An Inverse Boundary Value Problem in 2D Transport

Alexandru Tamasan*

Abstract: Homogeneous a priori bounded scattering is uniquely determined by the albedo operator. As a consequence of the inversion of the attenuated X-ray transform, we reconstruct isotropic scattering in two different ways.

1 Introduction

Consider the stationary transport of particles, in a source free scattering media, modeled by the equation

$$v \cdot \nabla_x u(x,v) + a(x,|v|)u(x,v) = b(x,|v|) \int_V \rho(x,v',v)u(x,v')dv',\ x \in \Omega,\ v \in V. \tag{1}$$

The domain Ω is convex, bounded with smooth boundary and the set of velocities V is $\{v : 0 < v_{min} \leq |v| \leq v_{max}\}$. The function $u(x,v)$ represents the density of particles at x, traveling in the direction $v/|v|$ with speed $|v|$. In particular $\int_\Omega \int_V u(x,v)dxdv$ is the total number of particles. The total macroscopic cross section $a(x,|v|)$ represents the likelihood per unit time for particles to either hit nuclei or be absorbed by the media. The scattering cross section $b(x,|v|)$ is dimensionless and represents the quota of collided particles which scatter. The kernel $\rho(x,v',v)$ represents the probability per unit path length for particles at x, traveling with velocity v' to scatter into the direction $v/|v|$ with speed $|v|$. The scattering kernel $k(x,v',v)$ is the product $b(x,|v|)\rho(x,v',v)$ measured in inverse unit length. Both a and k are intrinsic features of the media. We have in mind two kinds of applications governed by small scattering. One of them describes the kinetics of molecules in gazes. The other is the transport of neutrons through thin plates. For more details on the physical model refer to [16].

The forward problem solves the equation for u when some boundary sources are specified. The inverse boundary value problem aims to reconstruct a and k from the boundary measurements. This problem appeared first for the mono-energetic case in the works of Anikonov and his collaborators [1], [2], [3], [4]. The formal determination of a and k has been completely solved by Choulli and Stefanov in three dimensional models [10]. They analyzed the operator which takes incoming boundary data into the outgoing boundary data (the so called albedo operator). The Schwartz kernel of this operator has the main singularity, a delta distribution, concentrated along straight lines. From the strength of this singularity one can determine the X-ray transform of a. The next singularity is concentrated along planes through x determined by the two scattering directions v and v'. The strength of this singularity recovers k. For three dimensional domains, the number of variables in the distributional kernel of our data exceeds the number of variables of k. The problem is over-determined and this is one of the reasons why numerical methods as in [14] developed earlier.

In the two dimensional case, this method still determines a but fails to reconstruct k. The problem is formally determined and it is still open whether $k(x,v,v')$ can be uniquely determined

*University of Washington, Dept of Mathematics, Box 354350, Seattle, WA 98195, USA (tamasan@math.washington.edu).

from boundary measurements. The best general result for unit speed velocities is due to Romanov [19]. He gives a stability estimate for small absorption and small scattering which depends on position x and the angle between the two scattering directions: $k = k(x, \frac{v}{|v|} \cdot \frac{v'}{|v'|})$. Other known results assume special forms of scattering. In the unit speed velocity case, Bal [9] reconstructs k as a trigonometric polynomial in the scattering angle. Numerical reconstruction was obtain by Dorn [13] in two and three dimensional domains.

This paper considers the question of determining k for plane domains. There are two kinds of results. **Theorem 1** shows unique determination of homogeneous and a priori bounded scattering $k(v, v')$. This type of scattering is not completely covered by the result of Romanov, as we allowed the kernel to depend on two independent velocities. Moreover, we do not need a to be small. The weakness of our result is the homogeneous assumption. However, unlike the case considered by Bal [9], a need not be constant and k need not be trigonometric polynomial.

The second result presents two different inversion formulae (**Proposition 2, 3**) for unit speed velocities and isotropic scattering. Isotropy assumes that the particles scatter in every direction with the same probability, (ρ is constant and we determine formulae for b). The first formula is a straightforward application to the Novikov's inversion formula of the attenuated x-ray transform [18]. The second formula is an application of the theory of A-analytic functions as developed by Bukhgeim in [11], [7]. Although different in appearance we explain why they are the same.

2 The albedo operator and its kernel

This section presents some preliminary results concerning the forward problem. Throughout the paper we assume the *admissibility conditions*

$$\begin{cases} 0 \leq a \in L^\infty(\Omega \times (v_{min}, v_{max})) \\ 0 \leq k \in L^\infty(\Omega \times V \times V)). \end{cases} \tag{2}$$

Further conditions will be imposed on k later on.

The total travel time τ is given by $\tau(x, v) = \tau_-(x, v) + \tau_+(x, v)$, where $\tau_\pm(x, v)$ is the travel time from x in the direction $\pm v$ until the boundary is met. For a convex domain with smooth boundary $\tau_\pm$ are smooth maps. Let d be the diameter of Ω. Since velocities are bounded away from zero, all the particles reach the boundary within a maximum time $T = d/v_{min}$.

The albedo operator (our data) takes the incoming flux to the outgoing flux at the boundary. The exact definition follows from the existence and uniqueness result of Proposition 1 and from the trace Lemma 1 below. Let $n(x)$ denote the outer unit normal at $x \in \partial\Omega$. We define the incoming and outgoing boundary by $\Gamma_\pm := \{(x, v) \in \partial\Omega \times V \ : \pm n(x) \cdot v > 0\}$.

