

STABILITY IN CONDUCTIVITY IMAGING FROM PARTIAL MEASUREMENTS OF ONE INTERIOR CURRENT

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ABSTRACT. We prove a stability result in the hybrid inverse problem of recovering the electrical conductivity from partial knowledge of one current density field generated inside a body by an imposed boundary voltage. The region of stable reconstruction is well defined by a combination of the exact and perturbed data. This work explains the high resolution and accuracy reconstructions in some existing numerical experiments that use partial interior data.

1. INTRODUCTION

We consider the stability question in the inverse problem of recovering the electrical conductivity of a body from partial interior measurements of one current density field generated by an imposed boundary voltage. This inverse problem belongs to the larger class of multi-view imaging methods for conductivity imaging as developed, e.g., in [5, 7, 8, 16], see also [3, 25, 28, 29, 31] for some recent reviews.

To date, the current density field can be obtained from magnetic resonance measurements as discovered in 1996 by the work of Scott, et al. [27]. The first attempt to use the interior current to determine an isotropic conductivity appeared in Zhang [32]. Several reconstruction methods that use some knowledge of two or more currents have been subsequently proposed in the literature, e.g., in [10, 12, 17]. The fact that multiple interior measurements stabilizes the reconstruction has been shown in [4, 14, 15]. Moreover, multiple measurements are necessary for the unique determination in the anisotropic case [6, 18, 19]; the latter work also shows stability in two dimensions from full knowledge of four current density fields.

Conductivity imaging methods that use some interior knowledge of just one current density field have also been proposed, e.g., in [13, 16,

2000 *Mathematics Subject Classification.* 35R30, 35J65, 65N21.

Key words and phrases. hybrid inverse problems, current density imaging, magnetic resonance, electrical impedance tomography, 1-Laplacian.

A. Tamasan was supported by the NSF Grant DMS 1312883.

22–24]. For unperturbed boundary data, knowledge of the magnitude of one current density field in the entire domain stably recovers the conductivity as shown by Montalto and Stefanov [20]. An equivalent curl-based formulation of the problem was used by Kim and Lee [13] to prove stable recovery using the whole current density. Both of these stability results assume perturbations in the interior data within the range of a current density field. If one assumes a general perturbations in the interior data, then the recent uniqueness result by Moradifard, Nachman and Tamasan [21] could be combined with the structural stability obtained by Nashed and Tamasan [26] to show that the voltage potential depends continuously on the interior data. However, it is not clear whether the method extends to show continuous dependence on the interior data for the conductivity itself.

In the case of partial interior data, unique determination of planar conductivity has been showed to be possible in certain subdomains (see the *injectivity region* defined below) by Nachman, Tamasan and Timonov [25], but the stability question has been left open. While the voltage potential can be stably recovered in certain subregions from partial interior data as notice by Veras and Tamasan [30], the regularity in the estimate does not translate into a stability for the conductivity.

In this paper we assume that knowledge (of certain component of) the current density field is only partially available inside. We identify a specific subregions where conductivity can be stably recovered from partial interior and boundary data. The boundary voltage potential imposed to generate the current density field is assumed unperturbed, but need not be known everywhere. In addition, we assume the conductivity is known on some subset of the boundary, see Theorem 2.1. In general, the stability region is a subset of the injectivity region as illustrated in Figure 2. However, if the accessible part of the boundary (where the voltage potential is known) is simply connected, then the injectivity region and the stability region coincide under some geometric (strict convexity) assumptions on the shape of the boundary.

Different from the approach in [20] we use a non-linear version of the decomposition of the data operator and identify a natural component of the current density field sufficient to yield stability. To authors' knowledge, this is the first result where either injectivity or stability is established from knowledge of just one component of the current density. The required component depends both on the measured and on the exact data.

Similar to all the works in the hybrids methods, we also work with voltages free of singular points inside the domain. In dimension two, the condition of no critical points can be satisfied under the assumption

that the boundary illumination is almost two-to-one [1, 22]. In dimensions $n \geq 3$, it is still unclear if similar conditions exist; the recent results in [2, 11] describing the set of singular points assume nonzero frequencies.

