

Therefore it suffices to verify that

$$\int_a^b s''(x)[f''(x) - s''(x)] dx = 0.$$

By partial integration formula, we obtain

$$\begin{aligned}
& \int_a^b s''(x)[f''(x) - s''(x)]dx \\
= \quad & s''(x)[f'(x) - s'(x)]\Big|_a^b - \int_a^b s'''(x)[f'(x) - s'(x)]dx \\
= \quad & \sum_{i=0}^{n-1} \alpha_i \int_{x_i}^{x_{i+1}} [f'(x) - s'(x)]dx = 0.
\end{aligned}$$

This completes the proof. □

4.3.3 Cubic Spline Approximation

Now let consider the approximating power of cubic splines. Suppose now that the set X of knots $\{x_0, x_1, \ldots, x_n\}$ satisfies $a = x_0 < x_1 < \cdots < x_n = b$. Given a continuous function f defined on $[a, b]$. Define the norm on X by

$$\delta = \delta(X) = \max_{1 \leq i \leq n} |x_i - x_{i-1}|.$$

Then we have

Theorem 4.3.4 *Suppose that $f \in C^2([a, b])$ and $s \in S_3(X)$ satisfies*

$$s(x_i) = f(x_i), \ 0 \leq i \leq n$$

and

$$s'(a) = f'(a), \ s'(b) = f'(b).$$

Then for all $x \in [0, 1]$,

$$\|f - s\|_\infty \leq 5\delta^2 \omega(f'', \delta).$$

The proof of Theorem 4.3.3 can be found in Rivlin's book "An introduction to the Approximation of Functions". We omit the detail here due to the complexity of the proof.

Corollary 4.3.5 *Given a function f in $C^2([a, b])$, and take sets $X_n, n \geq 1$ of knots. If $\delta(X_n) \to 0$ as $n \to \infty$, then the interpolating cubic spline p_n to f at the knots of X_n converges uniformly to f.*

Exercises

1. Let G be an n-dimensional linear space of continuous functions on $[a, b]$, and let $x_1, \ldots, x_n$ be distincts points in $[a, b]$. Show that if

$$D\begin{pmatrix} g_1 & g_2 & \cdot & g_n \\ x_1 & x_2 & \cdots & x_n \end{pmatrix} \neq 0$$

for a basis $\{g_1, \ldots, g_n\}$ of G, then it holds for any basis of G.

2. Find a trigonometric polynomial of degree at most one to solve the Lagrange interpolation problem to the function $f(x) = e^x, x \in [-\pi, \pi]$ at the knots $-\pi/2, 0, \pi/2$.

3. Find a quadratic polynomial to solve the Lagrange interpolation problem to the function $f(x) = \cos \pi x, x \in [-1, 1]$ at knots $-1/2, 0, 1$.

4. Let G be the linear space spanned by $1, e^x, e^{2x}$. Find a function in G to solve the Lagrange interpolation problem to the function $f(x) = x^2, x \in [-1, 1]$ at the knots $-1, 0, 1$.

5. Consider a polynomial p of degree at most n, and the knots $x_1 < x_2 < \ldots < x_n < x_{n+1}$. Show that the polynomial p is the only polynomial solving the Lagrange interpolation problem to p at the above knots.

6. Consider the function $f(x) = x^5 + 1$ and the knots $-2, -1, 0, 1$. Compute the following divided differences,

$$f[-2, -1], f[-2, -1, 0], f[-1, 0, 1], f[-2, -1, 0], f[-2, -1, 0, 1],$$

and use Newton form to construct the Lagrange interpolation polynomial to f at the above four knots.

7. Let $f(x) = |x|, x \in [-1, 1]$, and let $q_n, 1 \leq n \leq 10$, be the unique Lagrange interpolating polynomials to f at the knots being roots of Chebeshev polynomials,

$$x_i^{(n)} = -\cos\frac{(2i-1)\pi}{2n}, i = 1, \ldots, n.$$

Compute the uniform norm of q_n numerically and plot q_n for all $, 1 \leq n \leq 10$.

8. Find the quadratic spline g at the knots $-1, -1/2, 0, 1/2, 1$, which interpolates $|\sin \pi x|$ at $-1, -3/4, -1/4, 1/4, 3/4, 1$.

9. Find the quadratic spline g at the knots $-1, 0, 1$, which interpolates $e^{|x|}$ at $-1, -1/2, 1/2, 1$.

10. Find the cubic spline g at the knots $-1, -1/2, 0, 1/2, 1$, which interpolates $\sin^2 \pi x/2, x \in [-1, 1]$ at $-1, -1/2, 0, 1/2, 1$ and has same dertivatives with $\sin^2 \pi x/2$ at the boundary knots $-1, 1$.

11. Consider the knot set $X := \{x_0, x_1, x_2\} = \{-1/2, 0, 1/2\}$.

(i) Construct cubic splines $p_i, 0 \leq i \leq 2$, and q_0, q_2 in $S_3(X)$ such that $p_i(x_i) = 1$ and $p_i(x_j) = 0$ for all $j \neq i$ and $p_i'(x_0) = p_i'(x_2) = 0$;

(ii) Construct cubic splines q_0, q_2 in $S_3(X)$ so that $q_s(x_i) = 0, 0 \leq i \leq 2$, $q_s'(x_s) = 1$ and $q_s'(x_t) = 0$ for $t \in \{0, 2\} \backslash \{s\}$.

(iii) Use the results in part (i) and (ii) to construct a cubic spline p such that $p(x_i) = f(x_i), 0 \leq i \leq 2$ and $p'(x_i) = f'(x_i), i = 0, 2$, where $f(x) = \cos \pi x + \sin \pi x, x \in [-1/2, 1/2]$.

Chapter 5

Approximation in Normed Linear Spaces

In this chapter, we study the approximation problem in a normed linear space. In particular, we consider the existence and uniqueness problem of the best approximation to a given function in a normed linear space out of a linear subspace.

5.1 Normed Linear Spaces

In this section, we introduce the concept of a normed linear space, and recall some basic properties of a normed linear space.

5.1.1 Linear Spaces

A *linear space* V is a set of vectors that has the following properties:

(i) There is an operation of *addition*, and the addition of any two vectors in the set produces another vector and satisfies:

- $u + (v + w) = (u + v) + w$ for all $u, v, w \in V$.
- $u + v = v + u$ for all $u, v \in V$.
- There is a zero vector 0 such that $u + 0 = u$ for all $u \in V$.
- Every vector u has a negative correspondence $-u$ such that $u + (-u) = 0$.

(ii) There is an operation of *multiplication*, and the multiplication of a real number gives another vector and satisfies:

- $\alpha(u+v) = \alpha u + \alpha v$ for all $\alpha \in \mathbf{R}$ and $u, v \in V$.
- $(\alpha+\beta)u = \alpha u + \beta u$ for all $\alpha, \beta \in \mathbf{R}$ and $u \in V$.
- $(\alpha\beta)u = \alpha(\beta u)$ for all $\alpha, \beta \in \mathbf{R}$ and $u \in V$.
- $1u = u$ for all $u \in V$.

One may verify that the following mathematical objects are linear spaces:

(a) The set $\mathbf{R}$ of all real numbers with usual addition and multiplication.

(b) The Euclidean space $\mathbf{R}^d = \{(x_1, \ldots, x_d) : \ x_i \in \mathbf{R}, 1 \le i \le d\}$ with coordinate addition and multiplication.

(c) The space $\Pi_r, r = 0, 1, \ldots$, of all trigonometric polynomials of degree at most r,

$$\begin{aligned} \Pi_r \quad = \quad & \Big\{ a_0 + \sum_{k=1}^{r} (a_k \cos k\pi x + b_k \sin n\pi x) : \\ & \quad a_0 \in \mathbf{R}, a_k, b_k \in \mathbf{R} \text{ for } k = 1, \ldots, r \Big\}. \end{aligned}$$

(d) The space P_r of all polynomials of degree at most r,

$$P_r = \left\{ \sum_{k=0}^{r} a_k x^k : \ a_k \in \mathbf{R}, 0 \le k \le r \right\}, \ r = 0, 1, \ldots.$$

(e) The space $L^2([a,b])$ of all square integrable functions on the interval $[a,b]$,

$$L^2([a,b]) = \left\{ f : \int_a^b |f(x)|^2 dx < \infty \right\};$$

(f) The space $C([a,b])$ of all continuous functions on a finite interval $[a,b]$.

(g) The shift-invariant space $V^2(\phi)$ spanned by the integer shifts of a compactly supported L^2 function ϕ on the real line $\mathbf{R}$,

$$V^2(\phi) = \left\{ \sum_{k \in \mathbf{Z}} c(k)\phi(x-k) : \ \sum_{k \in \mathbf{Z}} |c(k)|^2 < +\infty \right\}.$$

5.1.2 Normed Linear Spaces

A *normed linear space* V is a linear space with a norm $\|\cdot\|$, a function from V to $\mathbf{R}$ that has the following properties:

(i) $\|v\| \geq 0$ for all $v \in V$, and $\|v\| = 0$ if and only if $v = 0$.

(ii) $\|\alpha v\| = |\alpha|\|v\|$ for all $\alpha \in \mathbf{R}$ and $v \in V$.

(iii) (Triangle inequality) $\|u + v\| \leq \|u\| + \|v\|$ for all $u, v \in V$.

The following are some examples of normed linear spaces:

(a) The space $\mathbf{R}$ of all real numbers with magnitude as the norm on $\mathbf{R}$.

(b) The d-dimensional Euclidean space $\mathbf{R}^d$ with the norm $|\cdot|_p, 1 \leq p \leq \infty$, defined by

$$|x|_p = \left(\sum_{i=1}^{d} |x_i|^p\right)^{1/p} \qquad \text{for } x = (x_1, \ldots, x_d) \in \mathbf{R}^d.$$

(The norm $|\cdot|_2$ is usual norm on $\mathbf{R}^d$ and is commonly denoted by $|x|$.)