The space of solutions $W = \{f \in L^1(\Omega \times V) : \tau^{-1} f, v \cdot \nabla f \in L^1(\Omega \times V)\}$. The gradient is understood in the sense of distributions. The norm on W is given by $||f||_W := ||v \cdot \nabla f|| + ||\tau^{-1} f||$, where $|| \cdot ||$ is the standard norm on $L^1(\Omega \times V)$.

On $\Gamma_\pm$ we consider the measure $d\xi(x, v) := |n(x) \cdot v| d\mu dv$, where $d\mu$ is the induced Lebesgue measure on the boundary. Following the method of Cessenat [12], Choulli and Stefanov showed that functions in W restrict to $L^1(\Gamma_\pm, d\xi)$. Throughout the paper we denote by R the trace on Γ_+, $Rf = f|_{\Gamma_+}$.

Lemma 1 ([10]). *For any $f \in W$ we have $||Rf||_{L^1(\Gamma_\pm, d\xi)} \leq ||f||_W$.*

Without the scattering contribution, the linear transport equation decouples into a family of first order ordinary differential equations indexed by the velocity v. A function $f_- \in L^1(\Gamma_-, d\xi)$

can be extended to the whole space $\Omega \times V$ satisfying these equations. More precisely,

$$Jf_-(x,v) = \exp\left(-\int_0^{\tau_-(x,v)} a(x-sv,|v|)ds\right) f_-(x-\tau_-(x,v)v,v) \tag{3}$$

is the extension of f_-, which satisfies the non-scattered transport equation. For an incoming flux $f_- \in L^1(\Gamma_-, d\xi)$ we have that Jf_- lies in W and the following result holds.

Lemma 2 ([10]). $||Jf_-||_W \leq ||f_-||_{L^1(\Gamma_-,d\xi)}$.

Consider the equation (1) together with the boundary condition $u|_{\Gamma_-} = f_-$. Integrating along the direction v and using $\exp\left(-\int_0^t a(x-sv,|v|)ds\right)$ as an integrating factor, we can reduce the boundary value problem to the integral equation

$$u = Ku + Jf_-, \tag{4}$$

where K is defined by

$$Ku(x,v) = \int_0^{\tau_-(x,v)} \exp\left(-\int_0^t a(x-sv,|v|)ds\right) \int_V k(x-tv,v',v)u(x,v')dv'dt. \tag{5}$$

This calculation is similar to the one used in getting the Peierls' integral equation in section 4.

Let $\sigma(x,v')$, defined by

$$\sigma(x,v') = \int_V k(x,v',v)dv, \tag{6}$$

represent the total contribution at x of particles traveling with velocity v'. The *sub-critical* condition

$$||\tau\sigma||_\infty < 1 \tag{7}$$

is sufficient for the forward problem to be well posed. The sub-critical condition can be satisfied in the experiment of neutrons penetrating thin plates, as $\tau(x,v)$ is controlled by the width of the plate. In the other model, due to the small density of gazes, the kernel k can be small. When k is large a diffusion approximation is considered instead [6].

Under this assumption, the solution of (4) is given by the Neumann expansion

$$u = Jf_- + KJf_- + K^2Jf_- + ... = Jf_- + (I-K)^{-1}KJf. \tag{8}$$

The series converges in W and the restriction of u to Γ_+ makes sense.

We summarize the properties which make the direct problem well posed and allow to define the albedo operator.

Proposition 1 ([10]). *If the admissible* (2) *and sub-critical* (7) *conditions hold, then*

a) K defined by (5) is a bounded operator from $L^1(\Omega \times V, \tau^{-1}dxdv)$ to itself, with the operator norm $||K|| \leq ||\tau\sigma||_\infty < 1$;

b) the integral equation (4) *has a unique solution $u = (I-K)^{-1}Jf_-$;*

c) the albedo operator $\mathcal{A}$ takes f_- to $u|_{\Gamma_+}$ and it is bounded from $L^1(\Gamma_-, d\xi)$ to $L^1(\Gamma_+, d\xi)$.

Consider now the equation governed by another admissible pair (a, k_0) and such that k_0 also satisfies the sub-critical condition. Let K_0 be the corresponding integral operator defined as in (5). The following estimate will be needed later.

Lemma 3. *The operator* $R[(I-K)^{-1}K^2 - (I-K_0)^{-1}K_0^2]J : L^1(\Gamma_-, d\xi) \to L^1(\Gamma_+, d\xi)$ *is bounded with the operator norm satisfying*

$$||R[(I-K)^{-1}K^2 - (I-K_0)^{-1}K_0^2]J|| \leq |\tau\sigma - \tau\sigma_0|_\infty \frac{2|\tau\sigma_0|_\infty^2 + 3|\tau\sigma_0|_\infty + |\tau\sigma|_\infty + |\tau\sigma_0|_\infty|\tau\sigma|_\infty}{(1-|\tau\sigma_0|_\infty)(1-|\tau\sigma|_\infty)}. \tag{9}$$

Proof. First apply the Lemmas 1 and 2 and part $a)$ of Proposition 1 to get

$$||R(I-K)^{-1}K^2J - R(I-K_0)^{-1}K_0^2J|| \leq |(I-K)^{-1}K^2 - (I-K_0)^{-1}K_0^2|.$$

The norm on the right hand side denotes the operator norm from $L^1(\Omega \times V, \tau^{-1}dxdv)$ to itself. Write $(I-K)^{-1}K^2 - (I-K_0)^{-1}K_0^2 = (I-K)^{-1}[K^2(I-K_0) - (I-K)K_0^2](I-K_0)^{-1}$ and estimate the middle part. In $K^2(I-K_0) - (I-K)K_0^2$ replace K_0 by $K + (K_0 - K)$ and a straightforward calculation shows that

$$|K^2(I-K_0) - (I-K)K_0^2| \leq |K_0 - K|(2|K|^2 + 3|K| + |K_0| + |K||K_0|).$$

Apply again part $a)$ of Proposition 1 to get the conclusion. □

The main ingredient in studying the inverse problem for (1) is the singular decomposition of the Schwartz kernel of the albedo operator. A similar singular decomposition was obtained in [5] for three dimensional problems with point sources and zero incoming fluxes. We denote by $\delta_{[x_0]}(x)$ the delta distribution on the boundary supported at $x_0 \in \partial\Omega$, i.e. for any $\phi \in C^\infty(\partial\Omega)$,

$$\langle \delta_{[x_0]}, \phi \rangle = \int_{\partial\Omega} \delta_{[x_0]}(x)\phi(x)d\mu(x) = \phi(x_0).$$