(c) The space P_r of all polynomials of degree r with the norm $\|\cdot\|$ defined by

$$\|p\| = \sum_{k=0}^{r} |a_k| \quad \text{for } p(x) = \sum_{k=0}^{r} a_k x^k.$$

(d) The space $L^2([a, b])$ of square-integrable functions on the interval $[a, b]$ with the norm $\|\cdot\|_2$ defined by

$$\|f\|_2 = \left(\int_a^b |f(x)|^2 dx\right)^{1/2}.$$

(e) The space $C([a, b])$ of all continuous function on the interval $[a, b]$ with the uniform norm $\|\cdot\|_\infty$ defined by

$$\|f\|_\infty = \sup_{x \in [a,b]} |f(x)|.$$

Remark 5.1.1 We remark that a nonegative function on a linear space may not be a norm. For instance, the function $|\cdot|_p$ on Euclidean space $\mathbf{R}^d$,

$$|x|_p = \left(\sum_{i=1}^{n} |x_i|^p\right)^{1/p} \qquad \text{for } x = (x_1, \ldots, x_d) \in \mathbf{R}^d,$$

is not a norm when $0 < p < 1$, because since the triangle inequality in the definition of a norm does not hold in general. For instance, for $x = (1, 0)$ and $y = (0, 1)$, we have

$$|x + y|_p = 2^{1/p} > 2 = |x|_p + |y|_p.$$

5.1.3 Strictly Convex Norms

In this section, we show that the L^p norm on the space of all p-integrable functions on the interval $[a, b]$ is a strictly convex norm, where $1 < p < \infty$. Here a norm $\|\cdot\|$ on a linear space V is said to be a *strictly convex norm* if the unit sphere S of V, $S = \{v \in V : \|v\| = 1\}$, contains no open line segment, i.e., if v_1, v_2 are two distinct element in V and satisfy $\|v_1\| = \|v_2\| = 1$, then $\|tv_1 + (1-t)v_2\| < 1$ for all $t \in (0, 1)$.

Theorem 5.1.2 *Let $L^p([a, b]), 1 < p < \infty$, be the space of all p-integrable functions f on the interval $[a, b]$ with finite $\|f\|_p$, where*

$$\|f\|_p = \Big(\int_a^b |f(x)|^p dx\Big)^{1/p}.$$

Then $\|\cdot\|_p$ is a strictly convex norm on $L^p([a, b])$.

Remark 5.1.3 For $1 \le p \le \infty$, we define the L^p norm $\|f\|_p$ of a measurable function f by

$$\|f\|_p = \begin{cases} \Big(\int_a^b |f(x)|^p dx\Big)^{1/p}, & 1 \le p < \infty, \\ \mathrm{esssup}_{x\in[a,b]}|f(x)|, & p = \infty. \end{cases}$$

and let $L^p([a, b])$ be the space of all measurabele functions f on the interval $[a, b]$ with finite L^p norm $\|f\|_p$. In Theorem 5.1.2, it is shown that the L^p norm is a strictly convex norm when $p \in (1, \infty)$. The above strict convexity property is no longer true for $p = 1, \infty$. In particular, we notice that if f_1, f_2 are nonnegative functions on the interval $[a, b]$ with $\|f_1\|_1 = \|f_2\|_1 = 1$, then

$$\|tf_1 + (1-t)f_2\|_1 = t\|f_1\|_1 + (1-t)\|f_2\|_1 = 1$$

for all $t \in [0,1]$. Therefore the L^1 norm on $L^1([a,b])$ is not a strictly convex norm. Similarly, the L^∞ norm on the space $L^\infty([a,b])$ of all bounded functions on $[a,b]$ is not a strictly convex norm too because for nonnegative monotonously-increasing functions f_3 and f_4 on $[a,b]$ with $f_3(b) = f_4(b) = 1$, we obtain from the nonnegativeness and monotonicity of the function $tf_3 + (1-t)f_4$ that

$$\|tf_3 + (1-t)f_4\|_\infty = tf_3(b) + (1-t)f_4(b) = 1$$

for all $t \in [0,1]$.

To prove Theorem 5.1.2, we need the following two lemmas.

Lemma 5.1.4 *If A and B are positive and $0 \le t \le 1$, then*

$$A^t B^{1-t} \le tA + (1-t)B,$$

and the equality holds only if $t = 0, 1$ or $A = B$.

Proof. Take $h(x) = -\ln x$. Then $h'(x) = -1/x$ and $h''(x) = 1/x^2 > 0$. Therefore by Taylor expansion, we have

$$\begin{aligned} h(A) &= h(tA + (1-t)B) + (1-t)(A-B)h'(tA + (1-t)B) \\ &\quad + \frac{h''(\xi)}{2}(1-t)^2(A-B)^2 \\ &\ge h(tA + (1-t)B) + (1-t)(A-B)h'(tA + (1-t)B) \end{aligned}$$

and

$$\begin{aligned} h(B) &= h(tA + (1-t)B) - t(A-B)h'(tA + (1-t)B)) \\ &\quad + \frac{h''(\eta)}{2}t^2(A-B)^2 \\ &\ge h(tA + (1-t)B) - t(A-B)h'(tA + (1-t)B) \end{aligned}$$

for some ξ, η between A and B. Multiplying t and $1-t$ to the above two estimates respectively, then summing up and subsitituting $h(x)$ by $-\ln x$, we obtain

$$\ln \frac{1}{tA + (1-t)B} \le t \ln \frac{1}{A} + (1-t) \ln \frac{1}{B}.$$

Moreover, we see that the above inequality becomes an equality holds only when $t = 0, 1$, or $A = B$. Then the lemma follows. $\square$

Lemma 5.1.5 *Let $1 < p < \infty, 0 < \lambda < 1$ and $f_1, f_2 \in L^p([a, b])$. Then*

$$\int_a^b |\lambda f_1(x) + (1-\lambda) f_2(x)|^p dx$$
$$\leq \lambda \int_a^b |f_1(x)|^p dx + (1-\lambda) \int_a^b |f_2(x)|^p dx, \qquad (5.1.1)$$

and the above equality holds only if $f_1 = f_2$.

Proof. Set $h = \lambda f_1 + (1-\lambda) f_2$. By Lemma 5.1.4, we have

$$|f_1||h|^{p-1} \leq \frac{1}{p}|f_1|^p + \frac{p-1}{p}|h|^p$$

and

$$|f_2||h|^{p-1} \leq \frac{1}{p}|f_2|^p + \frac{p-1}{p}|h|^p.$$

Combining the above two estimates with

$$|h| \leq \lambda |f_1| + (1-\lambda)|f_2|,$$

we obtain

$$\begin{aligned}
& \int_a^b |h(x)|^p dx \\
\leq\ & \lambda \int_a^b |f_1(x)||h(x)|^{p-1} dx + (1-\lambda) \int_a^b |f_2(x)||h(x)|^{p-1} dx \\
\leq\ & \frac{\lambda}{p} \int_a^b |f_1(x)|^p dx + \frac{\lambda(p-1)}{p} \int_a^b |h(x)|^p dx \\
& + \frac{1-\lambda}{p} \int_a^b |f_2(x)|^p dx + \frac{(1-\lambda)(p-1)}{p} \int_a^b |h(x)|^p dx \\
=\ & \frac{\lambda}{p} \int_a^b |f_1(x)|^p dx + \frac{1-\lambda}{p} \int_a^b |f_2(x)|^p dx \\
& + \frac{p-1}{p} \int_a^b |h(x)|^p dx.
\end{aligned}$$

This proves (5.1.1). Moreover, we see from the above proof that the inequality in (5.1.1) becomes an equality only if $f_1(x) f_2(x) \geq 0$, $|f_1(x)| = |\lambda f_1(x) + (1-\lambda) f_2(x)|$ and $|f_2(x)| = |\lambda f_1(x) + (1-\lambda) f_2(x)|, x \in \mathbf{R}$. Therefore $f_1 = f_2$. $\square$

Now we start to prove Theorem 5.1.2.

Proof of Theorem 5.1.2. The strictly convexity property of the norm $\|\cdot\|_p$ follows easily from Lemma 5.1.5. Therefore it suffices to prove that $\|\cdot\|_p$ is a norm. Clearly for any $\alpha \in \mathbf{R}$ and $f \in L^p([a,b])$, we have that $\|\alpha f\|_p = |\alpha|\|f\|_p$, $\|f\|_p \geq 0$, and the above inequality becomes an equality only if $f = 0$. Then it remains to prove

$$\|f+g\|_p \leq \|f\|_p + \|g\|_p$$

for any nonzero functions f, g in $L^p([a,b])$. Applying Lemma 5.1.5 with $f_1 = f/\|f\|_p, f_2 = g/\|g\|_p$ and $\lambda = \|f\|_p/(\|f\|_p + \|g\|_p)$, we obtain,

$$\begin{aligned}
&\int_a^b \left(\frac{|f(x)+g(x)|}{\|f\|_p + \|g\|_p}\right)^p dx \\
\leq\ & \frac{\|f\|_p}{\|f\|_p + \|g\|_p}\int_a^b \left(\frac{|f(x)|}{\|f\|_p}\right)^p dx + \frac{\|g\|_p}{\|f\|_p + \|g\|_p}\int_a^b \left(\frac{|g(x)|}{\|g\|_p}\right)^p dx \\
=\ & \frac{\|f\|_p}{\|f\|_p + \|g\|_p} + \frac{\|g\|_p}{\|f\|_p + \|g\|_p} = 1.
\end{aligned}$$

This proves that $\|f+g\|_p \leq \|f\|_p + \|g\|_p$ for any nonzero functions $f, g \in L^p([a,b])$, and hence that $\|\cdot\|_p$ is a norm. □

5.2 Existence of Best Approximation

In this section, we consider the existence problem of best approximations out of a finite-dimensional linear subspace.

We start this section with the best approximation problem of the hat function $h(x) = \max(1-|x|, 0), x \in [-1,1]$, out of the linear space of constant functions.

Example 5.2.1 *Let h be the hat function. Find all best uniform approximating constants C_0, i.e., $\|h - C_0\|_\infty = \min_{C\in\mathbf{R}} \|h - C\|_\infty$.*

Solution. Noting that $0 \leq h(x) \leq 1$, we have

$$|h(x) - C| \geq |h(x)| \quad \forall\ x \in [-1,1]$$

for $C < 0$ and

$$|h(x) - C| \geq |h(x) - 1| \quad \forall\ x \in [-1,1]$$

for $C > 1$. Thus,

$$\min_{C \in \mathbf{R}} \|h - C\|_\infty = \min_{C \in [-1,1]} \|h - C\|_\infty.$$

Also noting that $h(x) - C$ is affine on $[-1, 0]$ and $[0, 1]$ for any $C \in \mathbf{R}$, we then obtain

$$\|h - C\|_\infty = \max(|h(-1) - C|, |h(0) - C|, |h(1) - C|) = \max(C, 1 - C)$$

for $0 \le C \le 1$. On the other hand,

$$\min_{C \in [0,1]} \max(C, 1 - C) = \frac{1}{2}.$$

Therefore,

$$\min_{C \in \mathbf{R}} \|h - C\|_\infty = \frac{1}{2},$$

and $C_0 = 1/2$ is the unique real number such that $\|h - C_0\|_\infty = 1/2$. $\square$

Remark 5.2.2 A general problem to find the best uniform approximating constant of a continuous function will be discussed in Chapter 5. In fact, it is shown that for a continuous function f on $[a, b]$, the best approximating constant is $(M + m)/2$, where M and m are the maximum and minimum of the function f on $[a, b]$ (see Corollary 3.5.3 for details).

From the above example, we see that there is a unique best uniform approximating constant to the hat function. In general, we have the following theorem about the existence of best approximation, which is also known as the fundamental theorem in approximation theory.

Theorem 5.2.3 *Let V be a normed linear space, and W be a finite dimensional linear subspace of V. Then, given any $v \in V$, there exists $w^* \in W$ such that*

$$\|v - w^*\| = \inf_{w \in W} \|v - w\|.$$

Remark 5.2.4 The requirement that the approximating space W is finite dimensional in Theorem 5.2.3 is essential. For example, let V be the space of all continuous functions on $[0, 1/2]$, and W be the space of all polynomials (without any restriction on their degrees). By Taylor's expansion,

$$\left| e^x - \sum_{k=0}^{n} \frac{x^k}{k!} \right| \le \frac{e x^{n+1}}{(n+1)!} \quad \forall\, x \in [0, 1],$$

which implies

$$\sup_{x\in[0,1]}\left|e^x-\sum_{k=0}^{n}\frac{x^k}{k!}\right|\le\frac{e}{(n+1)!}\to 0\quad\text{as } n\to\infty.$$

Therefore $\inf_{w\in W}\|f-w\|_\infty=0$, while e^x is obviously not a polynomial. This shows that there does not exist $w^*\in W$ such that $\sup_{x\in[0,1]}|e^x-w^*(x)|=0$.

To prove Theorem 5.2.3, we need two lemmas.

Lemma 5.2.5 *Let V be a finite dimensional linear space, and $x_1,\dots,x_n\in V$. Then the function f on $\mathbf{R}^n$ defined by*

$$f(\lambda_1,\dots,\lambda_n)=\|\lambda_1x_1+\dots+\lambda x_n\|$$

is uniformly continuous.

Proof. By the triangle inequality of the norm $\|\cdot\|$ on V,

$$-\|x-y\|\le\|x\|-\|y\|\le\|x-y\|\quad\forall\ x,y\in V.$$

Therefore for any vectors $\lambda=(\lambda_1,\dots,\lambda_n)$ and $\lambda^0=(\lambda_1^0,\dots,\lambda_n^0)\in\mathbf{R}^n$,

$$\begin{aligned}
&|f(\lambda_1,\dots,\lambda_n)-f(\lambda_1^0,\dots,\lambda_n^0)|\\
\le\ &\Big|\|\lambda_1x_1+\dots+\lambda_nx_n\|-\|\lambda_1^0x_1+\dots+\lambda_n^0x_n\|\Big|\\
\le\ &\|(\lambda_1-\lambda_1^0)x_1+\dots+(\lambda_n-\lambda_n^0)x_n\|\\
\le\ &|\lambda_1-\lambda_1^0|\|x_1\|+\dots+|\lambda_n-\lambda_n^0|\|x_n\|\\
\le\ &\Big(\sum_{i=1}^n|\lambda_i-\lambda_i^0|^2\Big)^{1/2}\times\Big(\sum_{i=1}^n\|x_i\|^2\Big)^{1/2}
\end{aligned}\tag{5.2.1}$$

This implies that for any given $\epsilon>0$,

$$|f(\lambda_1,\dots,\lambda_n)-f(\lambda_1^0,\dots,\lambda_n^0)|\le\epsilon$$

holds for all vectors $(\lambda_1,\dots,\lambda_n)$ and $(\lambda_1^0,\dots,\lambda_n^0)\in\mathbf{R}^n$ that satisfy

$$\Big(\sum_{i=1}^n|\lambda_i-\lambda_i^0|^2\Big)^{1/2}\le\Big(1+\sum_{i=1}^n\|x_i\|^2\Big)^{-1/2}\epsilon.$$

Then the uniform continuity of f is proved. □

We say that $x_1,\dots,x_n$ in a linear space V is *linearly independent* if $\lambda_1x_1+\ldots+\lambda_nx_n=0$ for some $\lambda_1,\dots,\lambda_n\in\mathbf{R}$ implies $\lambda_1=\dots=\lambda_n=0$..

Lemma 5.2.6 *Let V be a finite-dimensional linear space, and $\|\cdot\|$ be a norm on V. If $x_1, \dots, x_n \in V$ are linear independent, then there exist two positive constants A and B such that*

$$A\left(\sum_{i=1}^{n}|\lambda_i|^2\right)^{1/2} \le \|\lambda_1 x_1 + \cdots + \lambda_n x_n\| \le B\left(\sum_{i=1}^{n}|\lambda_i|^2\right)^{1/2}. \tag{5.2.2}$$

Proof. For any $(\lambda_1, \dots, \lambda_n) \in \mathbf{R}^n$, by the definition of a norm, we have

$$\begin{aligned} & \|\lambda_1 x_1 + \cdots + \lambda_n x_n\| \le \|\lambda_1 x_1\| + \cdots + \|\lambda_n x_n\| \\ = \ & |\lambda_1|\|x_1\| + \cdots + |\lambda_n|\|x_n\| \le \Big(\sum_{i=1}^{n}|\lambda_i|^2\Big)^{1/2} \times \Big(\sum_{i=1}^{n}\|x_i\|^2\Big)^{1/2}. \end{aligned}$$

Therefore the right estimate of (5.2.2) follows by letting $B = (\sum_{i=1}^{n}\|x_i\|^2)^{1/2}$.

Now we prove the estimate on the left-hand side of (5.2.2). Clearly it holds for the trivial case $\lambda_1 = \cdots = \lambda_n = 0$. So we may assume that $(\lambda_1, \dots, \lambda_n) \neq 0$ hereafter. Let

$$S^{n-1} = \left\{(\lambda_1, \dots, \lambda_n) \in \mathbf{R}^n : \sum_{i=1}^{n}|\lambda_i|^2 = 1\right\}$$

be the unit sphere in $\mathbf{R}^n$, and define a function f on the unit sphere S^{n-1} by

$$f(\lambda_1, \dots, \lambda_n) = \|\lambda_1 x_1 + \cdots + \lambda_n x_n\|.$$

Denote the minimum of f on the unit sphere S^{n-1} by

$$m = \inf_{(\lambda_1,\dots,\lambda_n)\in S^{n-1}} f(\lambda_1, \dots, \lambda_n).$$

Now we claim that $m > 0$. By the fact that $f(\lambda_1, \dots, \lambda_n) \ge 0$ for all $(\lambda_1, \dots, \lambda_n) \in S^{n-1}$, we have $m \ge 0$. Therefore it suffices to prove that $m \neq 0$. Suppose, on the contrary, that $m = 0$. Then there exists $(\lambda_1^0, \dots, \lambda_n^0) \in S^{n-1}$ by Lemma 5.2.5 so that

$$0 = m = f(\lambda_1^0, \dots, \lambda_n^0) = \left\|\lambda_1^0 x_1 + \cdots + \lambda_n^0 x_n\right\|,$$

which implies that $\lambda_1^0 x_1 + \cdots + \lambda_n^0 x_n = 0$ by the norm property. This together with the linear independent assumption yields $\lambda_1^0 = \cdots = \lambda_n^0 = 0$, which contradicts $(\lambda_1^0, \dots, \lambda_n^0) \in S^{n-1}$. The claim $m > 0$ is proved.

By the above claim $m > 0$,

$$\|\lambda_1 x_1 + \cdots + \lambda_n x_n\| \geq m \tag{5.2.3}$$

for any $(\lambda_1, \ldots, \lambda_n) \in S^{n-1}$. For any $0 \neq (\lambda_1, \ldots, \lambda_n)$, we have that $(\lambda'_1, \ldots, \lambda'_n) \in S^{n-1}$ where $\lambda'_i = \lambda_i/\Lambda, 1 \leq i \leq n$ and $\Lambda = (\sum_{i=1}^n |\lambda_i|^2)^{1/2}$. Thus,

$$\begin{aligned} & \|\lambda_1 x_1 + \cdots + \lambda_n x_n\| = \|\Lambda(\lambda'_1 x_1 + \cdots + \lambda'_n x_n)\| \\ = \ & \Lambda\|\lambda'_1 x_1 + \cdots + \lambda'_n x_n\| \geq \Lambda m \\ \geq \ & m\Big(\sum_{i=1}^n |\lambda_i|^2\Big)^{1/2}. \end{aligned}$$

Hence the left estimate of (5.2.2) follows by letting $A = m$. $\square$

Now we start to prove Theorem 5.2.3.