In order to simplify notation we define a map $E(s, x, v, v')$ by

$$E(s,x,v,v') := e^{-\int_0^s a(x-pv,|v|)dp} e^{-\int_0^{\tau_-(x-sv,v')} a(x-sv-pv',|v'|)dp}. \tag{10}$$

Lemma 4 (Singular Decomposition, [10]). *Assume that the admissible* (2) *and sub-critical* (7) *assumptions hold. The distribution kernel of the albedo operator* $\mathcal{A}$ *decomposes into* $\alpha_1+\alpha_2+\alpha_3$*, where*

$$\alpha_1 = e^{-\int_0^{\tau_-(x,v)} a(x-sv,v)ds} \delta_{[x-\tau_-(x,v)v]}(x')\delta(v-v'), \tag{11}$$

$$\alpha_2 = \int_0^{\tau_-(x,v)} E(s,x,v,v')k(x-sv,v',v)\delta_{[x-sv-\tau_-(x-sv,v')v']}(x')ds, \tag{12}$$

$$|v' \cdot n(x')|^{-1}\alpha_3 \in L^\infty(\Gamma_-; L^1(\Gamma_+, d\xi)). \tag{13}$$

3 Uniqueness of homogeneous a priori bounded scattering

In this section we prove that scattering $k = k(v, v')$, which also satisfy an a priori bound is uniquely determined by the albedo operator. This bound depends on the geometry of the domain and the range of velocities as follows. For any $v \in V$, let $\Gamma_+(v)$ denote the set $\{x \in \partial\Omega : (x, v) \in \Gamma_+\}$ and let $L := \inf_{v\in V} \text{length}(\Gamma_+(v))$. As $\partial\Omega$ is smooth and convex, L/π is always greater then the inverse of the maximum curvature. In particular L is positive (e.g. if Ω is a unit disc then $L = \pi$).

Recall that T was the maximum travel time and v_{max} is the fastest speed. As before, to any k we associate the map σ by (6). We prove our result in the class of scatterings k that satisfy

$$||\tau\sigma||_\infty \frac{3||\tau\sigma||_\infty + 4}{(1-||\tau\sigma||_\infty)^2} < \frac{Le^{-T||a||_\infty}}{Tv_{max}}. \tag{14}$$

Consider now two problems with the admissible coefficients (a, k) and (a_0, k_0). Let σ and σ_0 be the maps corresponding to k and k_0 defined as in (6) and K and K_0 be the operators given by (5). Let α_2 and α_2^0 be the Schwartz kernels of RKJ and RK_0J, and α_3 and α_3^0 be the Schwartz kernels of $R(I-K)^{-1}K^2J$ and $R(I-K_0)^{-1}K_0^2J$, respectively.

The uniqueness result is the following.

Theorem 1. *Let (a, k) and (a_0, k_0) be two admissible pairs, such that k and k_0 are homogeneous and satisfy* (14)*. If the corresponding albedo kernels of $\mathcal{A}$ and $\mathcal{A}_0$ coincide, then $a = a_0$ and $\sigma = \sigma_0$.*

Corollary 1. *Let (a, k) and $(a_0, 0)$ be two admissible pairs with k homogeneous. If the corresponding albedo kernels of $\mathcal{A}$ and $\mathcal{A}_0$ coincide, then $a = a_0$ and $k = 0$.*

Proof. Apply the theorem to get $\sigma(x, v') = \tau(x, v') \int_V k(v', v)dv = 0$ a.e. $v' \in V$. Now use the fact that k is non-negative. □

In order to use the singular structure of the kernel of $\mathcal{A}$, we test it against some functions ϕ_ϵ defined as follows. For $\phi \in C_0^\infty(\mathbb{R})$, $\phi(0) = 1$, $\int \phi(s)ds = 1$, with supp$\phi \subset (-1, 1)$ and small $\epsilon > 0$,

$$\phi_\epsilon(x, v, v') := \frac{1}{\epsilon}\phi\left(\frac{x \cdot m(v')}{\epsilon v \cdot m(v')}\right),$$

where $m(v')$ is the unit vector orthogonal to v', for some chosen orientation (see figure 1).

Let us denote the algebraic variety of the linearly dependent velocities by D. As in [10], we work with pairings of velocities outside D. The key calculation is given in the lemma below. Since no homogeneity or smallness of scattering is needed and we formulate the result and give the proof in full generality.