Proof of Theorem 5.2.3. Set $M = \|v\|$ and let $m = \inf_{w \in W} \|v - w\|$. By $0 \in W$ and the triangle inequality of the norm $\|\cdot\|$, we have

$$m \leq \|v - 0\| = M,$$

and

$$\|v - w\| \geq \|w\| - \|v\| > 2M - M \geq m$$

for any $w \in W$ with $\|w\| > 2M$. Therefore it suffices to find the best approximation w^* in the set $\{w : \|w\| \leq 2M\}$, i.e.,

$$m = \inf_{w \in W, \|w\| \leq 2M} \|v - w\|. \tag{5.2.4}$$

Select a basis $e_1, \ldots, e_n$ of the space W, where n is the dimension of W. By Lemma 5.2.6, there exists two positive constants A and B independent of $(\lambda_1, \ldots, \lambda_n) \in \mathbf{R}^n$ so that

$$A\Big(\sum_{i=1}^n |\lambda_i|^2\Big)^{1/2} \leq \|\lambda_1 e_1 + \cdots + \lambda_n e_n\| \leq B\Big(\sum_{i=1}^n |\lambda_i|^2\Big)^{1/2}.$$

Note that every element $w \in W$ can uniquely be written as

$$w = \lambda_1 e_1 + \ldots + \lambda_n e_n.$$

Therefore for any $w \in W$ with $\|w\| \le 2M$, the corresponding vector $(\lambda_1, \ldots, \lambda_n) \in \mathbf{R}^n$ satisfies $(\sum_{i=1}^n |\lambda_i|^2)^{1/2} \le 2A^{-1}M$. This leads to the following inclusion,

$$\{w \in W : \|w\| \le 2M\} \subset \left\{ \lambda_1 e_1 + \cdots + \lambda_n e_n : \Big(\sum_{i=1}^n |\lambda_i|^2 \Big)^{1/2} \le 2A^{-1}M \right\}. \tag{5.2.5}$$

By (5.2.4) and (5.2.5), we have

$$m = \inf_{(\sum_{i=1}^n |\lambda_i|^2)^{1/2} \le 2A^{-1}M} f(\lambda_1, \ldots, \lambda_n),$$

where we set

$$f(\lambda_1, \ldots, \lambda_n) = \|v - \lambda_1 e_1 - \ldots - \lambda_n e_n\|.$$

By Lemma 5.2.5, f is a continuous function on $\mathbf{R}^n$. Hence $m = f(\lambda_1^*, \ldots, \lambda_n^*)$ for some vector $(\lambda_1^*, \ldots, \lambda_n^*) \in \mathbf{R}^n$ with $(\sum_{i=1}^n |\lambda_i^*|^2)^{1/2} \le 2A^{-1}M$, which proves that the element $w^* := \sum_{i=1}^n \lambda_i^* e_i \in W$ is a best approximation to v out of W. $\square$

5.3 Uniqueness of Best Approximation

In this section, we consider the uniqueness problem of best approximations, and establish the following result.

Theorem 5.3.1 *Let V be a normed linear space with a norm $\|\cdot\|$, and W be a linear subspace of V. If the norm $\|\cdot\|$ is strictly convex, then for any given $v \in V$ there exists at most one best approximation out of W.*

Clearly, combining Theorems 5.2.3 and 5.3.1, we obtain

Corollary 5.3.2 *Let V be a norm linear space with a norm $\|\cdot\|$ and W be a finite-dimensional linear subspace of V. If the norm $\|\cdot\|$ is a strictly convex norm on V, then for any given $v \in V$ there is a unique best approximation w^* to v out of W.*

Remark 5.3.3 The assumption on the strict convexity of the norm $\|\cdot\|$ in Theorem 5.3.1 cannot be dropped in general. For instance, let $V := L^1([-1,1])$, the space of all integrable functions on $[-1,1]$, and let $W := \{c \sin \pi x : c \in \mathbf{R}\}$, an one-dimensional subspace of $L^1([-1,1])$. It is known that the L^1 norm on the space $L^1([-1,1])$ is not a strictly convex norm(see

Section 2.1.4). For the constant function $f = 1 \in L^1([-1,1])$, we claim that functions $c \sin \pi x$ are best approximations to f out of W for any $c \in [-1,1]$, i.e.,

$$\|1 - c\sin \pi x/2\|_1 = \inf_{w \in W} \|v - w\|_1 \quad \forall\ c \in [-1,1]. \tag{5.3.1}$$

Now we prove the claim (5.3.1). For $c \in [-1,1]$,

$$\|1 - c\sin \pi x/2\|_1 = \int_{-1}^{1} \Big(1 - c\sin\frac{\pi x}{2}\Big)dx = 2, \tag{5.3.2}$$

since $c\sin \pi x/2 \leq 1$ for all $x \in [-1,1]$. For $c \geq 1$, let $\xi_0 \in [0,1]$ be so chosen that $c \sin \pi\xi_0/2 = 1$. Then

$$\|1 - c\sin \pi x/2\|_1 = \int_{-1}^{\xi_0} \Big(1 - c\sin\frac{\pi x}{2}\Big)dx + \int_{\xi_0}^{1} \Big(c\sin\frac{\pi x}{2} - 1\Big)dx$$

$$= \quad 2 + 2\int_{\xi_0}^{1} \Big(c\sin\frac{\pi x}{2} - 1\Big)dx \geq 2. \tag{5.3.3}$$

Similarly for $c \leq -1$, we have

$$\|1 - c\sin \pi x/2\|_1 \geq 2. \tag{5.3.4}$$

Hence the claim (5.3.1) follows from (5.3.2), (5.3.3) and (5.3.4).

Remark 5.3.4 On the other hand, the strict convexity of the norm in Theorem 5.3.1 is not necessary to gurantee the uniqueness of the best approximation. In Section 2.2.1, the best uniform approximating constant to the hat function is unique even though the L^∞ is not strictly convex. In general, it will be shown later that the best uniform approximation to a continuous function on a finite interval out of P_r, the space of all polynomials of degree at most r, is unique, where $r \geq 0$ (see Theorem 3.5.4 for details).

We say a set E is *convex* if for any $w_1, w_2 \in E$, all elements on the line segment $tw_1 + (1-t)w_2, t \in [0,1]$, belong to E. For instance, the unit ball $B := \{v \in V : \ \|v\| \leq 1\}$ of a normed linear space V is a convex set. To prove Theorem 5.3.1, we consider the convext property of the set of all best approximations out of a linear space.

Lemma 5.3.5 *Let V be a normed linear space, and W be a linear subspace of V. Suppose that $v \in V$, and denote the set of all best approximations to v out of W by W^*. Then W^* is a convex set.*

Proof. Take any $w_0^*, w_1^* \in W^*$, we have

$$\|v - w_0^*\| = \|v - w_1^*\| = \inf_{w \in W} \|v - w\|. \tag{5.3.5}$$

Set $w_t = tw_1^* + (1-t)w_0^*, t \in [0,1]$. Then $w_t \in W$ and

$$\|v - w_t\| \geq \inf_{w \in W} \|v - w\|. \tag{5.3.6}$$

By (5.3.5) and the triangle inequality of the norm $\|\cdot\|$, we obtain

$$\begin{aligned} & \|v - w_t\| = \|t(v - w_1^*) + (1-t)(v - w_0^*)\| \\ \leq \; & t\|v - w_1^*\| + (1-t)\|v - w_0^*\| = \inf_{w \in W} \|v - w\|. \end{aligned} \tag{5.3.7}$$

Combining (5.3.6) and (5.3.7) leads to $\|v - w_t\| = \inf_{w \in W} \|v - w\|$. Thus w_t is also a best approximation to v out of W, and hence belongs to W^*. $\square$

An easy application of Lemma 5.3.5 leads to the following interesting result.

Corollary 5.3.6 *Let V be a normed linear space, and W be a linear subspace of V. Suppose that $v \in V$. Then either there is only one best approximation to v out of W, or there are infinitely many best approximation to v out of W.*

Now we start to prove Theorem 5.3.1.

Proof of Theorem 5.3.1. Suppose, on the contrary, that w_0^* and w_1^* are two distinct best approximations to v out of W. By Lemma 5.3.5, $w^* = (w_0^* + w_1^*)/2$ is also a best approximation to v out of W. Therefore

$$\|v - w_0^*\| = \|v - w_1^*\| = \|v - w^*\| = \rho, \tag{5.3.8}$$

where we set $\rho = \inf_{w \in W} \|v - w\|$. In the case $\rho = 0$, it follows from (5.3.8) that $v = w_0^*$ and $v = w_1^*$, which contradicts $w_0^* \neq w_1^*$. So we may assume $\rho \neq 0$ hereafter. Still by (5.3.8), we obtain

$$\|\rho^{-1}(v - w_0^*)\| = \|\rho^{-1}(v - w_1^*)\| = 1 \quad \text{and} \quad \|\rho^{-1}(v - w^*)\| = 1,$$

which is a contradiction since

$$\|\rho^{-1}(v - w^*)\| = \left\| \frac{1}{2} \left(\rho^{-1}(v - w_1^*) + \rho^{-1}(v - w_1^*) \right) \right\| < 1$$

by the strict convexity property of the norm $\|\cdot\|$. $\square$

Exercises

1. Define $\|f\|_1 = \int_a^b |f(x)|dx$ for any measurable function f on $[a,b]$ and let $L^1([a,b])$ be the space of all measurable functions f with finite $\|f\|_1$. Prove that $\|\cdot\|_1$ is a norm on $L^1([a,b])$.

2. Let $C([a,b])$ be the space of all continuous functions f on a finite interval $[a,b]$ with $\|f\|_\infty < \infty$, where $\|f\|_\infty = \sup_{x\in[a,b]} |f(x)|$. Prove that $\|\cdot\|_\infty$ is a norm.

3. Let P_r be the space of all polynomials of degree at most $r, r \geq 0$. Define

$$|||p||| = \sum_{k=0}^{r} |a_k| \quad \text{for } p(x) = \sum_{k=0}^{r} a_k x^k \in P_r.$$

Prove that $|||\cdot|||$ is a norm but not a strictly convex norm.

4. Let $\mathbf{R}^d$ be the d-dimensional Euclidean space, and define

$$|x|_p = \Big(\sum_{i=1}^{d} |x_i|^p\Big)^{1/p},\ x = (x_1,\ldots,x_d) \in \mathbf{R}^d,$$

where $1 \leq p < \infty$. Prove that $|\cdot|_p$ is a strictly convex norm on $\mathbf{R}^d$ for $1 < p < \infty$.

5. Let V be a finite-dimensional linear space, and $\|\cdot\|$ and $|||\cdot|||$ be two norms on V. Prove that there exist two positive constants A, B such that

$$A\|x\| \leq |||x||| \leq B\|x\| \quad \forall\ x \in V.$$

6. Let $f(x) = x, x \in [0,1]$, and $1 \leq p < \infty$. Find all constants c_0 so that

$$\|f - c_0\|_p = \min_{c\in\mathbf{R}} \|f - c\|_p$$

and evaluate $\|f - c_0\|_p$, where we define

$$\|g\|_p = \Big(\int_0^1 |g(x)|^p dx\Big)^{1/p},\ g \in L^p([0,1]).$$

7. Let $f(x) = e^x, x \in [0,1]$. Find all affine functions p^* so that

$$\|f - p^*\|_2 = \min_{p \in P_1} \|f - p^*\|_2,$$

where $\|\cdot\|_2$ is the usual L^2 norm on $L^2([0,1])$ and P_1 is the space of all affine functions.

8. Let f be the sign function on $[-1,1]$, that is,

$$f(x) = \begin{cases} 1 & \text{if } x > 0 \\ 0 & \text{if } x = 0 \\ -1 & \text{if } x < 0, \end{cases}$$

and W be the space spanned by $f(x) + 1$. Find all best uniform approximations g_0 to f out of W, i.e.,

$$\|f - g_0\|_\infty = \inf_{g \in W} \|f - g\|_\infty.$$

9. Construct a finitely-dimensional subspace W of $C([-1,1])$ and find a continuous function f on $[-1,1]$ so that the set of all best approximation to f out of W has more than one elements.

10. Let V be a normed linear space, and W be a closed linear subspace of V. For any given $v \in V$, denote the set of all best approximations to v out of W by W^*. Show that W^* is closed and bounded.

11. Let V be a normed linear space of functions on $[-a,a]$ so that $\|f\| = \|f(-\cdot)\|$ for all $f \in V$, and W be a finite dimensional linear subspace of V. Prove that for any given even function $f \in V$, there exists a even function $g^* \in W$ such that

$$\|f - g^*\| = \inf_{g \in V} \|f - g\|.$$

(Hint: To show that $g(-\cdot)$ is also a best approximation to the even function f when g is).

Chapter 6

Approximation in Hilbert Spaces

In this chapter, we consider the approximation problem in a Hilbert space. We start from recalling some basic properties of Hilbert spaces, then establish the existence and uniqueness of the best approximation of a function in a Hilbert space out of its finite-dimensional subspace, and finally apply the above existence and uniqueness result to the least square approximation of finite points on the plane by functions on the line.

6.1 Hilbert Spaces

In this section, we recall the definition of an inner product, introduce a strictly convex norm associated with an inner product, establish the Cauchy-Schwartz inequality, construct orthogonal projection, recall the Gram-Schmidt orthogonalization procedure and apply the procedure to construct orthogonal polynomials explicitly.

6.1.1 Inner Product

Let H be a linear space. An *inner product* on H is a function $\langle \cdot, \cdot \rangle : H \times H \longmapsto \mathbf{R}$, that has the following properties:

(i) $\langle u, u \rangle \geq 0$ for all $u \in H$, and the equality holds only if $u = 0$.

(ii) $\langle u, v \rangle = \langle v, u \rangle$ for all $u, v \in H$.

(iii) $\langle \alpha u + \beta v, w \rangle = \alpha \langle u, w \rangle + \beta \langle v, w \rangle$ for all $\alpha, \beta \in \mathbf{R}$ and $u, v, w \in H$.

An *inner product space* is a linear space H with an inner product $\langle \cdot, \cdot \rangle$, to be denoted by $(H, \langle \cdot, \cdot \rangle)$. A *Hilbert space* is a complete inner product space, that is, an inner product space such that every Cauchy sequence in H has a limit in H.

The following are some examples of inner product spaces:

(a) The Euclidean space $\mathbf{R}^d$ with inner product $\langle \cdot, \cdot \rangle$ defined by

$$\langle x, y \rangle = \sum_{i=1}^{d} x_i y_i \quad \forall\ x = (x_1, \ldots, x_d) \text{ and } y = (y_1, \ldots, y_d) \in \mathbf{R}^d.$$

(b) The space ℓ^2 of all square-summable sequence $(x_k)_{k \in \mathbf{Z}}$ with inner product defined by

$$\langle x, y \rangle = \sum_{k \in \mathbf{Z}} x_k y_k \quad \forall\ x = (x_k)_{k \in \mathbf{Z}} \text{ and } y = (y_k)_{k \in \mathbf{Z}} \in \ell^2.$$

(c) The space

$$L^2_w([a, b]) := \left\{ f : \int_a^b |f(x)|^2 w(x) dx < \infty \right\}$$

of all weighted L^2 functions on the interval $[a, b]$ with inner product defined by

$$\langle f, g \rangle = \int_a^b f(x) g(x) w(x) dx \quad \forall\ f, g \in L^2_w([a, b]),$$

where w is a weight function on $[a, b]$, i.e., $0 < w(x) < \infty$ for almost all $x \in [a, b]$.

6.1.2 Cauchy-Schwartz Inequality

For any inner product on a linear space, we have the following Cauchy-Schwartz inequality.

Theorem 6.1.1 *Let $(H, \langle \cdot, \cdot \rangle)$ be an inner product space. Then*

$$|\langle u, v \rangle|^2 \leq \langle u, u \rangle \langle v, v \rangle \quad \forall\ u, v \in H. \tag{6.1.1}$$

Moreover the inequality in (6.1.1) becomes an equality if and only if either $v = 0$ or $u = \lambda v$ for some $\lambda \in \mathbf{R}$.

Proof. Take any nonzero vector $u, v \in \mathbf{R}$. We may assume that both u, v are nonzero vector since (6.1.1) is trivial if either u or v is a zero vector. By the definition of an inner product, $\langle u + tv, u + tv\rangle \geq 0$ for all $t \in \mathbf{R}$, i.e.,

$$\langle u, u\rangle + 2t\langle u, v\rangle + t^2\langle v, v\rangle \geq 0 \quad \forall\ t \in \mathbf{R}. \tag{6.1.2}$$

Hence the inequality (6.1.1) follows by taking $t = -\langle u, v\rangle/\langle v, v\rangle$ in the above inequality (6.1.2).

For $v = 0$ or $u = \lambda v$, one may easily verify that the Cauchy-Schwartz inequality (6.1.1) becomes an equality. Conversely if the Cauchy-Schwartz inequality (6.1.1) becomes an equality, then it follows from (6.1.2) that either $v = 0$ or $\langle u + tv, u + tv\rangle = 0$ for some $t \in \mathbf{R}$. For the late case, $u + tv = 0$ by the first property of an inner product, and hence $u = \lambda v$ for some $\lambda \in \mathbf{R}$. $\square$

For an inner product $\langle \cdot, \cdot\rangle$ on a linear space H, we define $\|\cdot\|$ on H by

$$\|u\| = \sqrt{\langle u, u\rangle},\ u \in H. \tag{6.1.3}$$

In the following theorem, we show that $\|\cdot\|$ is a strictly convex norm on H.

Theorem 6.1.2 *Let $(H, \langle \cdot, \cdot\rangle)$ be an inner product space. Then the function $\|\cdot\|$ defined in (6.1.3) is a strictly convex norm.*

Proof. Clearly $\|u\| \geq 0$ for all $u \in V$, $\|u\| = 0$ only if $u = 0$, and $\|\alpha u\| = |\alpha|\|u\|$ for all $\alpha \in \mathbf{R}$ and $u \in V$. By Cauchy-Schwartz inequality, we have

$$\|u + v\|^2 = \langle u + v, u + v\rangle \leq \langle u, u\rangle + \langle v, v\rangle + 2\|u\|\|v\| = (\|u\| + \|v\|)^2$$

for all $u, v \in V$. This concludes that the map $\|\cdot\|$ from H to $\mathbf{R}$ is a norm. Now we prove that the norm $\|\cdot\|$ is a strictly convex norm. Take two distinct elements $u, v \in H$ with $\|u\| = \|v\| = 1$, and take $t \in (0, 1)$. Then it suffices to prove that

$$\|tu + (1 - t)v\| < 1. \tag{6.1.4}$$

Suppose, on the contrary, that $\|tu + (1 - t)v\| \geq 1$. Recalling that

$$\|tu + (1 - t)v\| \leq t\|u\| + (1 - t)\|v\| = 1$$

by the triangle inequality of a norm, we then have that $\|tu + (1 - t)v\| = 1$. By direct computation,

$$\langle tu + (1 - t)v, tu + (1 - t)v\rangle = t^2\|u\|^2 + (1 - t)^2\|v\|^2 + 2t(1 - t)\langle u, v\rangle,$$

which together with

$$\|u\| = \|v\| = \|tu + (1-t)v\| = 1 \tag{6.1.5}$$

implies that

$$\langle u, v\rangle = 1 = \|u\|\|v\|.$$

Therefore $u = \lambda v$ by Theorem 6.1.2. Substituting the above relation into (6.1.5) leads to $\lambda = 1$, which contradicts $u \neq v$. □

6.1.3 Orthogonal Projection

Let H be an inner product space with inner product $\langle \cdot, \cdot \rangle$ and M be a linear subspace of H. For $u \in H$, we say that u is *perpendicular* to M, to be denoted by $u \perp M$, if $\langle u, v\rangle = 0$ for all $v \in M$.

For every $u \in H$, the *orthogonal projection* of u onto M is the vector u^* in M such that $u - u^*$ is perpendicular to M.

Recalling that the only element in M perpendicular to M is zero element, we see that the orthogonal projection of u must be unique if there exists. Moreover, the map from u to the orthogonal projection u^* is a linear operator, to be denoted by P and to be called *orthogonal projector*, from H to M. For the orthogonal operator P, we have

$$P^2 = P \quad \text{and} \quad (I-P)^2 = I - P. \tag{6.1.6}$$

For the case that M is a finite-dimensional linear space, the orthogonal projection onto M always exists and can be constructed explicitly.