Lemma 5. *For almost every $(x, v', v) \in \Omega \times (V^2 - D)$,*

$$\lim_{\epsilon\to 0} \int_{\partial\Omega} \alpha_2(x + \tau_+(x, v)v, v, x', v')\phi_\epsilon(x' - x, v, v')d\mu(x') = E(\tau_+(x, v), x + \tau_+(x, v)v, v, v')k(x, v', v). \tag{15}$$

and

$$\lim_{\epsilon\to 0} \int_{\partial\Omega} \alpha_3(x + \tau_+(x, v)v, v, x', v')\phi_\epsilon(x' - x, v, v')d\mu(x') = \alpha_3(x + \tau_+(x, v)v, v, x - \tau_-(x, v)v', v')(v \cdot m). \tag{16}$$

Proof. For ϵ small enough, both integrations are on the part of the boundary for which v' is an incoming direction. Indeed, for $x^- = x - \tau_-(x, v')v'$, we have

$$\text{supp}\phi_\epsilon(\cdot - x^-, v, v') \subset \{x' \in \partial\Omega : |(x' - x^-) \cdot m| < \epsilon|v|\},$$

and $(x^-, v') \in \Gamma_-$. Using the fact that $\tau_-(x + \tau_+(x, v)v, v) = \tau(x, v)$ and the formula for α_2, the first integral becomes

$$\int_{\partial\Omega} \int_0^{\tau(x,v)} E(s, x + \tau_+(x, v)v, v, v')k(x + \tau_+(s, v)v - sv, v', v)\times$$
$$\delta_{[x+(\tau_+(x,v)-s)v-\tau_-(x+(\tau_+(x,v)-s)v,v')v']}(x')\frac{1}{\epsilon}\phi\left(\frac{(x'-x)\cdot m}{\epsilon v\cdot m}\right) d\mu(x') =$$
$$\int_0^{\tau(x,v)} E(s, x + \tau_+(x, v)v, v, v')k(x + (\tau_+(x, v) - s)v, v', v)\frac{1}{\epsilon}\phi\left(\frac{-s + \tau_+(x, v)}{\epsilon}\right) ds.$$

For the last equality we used the orthogonality of $m(v')$ on v' and

$$\langle x + (\tau_+(x, v) - s)v - \tau_-(x + (\tau_+(x, v) - s)v, v')v' - x\, , m(v')\rangle = (\tau_+(x, v) - s)(v \cdot m).$$

Since the map $s \to E(s, x+\tau_+(x, v)v, v, v')k(x+(\tau_+(x, v)-s)v, v', v)$ is integrable in a neighborhood of zero, the last term converges to $E(\tau_+(x, v), x + \tau_+(x, v)v, v, v')k(x, v', v)$ as $\epsilon \to 0$.

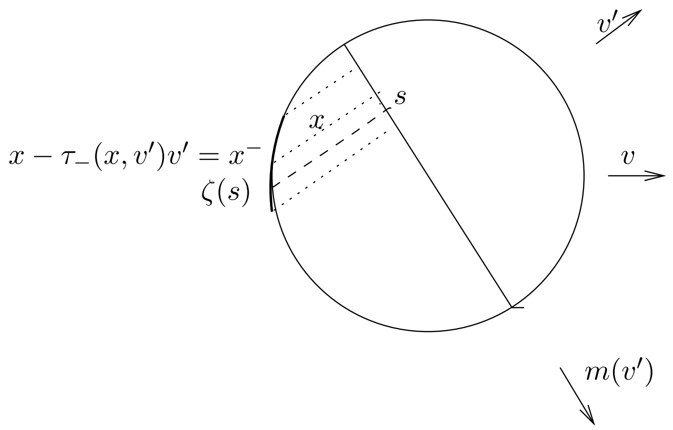

Figure 1: $\zeta(s) = (x \cdot m + s)m - \sqrt{R^2 - (x \cdot m + s)^2}v'/|v'|$

In order to compute the second limit, we parameterize the boundary such that the tangent vector at $x^+ = x + \tau_+(x, v)$ has unit length. In the case of a disc of radius R, such a parameterization is $\zeta(s) = (x \cdot m + s)m - \sqrt{R^2 - (x \cdot m + s)^2}v'/|v'|$ with $s = (x' - x) \cdot m = \langle x' - \tau_-(x, v')v', m\rangle$. We have

$$\int_{\partial\Omega} \alpha_3(x + \tau_+(x, v)v, v, x', v')\frac{1}{\epsilon v \cdot m}\phi\left(\frac{(x'-x)\cdot m}{\epsilon v\cdot m}\right) v \cdot m d\mu(x') = \quad (17)$$
$$\int_{-\epsilon_0 v\cdot m}^{\epsilon_0 v\cdot m} \alpha_3(x + \tau_+(x, v)v, v, \zeta(s), v')\frac{1}{\epsilon v \cdot m}\phi\left(\frac{s}{\epsilon v\cdot m}\right) (v \cdot m)||\zeta'(s)||ds. \quad (18)$$

From lemma 4 we know that α_3 is integrable in all the arguments. Applying Fubini's theorem we get that the map $s \to \alpha_3(x + \tau_+(x, v)v, v, \zeta(s), v')(v \cdot m)||\zeta'(s)||$ is integrable in a neighborhood of zero. Since $||\zeta'(0)|| = 1$, the result follows from the approximation of the identity. □

Proof of Theorem 1. The fact $a = a_0$ is the result of Choulli and Stefanov, still valid in two dimensional domains. Thus we do not have to distinguish E in (10) for the two cases. Moreover, when T is the maximum time required to reach the boundary, the lower bound

$$e^{-T||a||_\infty} \leq |E(0, x, v, v')| \quad (19)$$

holds point-wise for $(x,v)\in\Gamma_+$ and $v'\in V$.