Theorem 6.1.3 *Let H be an inner product space and M be a finite-dimensional linear subspace of H. Then for any $u \in H$ there exists an orthogonal projection Pu onto M. Moreover if $e_1, \ldots, e_n$ is a basis of the linear space M, then the orthogonal projection Pu is given by*

$$Pu = \sum_{k=1}^{n} \alpha_i e_i, \tag{6.1.7}$$

where $\alpha_1, \ldots, \alpha_n$ satisfies the following linear system:

$$\begin{pmatrix} \langle e_1, e_1\rangle & \cdots & \langle e_n, e_1\rangle \\ \vdots & \ddots & \vdots \\ \langle e_1, e_n\rangle & \cdots & \langle e_n, e_n\rangle \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix} = \begin{pmatrix} \langle u, e_1\rangle \\ \vdots \\ \langle u, e_n\rangle \end{pmatrix}. \tag{6.1.8}$$

Remark 6.1.4 The $n \times n$ square matrix in (6.1.8) is known as the *Gram matrix* of $\{e_1, \ldots, e_n\}$. We say that $\{e_1, \ldots, e_n\}$ is an orthonormal basis of a linear space M if $\{e_1, \ldots, e_n\}$ is a basis of the linear space W and if the corresponding Gram matrix is the identity matrix. By Theorem 6.1.3, we conclude that if $\{e_1, \ldots, e_n\}$ is an orthonormal basis of the linear space M, then the orthogonal projection Pu of u onto M is given by

$$Pu = \sum_{i=1}^{n} \langle u, e_i \rangle e_i. \tag{6.1.9}$$

Now we prove Theorem 6.1.3.

Proof of Theorem 6.1.3. First we prove that the $n \times n$ square matrix A with $\langle e_i, e_j \rangle$ as entries is nonsingular. Suppose, on the contrary, that A is singular. Then its rows are linearly dependent, which implies that

$$\sum_{i=1}^{n} \beta_i \langle e_i, e_j \rangle = 0 \quad \forall\, j = 1, \ldots, n$$

for some nonzero vector $(\beta_1, \ldots, \beta_n) \in \mathbf{R}^n$. Hence for $w := \sum_{i=1}^{n} \beta_i e_i$, we have

$$\langle w, w \rangle = \sum_{j=1}^{n} \beta_j \langle w, e_j \rangle = 0,$$

which implies that $w = 0$. This contradicts to $(\beta_1, \ldots, \beta_n) \neq 0$, and hence proves that A is nonsingular. Therefore the linear system (6.1.8) is solvable and has a unique solution.

Then we prove that Pu defined by (6.1.7) is perpendicular to M. Recall that $e_1, \ldots, e_n$ is a basis of M. Then it suffices to prove that

$$\langle Pu, e_j \rangle = \langle u, e_j \rangle \quad j = 1, \ldots, n,$$

which follows from easily from (6.1.7) and (6.1.8). □

6.1.4 Gram-Schmidt Orthonormalization Procedure

From Remark 6.1.4, we see that the orthogonal projection has simple representation when an orthonormal basis of M is constructed. This inspires us to consider the orthogonalization of any given basis, that is, given a linear independent set $\{u_1, \ldots, u_n\}$ of an inner product space H, construct an orthonormal set $\{v_1, \ldots, v_n\}$ such that the space generated by $u_1, \ldots, u_n$ is the same as the space generated by $v_1, \ldots, v_n$.

Gram-Schmidt orthonormalization procedure.

At first, we set

$$v_1 = \frac{u_1}{\|u_1\|}.$$

Clearly the space spanned by v_1 is the same as the one spanned by u_1.

Secondly we subtract away the orthogonal projection of u_2 onto the space spanned by v_1,

$$\hat{v}_2 = u_2 - \langle u_2, v_1 \rangle v_1,$$

and set

$$v_2 = \frac{\hat{v}_2}{\|\hat{v}_2\|}.$$

The above procedure works because $\hat{v}_2 \neq 0$, since otherwise u_1 and u_2 would be linearly dependent, which contradicts the linearly independent assumption on $u_1, u_2, \ldots, u_n$. From the above construction we also see that the space spanned by v_1, v_2 is the same as the one spanned by u_1, u_2.

Inductively, we assume that we have constructed orthonormal basis $v_1, \ldots, v_k$ from $u_1, \ldots, u_k$ such that the space V_k spanned by $v_1, \ldots, v_k$ is the same as the one spanned by $u_1, \cdots, u_k$, and such that $v_1, \ldots, v_k$ is an orthonormal basis of V_k. Now we define

$$\hat{v}_{k+1} = u_{k+1} - \langle u_{k+1}, v_1 \rangle v_1 - \cdots - \langle u_{k+1}, v_k \rangle v_k$$

and set

$$v_{k+1} = \frac{\hat{v}_{k+1}}{\|\hat{v}_{k+1}\|}.$$

Here we have used the observation that $\hat{v}_{k+1} \neq 0$, since otherwise $u_1, \ldots, u_{k+1}$ are linearly dependent which is a contradiction. Also one may easily verify that the space V_{k+1} spanned by $v_1, \ldots, v_{k+1}$ is the same as the one spanned by $u_1, \cdots, u_{k+1}$, and such that $v_1, \ldots, v_{k+1}$ is an orthonormal basis of V_{k+1}.

We continue the above process until $k = n$.

6.1.5 Orthogonal Polynomials

Given a finite interval $[a, b]$ and an integrable weight w, we may use the Gram-Schmidt orthonormalization procedure to construct an orthonormal basis $\{p_0, \ldots, p_n\}$ from the natural basis $\{1, \ldots, x^r\}$ of P_r under the inner product $\langle \cdot, \cdot \rangle_w$:

$$\hat{p}_0 = 1,\ p_0 = \hat{p}_0/\|\hat{p}_0\|$$

and inductively we define

$$\hat{p}_k := x^k - \sum_{j=0}^{k-1} \langle x^k, p_j \rangle_w p_j \tag{6.1.10}$$

and

$$p_k := \hat{p}_k/\|\hat{p}_k\| \tag{6.1.11}$$

until $k = r$.

For the case that the interval is the unit interval $I = [-1, 1]$ and the weight is defined by $w_{\alpha,\beta}(x) = (1-x)^\alpha (1+x)^\beta$, where $\alpha, \beta > -1$, the associated set of orthogonal polynomials obtained from Gram-Schmidt orthonormalization procedure of the polynomials $\{1, x, x^2, \ldots\}$ under the inner product $\langle \cdot, \cdot \rangle_{w_{\alpha,\beta}}$, is denoted by $\{P_n^{(\alpha,\beta)}(x)\}_{n=0}^\infty$, and usually called the *Jacobi polynomials.*

If we specify the indices α and β and consider the case $\alpha = \beta = 0$, the resulting Jacobi polynomials $P_n^{(0,0)}$ after renormalization at 1 are called *Legendre polynomials* P_n, $P_n = c_n P_n^{(0,0)}$ for some $c_n \in \mathbf{R}$ and $P_n(1) = 1$. In this case, the expression of Legendre polynomials up to degree 4:

$$\begin{aligned}
P_0(x) &= 1,\\
P_1(x) &= x,\\
P_2(x) &= \frac{3}{2}x^2 - \frac{1}{2},\\
P_3(x) &= \frac{5}{2}x^3 - \frac{3}{2}x,\\
P_4(x) &= \frac{35}{8}x^4 - \frac{15}{4}x^2 + \frac{3}{8},
\end{aligned}$$

while for $n \geq 2$, P_n is defined by

$$nP_n(x) = (2n-1)xP_{n-1}(x) - (n-1)P_{n-2}.$$

If we consider the case that $\alpha = \beta = -1/2$, the resulting Jacobi polynomials $P_n^{(-1/2,-1/2)}$ after renormalization at 1 are *Chebyshev polynomials* T_n,

$$T_n(x) = \cos n\theta,$$

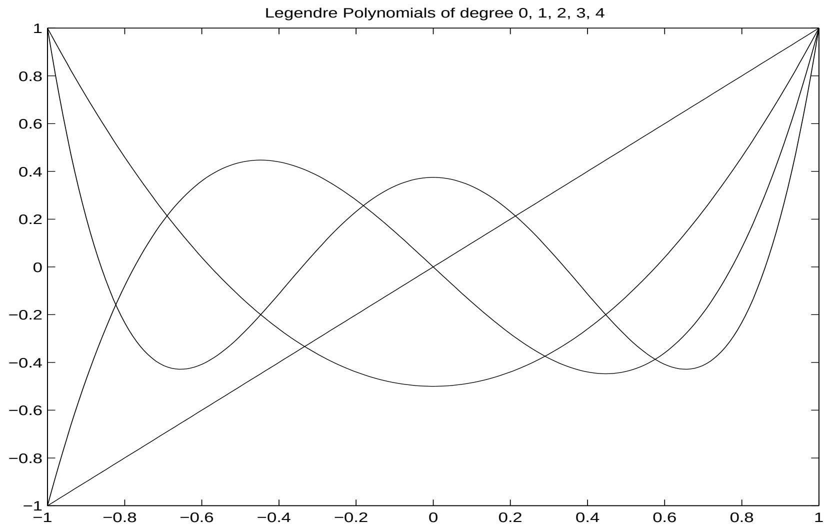

Figure 3.1: Legendre Polynomials of degrees $0, 1, 2, 3, 4$

where $x = \cos\theta$. Here we give the explicit expression of Chebyshev polynomials with degrees up to 4.

$$\begin{aligned}
T_0(x) &= 1,\\
T_1(x) &= x,\\
T_2(x) &= 2x^2 - 1,\\
T_3(x) &= 4x^3 - 3x,\\
T_4(x) &= 8x^4 - 8x^2 + 1.
\end{aligned}$$