The preceding lemma ensures that for almost every $(x,v,v')\in\Omega\times(V^2-D)$,

$$E(\tau_+(x,v),x+\tau_+(x,v)v,v,v')(k-k_0)(v',v)=\\ -(\alpha_3-\alpha_3^0)(x+\tau_+(x,v)v,v,x-\tau_-(x,v')v',v')(v\cdot m).$$

If $x^+=x+\tau_+(x,v)$, then for every $(x^+,v)\in\Gamma_+$ we have

$$E(0,x^+,v,v')(k-k_0)(v',v)=-(\alpha_3-\alpha_3^0)(x^+,v,x^+-\tau_-(x^+,v')v',v')(v\cdot m).$$

For any $y\in\Omega$ multiply both sides by $\tau(y,v')$, take absolute values on both sides and use the lower bound (19) and the upper bound $|\tau(y,v')(v\cdot m)|\le Tv_{max}$ to get

$$e^{-T||a||_\infty}|\tau(y,v')(k-k_0)(v',v)|\le|(\alpha_3-\alpha_3^0)(x^+,v,x^+-\tau_-(x^+,v')v',v')|Tv_{max}.$$

Notice that the left hand side is independent of x^+. Integrate the inequality in $(x^+,v)\in\Gamma_+$ and use the fact that x^+ spans a boundary arc of length no less than L, to get

$$\begin{aligned}Le^{-T||a||_\infty}\tau(y,v')\int_V|k-k_0|(v',v)dv&\le Tv_{max}\int_{\Gamma_+}|(\alpha_3-\alpha_3^0)(x^+,v,x^+-\tau_-(x^+,v')v',v')|d\mu dv\\ &\le Tv_{max}\sup_{(x',v')\in\Gamma_-}\int_{\Gamma_+}|(\alpha_3-\alpha_3^0)(x^+,v,x',v')|d\mu dv.\end{aligned}$$

Now take the supremum over $(y,v')\in\Omega\times V$. We end up with

$$Le^{-T||a||_\infty}|\tau\sigma-\tau\sigma_0|_\infty\le\sup_{(x',v')\in\Gamma_-}||(\alpha_3-\alpha_3^0)(\cdot,\cdot,x',v')||_{L^1(\Gamma_+)}Tv_{max}. \tag{20}$$

The right hand side is the operator norm of $R[(I-K)^{-1}K^2-(I-K_0)^{-1}K_0^2]J$ and the estimate (9) gives an upper bound. After dividing by Tv_{max} we conclude with

$$|\tau\sigma-\tau\sigma_0|_\infty\left(\frac{Le^{-T||a||_\infty}}{Tv_{max}}-\frac{2|\tau\sigma_0|_\infty^2+3|\tau\sigma_0|_\infty+|\tau\sigma|_\infty+|\tau\sigma_0|_\infty|\tau\sigma|_\infty}{(1-|\tau\sigma_0|_\infty)(1-|\tau\sigma|_\infty)}\right)\le 0. \tag{21}$$

The smallness assumption (14) gives the reverse inequality. Therefore $\tau\sigma$ must equal $\tau\sigma_0$ almost everywhere. ☐

Notice that this method still works in the case of slow particles, when velocities can go to zero and T gets large. The factor $L/(Tv_{max})$ stays bounded away from zero but not $\exp(-T||a||_\infty)$. However, we can replace the bound (19) by

$$\exp\left(-\sup_{(x^+,v)\in\Gamma_+}\int_0^\infty a(x^+-sv,|v|)ds\right)<|E(0,x^+,v,v')|$$

and ask that the integral be finite. This is always the case in practice since a is compactly supported.

Unlike the sub-critical assumption, the smallness assumption is due to the technicalities of the method and not to the physical phenomenon. To compare it with the sub-critical assumption we consider the example of the one velocity model in the unit disc. We have $v_{max}=1$, $T=2$, $L=\pi$ and the right hand side of (14) becomes $\epsilon_0=\pi e^{-2||a||_\infty}/2$. As long as a is such that $\epsilon_0<3$, a simple calculation reduces the smallness assumption to

$$||\tau\sigma||_\infty<\frac{1}{3-\epsilon_0}\left(-2-\epsilon_0+\sqrt{(2+\epsilon_0)^2+\epsilon_0(3-\epsilon_0)}\right).$$

For the applications we had in mind we can consider $||a||_\infty = 1$ and then the method guarantees uniqueness only for those scattering kernels k for which $||\tau\sigma||_\infty < 0.2762$.

The question remains open whether sub-criticality suffices to get uniqueness. There are no counterexamples in this case. As we shall see in the next section, for isotropic scattering, just sub-criticality suffices.

4 Isotropic Scattering

This section deals with a special case of (1). All the velocities have unit speed and the scattering kernel depends only on position, $k = k(x)$. In this case the inverse problem becomes over-determined and two formulae are provided for reconstruction. For convenience we rewrite the transport equation (1) as

$$\theta \cdot \nabla_x u(x,\theta) + a(x)u(x,\theta) = b(x) \int_{\mathbf{S}^1} u(x,\theta')d\theta', \quad x \in \Omega \subset \mathbb{R}^2,\ \theta \in \mathbf{S}^1. \tag{22}$$

The admissibility condition reduce to $0 \leq a, b \in L^\infty(\Omega)$ and the sub-criticality condition reduce to $\mathrm{diam}(\Omega)||\sigma||_\infty < 1$. Throughout this section a is known.

The first reconstruction method is based on the Novikov's inversion formula for the attenuated x-ray transform. For a fixed $\theta \in \mathbf{S}^1$ and $x \in \partial\Omega$ recall that $\tau_\pm(x,\theta)$ denotes the travel time from x in the direction $\pm\theta$, until the boundary is met. The divergence beam of a is a map on $\Omega \times \mathbf{S}^1$ given by

$$Da(x,\theta) = \int_0^{\tau_+(x,\theta)} a(x+s\theta)ds. \tag{23}$$

Also let $f(x)$ denote the right hand side of (22) and treat it as a source term. Evaluate (22) at $x + t\theta$ and multiply both sides by $\exp\left(-Da(x+t\theta,\theta)\right)$ to get

$$\frac{d}{dt}\left(\exp(-Da(x+t\theta,\theta))u(x+t\theta,\theta)\right) = \exp(-Da(x+t\theta,\theta))f(x+t\theta). \tag{24}$$

Integrating in t from 0 to $\tau_+(x,\theta)$ and using the compact support of a and b, we get

$$\exp(-Da(x+\tau_+(x,\theta)\theta,\theta))u(x+\tau_+(x,\theta)\theta,\theta) - u(x,\theta) = \int_{-\infty}^{\infty} \exp(-Da(x+t\theta,\theta))f(x+t\theta)dt. \tag{25}$$

The left hand side in (25) is known from boundary measurements. Since it is a function on the tangent bundle to the unit circle, we denote

$$g(x\cdot\theta^\perp,\theta) = \exp(-Da(x+\tau_+(x,\theta)\theta,\theta))u(x+\tau_+(x,\theta)\theta,\theta) - u(x,\theta).$$

The right hand side of (25) is the attenuated x-ray transform of $f(x)$. Novikov's inversion formula [18] reconstructs f from g. The form below is due to Natterer [17].

Lemma 6. *Assume that $f \in L_c^\infty(\mathbb{R}^2)$ with support in $\bar{\Omega}$ then*

$$f(x) = -\frac{1}{4\pi}\mathbf{Re}\ div \int_{\mathbf{S}^1} \theta e^{-Da(x,\theta)} e^h H e^{\bar{h}} g(x\cdot\theta^\perp,\theta)d\theta, \tag{26}$$

where $h(s,\theta) = (1/2)(I + iH)Pa(s\theta^\perp,\theta^\perp)$ with $Pa(x,\theta)$ is the X-ray transform of a, and H is the Hilbert transform $Hg(s) = \frac{1}{\pi}\int_{\mathbb{R}} g(t)/(s-t)dt$.

Next we deduce the Peierls' integral equation [1]. For brevity, let us denote by u_0 the total flux through a point; $u_0(x) = \int_{\mathbf{S}^1} u(x,\theta)d\theta$. Consider again (22) evaluated at $x - t\theta$ and multiply by the integrand factor $\exp(-\int_0^t a(x-s\theta)ds)$ to get an alternate form of (24)

$$\frac{d}{dt}\left\{\exp\left(-\int_0^t a(x-s\theta)ds\right)u(x-t\theta,\theta)\right\} = \exp\left(-\int_0^t a(x-s\theta)ds\right)f(x-t\theta).$$

Integrate first in t from 0 to $\tau_-(x,\theta)$ and then integrate in θ over $\mathbf{S}^1$ to get

$$u_0(x) = \int_{\mathbf{S}^1} \exp\left(-\int_0^{\tau_-(x,\theta)} a(x-s\theta)ds\right) u(x-\tau_-(x,\theta)\theta,\theta) +$$

$$\int_{\mathbf{S}^1}\int_0^\infty f(x-t\theta)\exp\left(-\int_0^t a(x-s\theta)ds\right)dtd\theta.$$

After a change to polar coordinates $t = |x-y|$ and $\theta = (x-y)/|x-y|$ we have that the second term is equal to

$$\int_{\mathbb{R}^2}\frac{f(y)}{|x-y|}\exp\left(-|x-y|\int_0^1 a(sx+(1-s)y)ds\right)dy.$$

We get the formula

$$u_0(x) = \int_{\mathbf{S}^1} \exp\left(-\int_0^{\tau_-(x,\theta)} a(x-s\theta)ds\right) u(x-\tau_-(x,\theta)\theta,\theta)d\theta + \tag{27}$$

$$\int_{\mathbb{R}^2}\frac{f(y)}{|x-y|}\exp\left(-|x-y|\int_0^1 a(sx+(1-s)y)ds\right)dy,$$

known as the Peierls' integral equation.

Notice that $u(x-\tau_-(x,\theta)\theta,\theta)$ is the incoming flux we apply to the boundary, hence it is known. With f given by (26), the second integral also involves known quantities. We summarize with the following.

Proposition 2. *Assume the admissible and sub-critical conditions. The albedo operator determines $a \in L^\infty(\Omega)$. One extra measurement, with a continuous incoming flux, determines $f \in L^\infty(\Omega)$ via (26) and then b by the formula*

$$b(x) = \frac{f(x)}{u_0(x)}, \tag{28}$$

with u_0 given in (27).

Next we take an alternative approach to finding a formula for an isotropic k. This time the main ingredient is the Cauchy integral formula from the theory of A-analytic functions as developed by Bukhgeim [11]. We use the complex z to identify a point in Ω. The absorption a, determined by the albedo operator, is assumed in $C^2(\bar{\Omega})$.

The theory of A-analytic functions is an extension of analyticity to functions on the complex plane with values in Banach spaces. We denote by l^2 the space of square sum-able sequences and by $l^{2,1}$ the Sobolev space of sequences $u = (u_0, u_1, u_2, ...)$ such that $\sum_{n=0}^{\infty}(1+n^2)|u_n|^2 < \infty$. Let $A : l^2 \to l^2$ be a bounded operator.

Definition 1. *A sequence valued function $\mathbf{v}(z) \in C^1(\Omega; l^2)$ is called A-analytic in Ω if $\mathbf{v}$ satisfies*

$$\bar{\partial}\mathbf{v} + A\partial\mathbf{v} = 0. \tag{29}$$

The following analog of the Cauchy integral formula holds.