If we consider the case that $\alpha = \beta = 1/2$, the resulting Jacobi polynomials $P_n^{(-1/2,-1/2)}$ after renormalization at 1 are the *Chebyshev polynomials of second kind* $U_n, n \geq 0$, by

$$U_n(x) = \frac{\sin(n+1)\theta}{\sin\theta},$$

where $x = \cos\theta$. The following are the explicit expression of Chebyshev polynomials of second kind of degree up to 4 and their figures.

$$U_0(x) = 1,$$

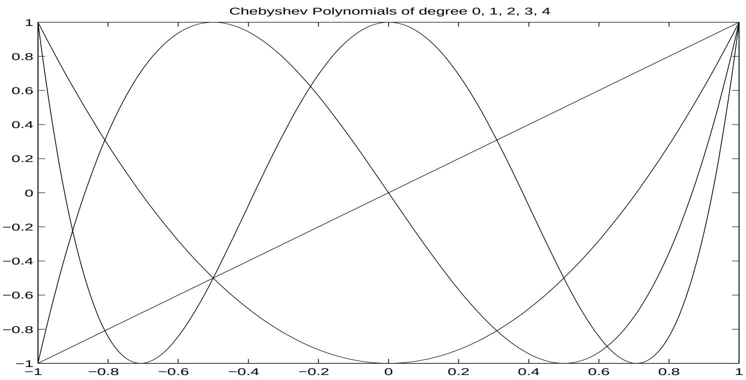

Figure 3.2: Chebyshev polynomials of degrees $0, 1, 2, 3, 4$

$$
\begin{aligned}
U_1(x) &= 2x, \\
U_2(x) &= 4x^2 - 1, \\
U_3(x) &= 8x^3 - 4x, \\
U_4(x) &= 16x^4 - 12x^2 + 1.
\end{aligned}
$$

6.2 Existence and Uniqueness of Best Approximations

In this section, we establish the following result about existence, uniqueness, and explicit construction of best approximations in Hilbert space.

Theorem 6.2.1 *Let H be an inner product space and M be a finite-dimensional subspace of H. Then for any $u \in H$ there exists a unique $u^* \in M$ such that*

$$\|u - u^*\| = \min_{v \in M} \|u - v\|.$$

Moreover, u^* is a best approximation to u out of M if and only if $u - u^* \perp M$, that is, $u^* = Pu$, where P is the orthogonal projection onto M.

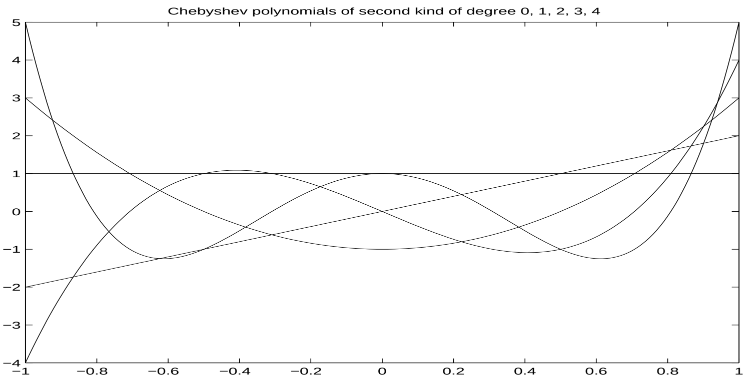

Figure 3.3: Chebyshev polynomials of second kind of degrees $0, 1, 2, 3, 4$

Proof. The existence of the best approximation follows from Theorem 5.2.3, while the uniqueness of best approximations is true because of Theorems 5.3.1 and 6.1.2.

Now we show that the best approximation is just the projection of u onto M. Let u^* be a best approximation and set $d = \|u - u^*\|$. Then for any $\lambda \in \mathbf{R}$ and $v \in M$,

$$d^2 \leq \|u - u^* - \lambda v\|^2 = d^2 - 2\lambda\langle u - u^*, v\rangle + \lambda^2\|v\|^2,$$

which yields

$$2\lambda\langle u - u^*, v\rangle \leq \lambda^2\|v\|^2. \tag{6.2.1}$$

Letting $\lambda \to 0+$ and $\lambda \to 0-$ in (6.2.1) respectively, we obtain

$$2\langle u - u^*, v\rangle \leq 0 \quad \text{and} \quad 2\langle u - u^*, v\rangle \geq 0$$

for all $v \in W$. Thus $\langle u - u^*, v\rangle = 0$ for all $v \in M$. This proves that $u - u^*$ is perpendicular to M.

Conversely if $w \in M$ is so chosen that $u - w \perp M$, then for any $v \in M$,

$$\|u - v\|^2 - \|u - w\|^2 = \|v - w\|^2 - 2\langle u - w, v - w\rangle = \|v - w\|^2 \geq 0,$$

which implies that w is a best approximation of u out of M. $\square$

6.3 Least Square Approximation in $\mathbf{R}^d$

In this section, we give an application of Theorem 6.2.1. More applications will be given in later chapters. By Theorems 6.1.3 and 6.2.1, we have the following result about the least square approximation of a vector in $\mathbf{R}^d$ out of a linear subspace W of $\mathbf{R}^d$.

Theorem 6.3.1 *Let W be a linear subspace of $\mathbf{R}^d$. Then for any given $u \in \mathbf{R}^d$ there exists unique $u^* \in W$ so that*

$$\|u - u^*\| = \inf_{v \in W} \|u - v\|.$$

Moreover if $e_1, \ldots, e_m$ is a basis of W, then $u^* = \sum_{i=1}^m \alpha_i e_i$ satisfies

$$\begin{pmatrix} \langle e_1, e_1 \rangle & \cdots & \langle e_m, e_1 \rangle \\ \vdots & \ddots & \vdots \\ \langle e_1, e_m \rangle & \cdots & \langle e_m, e_m \rangle \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_m \end{pmatrix} = \begin{pmatrix} \langle u, e_1 \rangle \\ \vdots \\ \langle u, e_m \rangle \end{pmatrix}. \tag{6.3.1}$$

Now we apply the above theorem to solve the following least square approximation of certain points on plane by functions in a finite-demensional space.

Example 6.3.2 *Let $X := \{(x_k, y_k) \in \mathbf{R}^2, 1 \le k \le n\}$ be some points in $\mathbf{R}^2$ and W be a finite-dimensional space of continuous functions. Find the least square approximation $f^* \in W$ to the points X, that is,*

$$\Big(\sum_{k=1}^n |y_k - f^*(x_k)|^2 \Big)^{1/2} = \min_{f \in W} \Big(\sum_{k=1}^n |y_k - f(x_k)|^2 \Big)^{1/2}.$$

Solution. Let m denote the dimension of the space W, and $f_1, \ldots, f_m$ be a basis of W. Then every function $f \in W$ can be written as $f = \sum_{i=1}^m \alpha_i f_i$ for some $(\alpha_1, \ldots, \alpha_m) \in \mathbf{R}^m$. Thus the problem reduces to finding $(\alpha_1^*, \cdots, \alpha_m^*) \in \mathbf{R}^m$ so that

$$\Big(\sum_{k=1}^n \Big| y_k - \sum_{i=1}^m \alpha_i^* f_i(x_k) \Big|^2 \Big)^{1/2} = \min_{(\alpha_1, \ldots, \alpha_m) \in \mathbf{R}^m} \Big(\sum_{k=1}^n \Big| y_k - \sum_{i=1}^m \alpha_i f_i(x_k) \Big|^2 \Big)^{1/2}.$$

In other words, it suffices to find u^* so that

$$\|u - u^*\| = \inf_{v \in W} \|u - v\|,$$

where $\|\cdot\|$ is usual Euclidean norm on $\mathbf{R}^n$, $u = (y_1, \ldots, y_n)$, and the linear space W is spanned by vectors

$$e_i := (f_i(x_1), \ldots, f_i(x_n)),\ 1 \le i \le m. \tag{6.3.2}$$

Let $\{u_1, \ldots, u_t\}$ be an orthonormal basis of the linear space, and write

$$u_j = \sum_{i=1}^{m} \alpha_{ji} e_i, \quad 1 \le j \le m.$$

By Theorem 6.3.1, the function $f = \sum_{i=1}^{m} \left(\sum_{j=1}^{t} \langle y, u_j \rangle \alpha_{ji} \right) f_i$ is a solution. $\square$

For the case that $W = P_{m-1}$, the space of all polynomials of degree at most $m-1$, the vectors $e_1, \ldots, e_m$ in (6.3.2) are linear independent if $m < n$ because any nonzero polynomial of degree at most $m-1$ has no more than n roots. Therefore there is a unique polynomial p^* of degree at most $m-1$ so that

$$\left(\sum_{k=1}^{n} |y_k - p^*(x_k)|^2 \right)^{1/2} = \min_{p \in P_{m-1}} \left(\sum_{k=1}^{n} |y_k - p(x_k)|^2 \right)^{1/2}.$$

If we further specify that $m = 2$, we have

$$p^*(x) = \alpha_0^* x + \alpha_1^*,$$

where

$$\begin{aligned} \alpha_0^* &= \frac{n \sum_{k=1}^{n} x_k y_k - (\sum_{k=1}^{n} x_k)(\sum_{k=1}^{n} y_k)}{n \sum_{k=1}^{n} x_k^2 - (\sum_{k=1}^{n} x_k)^2}, \\ \alpha_1^* &= \frac{(\sum_{k=1}^{n} y_k)(\sum_{k=1}^{n} x_k^2) - (\sum_{k=1}^{n} y_k x_k)(\sum_{k=1}^{n} x_k)}{n \sum_{k=1}^{n} x_k^2 - (\sum_{k=1}^{n} x_k)^2}. \end{aligned}$$

Exercises

1. Use Gram-Schmidt orthogonal process to orthonormalize the following two bases of the 3-dimensional Euclidean space with standard inner product:

(i) $e_1 = (1, 1, 1),\ e_2 = (2, 0, 3),\ e_3 = (4, 2, 0)$;

(ii) $e_1 = (4,2,0),\ e_2 = (1,1,1),\ e_3 = (2,0,3)$.