Theorem 2. *Assume* $\mathbf{u}(z) \in C(\bar{\Omega}; l^{2,1}) \cap C^1(\Omega; l^2)$ *is A-analytic. Then, for all* $z \in \Omega$,

$$\mathbf{u}(z) = \frac{1}{2\pi i} \int_{\partial\Omega} K(\zeta - z)(d\zeta + A d\bar{\zeta})\mathbf{u}(\zeta), \tag{30}$$

where $K(z) = (zI + \bar{z}A)^{-1}$.

Outside the unit disk the operator valued function $K(z)$ is bounded from l^2 to l^2. Moreover K extends continuously to the unit circle $\{z : |z| = 1\}$ as an unbounded operator from $l^{2,1}$ to l^2. For us it plays the role of $1/z$, see [11] for further details. In our case A is the double shift to the left, denoted by $\mathcal{L}^2$. We briefly describe the reduction of the transport equation (1) to a Beltrami type equation for operator valued coefficients.

Let $\theta = (\cos\varphi, \sin\varphi)$, $\bar{\partial} = (\partial_x + i\partial_y)/2$ and $\partial = (\partial_x - i\partial_y)/2$. The advection operator $\theta \cdot \nabla$ becomes $e^{-i\varphi}\bar{\partial} + e^{i\varphi}\partial$. Expand $u(z,\varphi) = \sum_{-\infty}^{\infty} u_n(z)e^{-in\varphi}$ and let $\mathbf{u}$ denote the sequence valued function $\mathbf{u}(z) = (u_0(z), u_1(z), u_2(z), ...)$. Notice the consistency of u_0 defined previously. Since u is real valued we have $u_n = \bar{u}_{-n}$. Plug into (1) the new differential operator and identify coefficients to get the infinite dimensional system

$$\bar{\partial}(u_n)(z) + \partial u_{n+2}(z) + a(z)u_{n+1}(0) = 0, \ n = 0, 1, 2, ... \tag{31}$$

which together with the $n = -1$ case yields

$$2\mathbf{Re}(\partial u_1(z)) + a(z)u_0(z) = b(z)u_0(z). \tag{32}$$

Let $\mathcal{L}$ denote the left translation operator $\mathcal{L}(u_0, u_1, u_2, ...) = (u_1, u_2, ...)$ and $\mathcal{R}$ denote the adjoint, right translation $\mathcal{R}(u_0, u_1, u_2, ...) = (0, u_0, u_1, u_2, ...)$. Powers of these operators are understood as compositions. The system (31) becomes the Beltrami type equation for $\mathbf{u}$,

$$\bar{\partial}\mathbf{u} + \mathcal{L}^2\partial\mathbf{u} + a(z)\mathcal{L}\mathbf{u} = 0. \tag{33}$$

The equation above has no b involved. Moreover, since $\mathcal{L}^2$ and $a(z)\mathcal{L}$ commute we are able to reduce (33) to an A-analytic equation for $A = \mathcal{L}^2$. This type of construction was done in [7] in connection with the attenuated x-ray transform.

Let us consider the (formal) Fourier expansion in the angle variable of the divergent beam (23), $Da(z,\theta) = \sum_{-\infty}^{\infty} b_n(z)e^{-in\varphi}$. The following lemma is a direct consequence of the regularity of a and the smoothness of τ_+.

Lemma 7. *If* $a \in C^2(\bar{\Omega})$ *then* $Da \in C^(\bar{\Omega} \times \mathbf{S}^1)$. *Consequently the series* $\sum_{n\in\mathbf{Z}}(1+n^2)^2|b_n(z)|^2$, $\sum_{n\in\mathbf{Z}}(1+n^2)\bar{\partial}|b_n(z)|^2$ *and* $\sum_{n\in\mathbf{Z}}(1+n^2)\partial|b_n(z)|^2$ *converge uniformly in* $\bar{\Omega}$.

The divergence beam Da solves $\theta \cdot \nabla_x Da(z,\theta) = -a(z)$ or equivalently

$$e^{-i\varphi}\bar{\partial}Da(z,\theta) + e^{i\varphi}\partial Da(z,\theta) = -a(z).$$

Using the power series expansion and the convergence properties of Lemma 7, we have that $\bar{\partial}b_{-2k-3}(z) + \partial b_{-2k-1}(z) = 0$, for $k = 0, 1, 2, ...$ and $\partial b_{-1}(z) = -a(z)$. Using the sequence of odd indices of Da we define $G(z)$ an operator valued function by

$$G(z) = \sum_{k=0}^{\infty} b_{-2k-1}(z)\mathcal{L}^{2k+1}. \tag{34}$$

Using the fact that $\mathcal{L}$ has the operator norm 1 and applying Lemma 7 we get that $G(z) \in C^1(\Omega; l^{2,1}) \cap C(\bar{\Omega}; l^{2,1})$. Moreover, a calculation shows that $\bar{\partial}G + \mathcal{L}^2 \partial G + a(z)\mathcal{L} = 0$. The operator e^{-G} is well defined by the power series expansion of G as an operator in $C^1(\Omega; l^{2,1}) \cap C(\bar{\Omega}; l^{2,1})$. If $\mathbf{v} = e^{-G}\mathbf{u}$ and $\mathbf{u}$ is a solution of the Beltrami equation (33) then $\mathbf{v}$ is $\mathcal{L}^2$ -analytic. Using the integral formula of Theorem 30 we obtain

$$\mathbf{u}(z) = \frac{1}{2\pi i} \int_{\partial\Omega} K(\zeta - z) e^{G(\zeta) - G(z)} (d\zeta - \mathcal{L}^2 d\bar{\zeta}) \mathbf{u}(\zeta), \tag{35}$$

The formula above requires knowledge of $\mathbf{u}$ on the boundary. This is given by the Fourier expansion (in the scattering angle) of boundary measurements. The incoming flux should be at least continuous in the space variable and C^2 regular in the angle variable to produce A-analytic extension by (35). We summarize with the following.