2. Let M be the space spanned by $1, x, x^2$. Use Gram-Schmidt orthogonal process to construct an orthonormal basis of M under the inner product$\langle \cdot, \cdot \rangle$ defined by
$$\langle f, g\rangle = \int_0^\infty f(x)g(x)e^x dx \quad \text{for } f, g \in M.$$

3. Let M be the space spanned by $1, \sin x, \cos x$. Use Gram-Schmidt orthogonal process to construct an orthonormal basis of M under the inner product$\langle \cdot, \cdot \rangle$ defined by
$$\langle f, g\rangle = \int_{-\pi}^{\pi} f(x)g(x)\sin^2 x dx \quad \text{for } f, g \in M.$$

4. Let M be the space spanned by $1, e^x, \sin \pi x$. Use Gram-Schmidt process to construct an orthonormal basis of M under the inner product$\langle \cdot, \cdot \rangle$ defined by
$$\langle f, g\rangle = \int_0^1 f(x)g(x)dx \quad \text{for } f, g \in M.$$

5. Justify that the Chebyshev polynomials T_n are even functions for even n and odd functions for odd n.

6. Justify that the Chebyshev polynomials T_n satisfy the following relation:
$$T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x),\ n \geq 1.$$

7. Let $X = \{(-2,-1),\ (-1,1),\ (0,-2),\ (1,3),\ (2,-1)\}$. Find the quadratic polynomial p with least square approximation error to the points in X.

8. Let $(x_k, y_k) = (\sin k\pi/100, \cos k\pi/200), 0 \leq k \leq 50$. Find the best approximating polynomial of the form $l^*(x) = a^* + b^* \sin x + b^* x^3$ numerically such that
$$\sum_{k=0}^{50} |y_k - l^*(x_k)|^2 = \min_{l(x)=a+b\sin x + cx^3, a,b,c\in\mathbf{R}} |y_k - l(x_k)|^2,$$
and compute the error $\sum_{k=0}^{50} |y_k - l^*(x_k)|^2$.

9. Find the least square approximating affine function $p(x)$ to x^2 on $[-1,1]$, and find out all simple zeros of $x^2 - p(x)$.

10. Find the best approximating affine function p to the function e^x on $[0,\infty)$ under the norm $\|\cdot\|$ defined by $\|f\| = \int_0^\infty |f(x)|^2 e^{-3x} dx$.

Chapter 7

Multiresolution Approximation

In this chapter, we introduce the cubic spline on any knots and B-spline on equally spaced knots, and study their approximation properties.

7.1 Cubic Spline Interpolation

In this section, we consider the approximating properties of the "smooth" piecewise *cubic function.* Such a function is called *cubic spline.*

Suppose that X_n denotes the set of real numbers $\{x_0, \ldots, x_n\}$, where

$$a \leq x_0 < x_1 < \ldots < x_n \leq b.$$

Let $S(X_n)$ be the set of all functions $s(X_n, x) = s(x) \in C^2([a,b])$ having the property that in each interval $[x_i, x_{i+1}], 0 \leq i \leq n-1$, $s(x)$ agrees with a polynomial of degree at most 3. We call the function $s \in S(X_n)$ a cubic spline, and the points $x_0, \ldots, x_n$ knots.

Claim: A polynomial of degree at most 3 is a cubic polynomial.

Example: Let $a = -1, b = 1$ and $x_i = -1 + i/3, 0 \leq i \leq 5$. Construct a cubic polynomial not in P_3.

Basis of $S(X_n)$ Both the families

$$1, x, x^2, x^3, (x - x_1)_+^3, \ldots, (x - x_{n-1})_+^3,$$

and

$$1, x, x^2, x^3, (x - x_1)_-^3, \ldots, (x - x_{n-1})_-^3,$$

are basis of $S(X_n)$.

Hint: (i) Show all those functions are cubic spline; (ii) show that any cubic spline is a linear combination of those basis.

Dimension of $S(X_n)$: $4 + (n-1) = n+3$.

Interpolating Problem Under what circumstances does there exist an interpolating spline?

First let us verify that: *If $\alpha < \beta$, then the unique polynomial p of degrees at most 3 that satisfies*

$$p(\alpha) = u_1, p(\beta) = u_2, p'(\alpha) = v_1, p'(\beta) = v_2$$

is

$$\begin{aligned} p(x) &= u_1 \left[\frac{(x-\beta)^2}{(\beta-\alpha)^2} + 2\frac{(x-\alpha)(x-\beta)^2}{(\beta-\alpha)^3} \right] \\ &\quad + u_2 \left[\frac{(x-\alpha)^2}{(\beta-\alpha)^2} - 2\frac{(x-\beta)(x-\alpha)^2}{(\beta-\alpha)^3} \right] \\ &\quad + v_1 \frac{(x-\alpha)(x-\beta)^2}{(\beta-\alpha)^3} + v_2 \frac{(x-\alpha)^2(x-\beta)}{(\beta-\alpha)^2}. \end{aligned}$$

For the case that $\alpha = 0$ and $\beta = 1$,

$$p(x) = u_1(x-1)^2(1+2x) + u_2 x^2(3-2x) + v_1 x(x-1)^2 + v_2 x^2(x-1).$$

Theorem *Given numbers $\tilde{s}_0$ and $\tilde{s}_n$, there exists a unique spline satisfying*

$$s(f, X_n, x_i) = f_i, \ 0 \le i \le n$$

and

$$s'(f, X_n, x_i) = \tilde{s}_i, \ i = 0, n.$$

How to solve the interpolation problem:

- Find a basis $e_1, \ldots, e_{n+3}$;
- Set $p(x) = \sum_{i=1}^{n+3} a_i e_i$;
- Solve the linear system

$$p(x_i) = f(x_i), \ i = 0, \ldots, n$$

  and

$$p'(x_0) = \tilde{s}_0 \quad \text{and} \quad p'(x_n) = \tilde{s}_n.$$

Examples

- Consider $f(x) = x^5, x \in [-1,1]$ and the knots $-1/2, 0, 1/2$. Find a cubic spline p such that $p(x_i) = f(x_i), 0 \leq i \leq 2$ and $p'(x_i) = f'(x_i), i = 0, 2$.

- Consider $f(x) = \cos \pi x/2, x \in [-1,1]$ and the knots $-1/2, 0, 1/2$. Find a cubic spline p such that $p(x_i) = f(x_i), 0 \leq i \leq 2$ and $p'(x_i) = f'(x_i), i = 0, 2$.

- Consider $f(x) = x^4 + \sin \pi x, x \in [-1,1]$ and the knots $-1/2, 0, 1/2$. Find a cubic spline p such that $p(x_i) = f(x_i), 0 \leq i \leq 2$ and $p'(x_i) = f'(x_i), i = 0, 2$.

7.1.1 Extreme Property

Theorem 7.1.1 *Suppose that $a = x_0 < x_1 < \ldots < x_n = b$ and $f \in C^2[a,b]$. If we take $f_i = f(x_i), 0 \leq i \leq n$ and consider the spline that satisfies*

$$\begin{aligned} s(x_i) &= f_i, 0 \leq i \leq n \\ s'(x_0) &= f'(x_0), s'(x_n) = f'(x_n). \end{aligned}$$

Then we have

$$\int_a^b [f''(x)]^2 dx - \int_a^b [s''(x)]^2 dx = \int_a^b [f''(x) - s''(x)]^2 dx.$$

Proof.

$$\begin{aligned} & \int_a^b [f''(x) - s''(x)]^2 dx \\ = & \int_a^b [f''(x)]^2 - \int_a^b [s''(x)]^2 dx \\ & -2\int_a^b s''(x)[f''(x) - s''(x)] dx. \end{aligned}$$

Therefore it suffices to verify that

$$\int_a^b s''(x)[f''(x) - s''(x)] dx = 0.$$

By partial integration formula, we have

$$\begin{aligned}
& \int_a^b s''(x)[f''(x) - s''(x)]dx \\
= \; & s''(x)[f'(x) - s'(x)]\Big|_a^b \\
& - \int_a^b s'''(x)[f'(x) - s'(x)]dx \\
= \; & - \int_a^b s'''(x)[f'(x) - s'(x)]dx \\
= \; & \sum_{i=0}^{n-1} \alpha_i \int_{x_i}^{x_{i+1}} [f'(x) - s'(x)]dx \\
= \; & 0.
\end{aligned}$$

□

The minimizing property shown above helps us to explain the origin of the name spline for interpolating piecewise cubics. Engineers have for a long time used this rods to fair curves through given points.

Exercise Consider $f(x) = \sin \pi x, x \in [-1, 1]$ and the knots $\{-1/2, 0, 1/2\}$. Find the cubic spline that satisfies the condition in the above theorem, and compute $\int_a^b |f''(x)|^2 dx$, $\int_a^b |s''(x)|^2 dx$ and $\int_a^b |f''(x) - s''(x)|^2 dx$.

7.1.2 Approximation Property

Now let consider the approximating power of cubic spline.

Suppose now that $a = 0, b = 1$ and the knots $\{x_0, x_1, \ldots, x_n\}$ satisfies $0 = x_0 < x_1 < \cdots < x_n = 1$. Given a continuous function f defined on $[0, 1]$. Put $\triangle x_i = x_{i+1} - x_i$ amd define the norm on X_n by

$$\delta = \delta(X_n) = \max_{0 \le i \le n-1} \triangle x_i.$$

Theorem 7.1.2 *Suppose that $f \in C^2(I)$ and $s \in S(X_n)$ satisfies*

$$s(x_i) = f(x_i), 0 \le i \le n$$

and

$$s'(0) = f'(0), s'(1) = f'(1).$$

Then for all $x \in [0, 1]$,

$$|f^{(r)}(x) - s^{(r)}(x)| \le 5\delta^{2-r}\omega(f'', [0, 1], \delta).$$

The above theorem shows that, if $\delta(X_n) \to 0$ as $n \to \infty$, the interpolating spline and its first derivatives converge uniformly to a given function in $C^2(I)$.

Similar result can be established for $f \in C^1(I), C^3(I)$.

Exercise Let $f = \sin \pi x$. By using Jackson's theorem and the above theorem, derive an upper bound for the estimate

$$\min_{s \in S(X_n)} \|f - s\|.$$

7.2 B-splines

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