Proposition 3. *Assume $a \in C^2(\bar{\Omega})$ is known. The scattering coefficient b can be reconstructed from one measurement by the formula*

$$b(z) = \frac{2\mathbf{Re}(\partial u_1(z)) + a(z)u_0(z)}{u_0(z)}, \tag{36}$$

where u_0 and u_1 are the first two components of $\mathbf{u}$ given by (35).

The second integral in Peierls' integral formula (27) is in fact $C^\alpha(\bar{\Omega})$ regular, for some $\alpha \in (0, 1)$. This is a classical result in the theory of weakly-singular integral operators applied to $L^\infty(\Omega)$ functions [15]. The function defined by the first integral in (27) depends on the boundary flux from the experiment as well as on the regularity of the boundary. For convex smooth boundary we get a smooth $\tau_\pm(x, \theta)$. If we apply a continuous incoming flux then $u_0(x)$ is also continuous. For $b(x) \geq 0$ and a positive incoming flux we get a positive $u_0(x)$. Due to the continuity remark above, we conclude that (28) holds point-wise.

Formulae (28) and (36) are different in appearance but they are closely related. The numerator $2\mathbf{Re}(\partial u_1(z)) + a(z)u_0$ is nothing but Bukhgeim's inversion formula for the attenuated x-ray transform of the function $f = bu_0$ [7]. In other words the two formulae differ only by the way the inversion of the right hand side in (22) is computed.

Formula (26) has been numerically implemented using a filter back projection algorithm [8], [17]. Consequently, the isotropic scattering given by (28) can be numerically reconstructed.

The A-analytic method cannot be applied to anisotropic scattering kernel, even in the case of homogeneous scattering, i.e. k is independent of x. The difficulty arises in constructing an integrating factor for the Beltrami type equation (33). The operator $a(z)\mathcal{L}$ changes into a more general Fourier multiplier $B\mathcal{L}$ where $B\mathbf{u} = (b_0u_0, b_1u_1, ...)$. The main impediment is the fact that $\mathcal{L}^2$ and $B\mathcal{L}$ do not commute.

References

[1] D. S. Anikonov 1975, The unique determination of the coefficients and the right side in a transport equation, *Differential Equations*, **11** 6-12.

[2] D. S. Anikonov 1984, Uniqueness of simultaneous determination of two coefficients of the transport equation, *Soviet Math. Dokl.*, **30** 149-151.

[3] D. S. Anikonov, I.V. Prokhorov, A. E. Kovtanyuk 1993, Investigation of scattering and absorbing media by the methods of x-ray tomography, *J. Inv. Ill-Posed Problems*, **1** 259-281.

[4] Yu. E. Anikonov, B. A. Bubnov 1988, Inverse problems in transport theory, *Soviet Math. Dokl.*, **37** 497-499.

[5] V. S. Antyufeev, A. N. Bondarenko 1996, X-ray tomography in scattering media, *SIAM J. Appl. Math.*, **56** 573-587.

[6] R. Dautray, J. L. Lions, 1993 *Mathematical Analysis and Numerical Methods for Science and Technology*, vol.6, Springer-Verlag, Berlin.

[7] E. V. Arzubov, A. L. Bukhgeim and S. G. Kazantsev 1998, Two dimensional tomography problem and the theory of A-analytic functions, *Siberian Advances in Mathematics*, **8** 1-20.

[8] L. A. Kunyanski 2001, New SPEC reconstruction algorithm based in the Novikov's explicit inversion formula, *Caltech Preprint.*

[9] G. Bal 2000, Inverse problems for homogeneous transport equations. Part II. Multidimensional case, *Inverse Problems*, **16** 1013–1028.

[10] M. Choulli, P. Stefanov 1999, An inverse boundary value problem for the stationary transport equation, *Osaka J. Math.*, **36** 87-104.

[11] A. L. Bukhgeim 1995, *Inversion Formulas in Inverse Problems*, supplement in M. M. Lavrentev and L. Ya. Savalev "Linear Operators and Ill-Posed Problems", Plenum, New York.

[12] M. Cessenat 1985, Théorems de trace pour des espaces de fonctions de la neutronique, *C. R. Acad. Sci. Paris*, Sér **I** **300** 89-92.

[13] O. Dorn 1998, A transport-backtransport method for optical tomography, *Inverse Problems*, **14** 1107–1130.

[14] N. J. McCormick 1986, Methods for solving inverse problems for radiation transport–an update, *Trans. Theor. Stat. Phys.*, **15** 759–772.

[15] S. G. Mikhlin 1965, *Multidimensional Singular Integrals And Integral Equations*, Oxford, New York, Pergamon Press.

[16] E. E. Lewis, W. F. Miller Jr. 1993 *Computational Methods of Neutron Transport*, American Nuclear Society, La Grange Park, Illinois.

[17] F. Natterer 2001, Inversion of the Attenuated Radon Transform, *Inverse Problems*, **17** 113-119.

[18] R. G. Novikov 2000, An inversion formula for the attenuated X-ray transform, *Preprint CNRS*, UMR 6629, Départment de Mathématique, Université de Nantes.

[19] V. G. Romanov 1996, A stability estimate in the problem of determining the dispersion index and the relaxation for the transport equation, *Siberian Mathematical Journal*, **37** 308-324.