2. STATEMENT OF THE RESULTS

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain $C^{2,\alpha}$ -diffeomorphic with the unit ball, $0 < \alpha < 1$. This assumption can be relaxed as explained in the proofs. Let σ be a positive function in $C^{2,\alpha}(\overline{\Omega})$ and let $f \in C^{2,\alpha}(\partial\Omega)$. From Schauder's regularity theory we know that the unique solution u of the Dirichlet problem

$$(1) \quad \nabla \cdot \sigma \nabla u = 0 \quad \text{in } \Omega, \quad u|_{\partial\Omega} = f.$$

is in $C^{2,\alpha}(\overline{\Omega})$, see e.g. [9, Theorem 6.18]. We refer to such a u as being σ -harmonic. The corresponding current density vector field $\mathbf{J} : C^{1,\alpha}(\overline{\Omega}) \rightarrow C^{1,\alpha}(\overline{\Omega})$ is defined by

$$(2) \quad \mathbf{J}(\sigma) = -\sigma \nabla u.$$

In order to state our partial data results we introduce the following notation. For some $u \in C^1(\overline{\Omega})$ arbitrary (not necessarily σ -harmonic) with $|\nabla u| > 0$ in $\overline{\Omega}$, and p an arbitrary point in Ω , let γ_p denote the integral curve starting at p in the direction of ∇u . Since $|\nabla u| > 0$, there exist a first time, denoted by t_p^+ (resp. t_p^-), such that the integral curve starting at p and moving in the direction $\pm \nabla u$ hits the boundary. We denote by

$$l_p^\pm = \{\gamma_p(t) : t \in [0, t_p^\pm]\}$$

the segment of the integral curve from $t = 0$ to $t = t_p^\pm$. Also, let

$$\Sigma_p = \{x \in \overline{\Omega} : u(x) = u(p)\}$$

denote the level curve of u passing through p , see Figure 1.

Let Γ be an open subset of the boundary to denote the accessible part. The following definitions specify the interior subdomains, where the unique and stable determination of the conductivity can be guaranteed.

Definition 2.1. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain $C^{2,\alpha}$ -diffeomorphic to the unit ball and let $u \in C^{2,\alpha}(\overline{\Omega})$, where $0 < \alpha < 1$.*

Injectivity region: *We say that a point $p \in \overline{\Omega}$ is visible from Γ along the equipotential set if*

$$\Sigma_p \cap \partial\Omega \subset \Gamma.$$

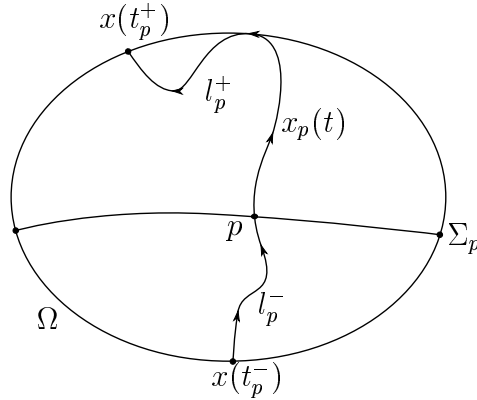


Figure 1. Illustration of the segment l_p and the surface Σ_p .

The set $\mathcal{I}(\Gamma, u)$ of points in $\bar{\Omega}$ that are visible from Γ along equipotential sets is called the injectivity region,

$$(3) \quad \mathcal{I}(\Gamma, u) := \{p \in \bar{\Omega} : \Sigma_p \cap \partial\Omega \subset \Gamma\}.$$

Stability region: We say that the trajectory through p along ∇u is visible from Γ if either $l_p^+ \subset \mathcal{I}(\Gamma, u)$ or $l_p^- \subset \mathcal{I}(\Gamma, u)$. The set $\mathcal{S}(\Gamma, u)$ of points in $\bar{\Omega}$ for which the corresponding trajectories along ∇u are visible from Γ is called the stability region, i.e.,

$$(4) \quad \mathcal{S}(\Gamma, u) := \{p \in \mathcal{I}(\Gamma, u) : l_p^+ \subset \mathcal{I}(\Gamma, u) \text{ or } l_p^- \subset \mathcal{I}(\Gamma, u)\}.$$

Clearly $\mathcal{S}(\Gamma, u) \subseteq \mathcal{I}(\Gamma, u)$ and, if Γ is connected, the equality can hold as illustrated in Figure 3. In Figure 2 we illustrate the injectivity and stability regions, when Γ is not connected.

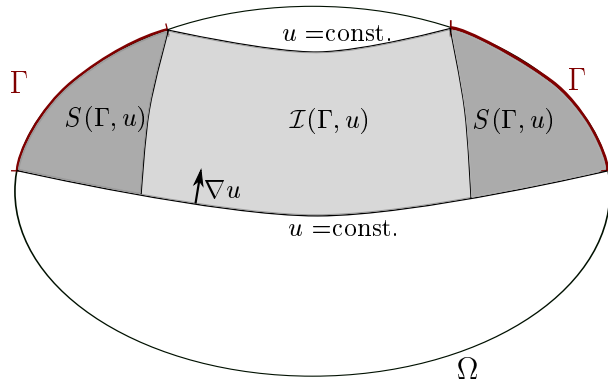


Figure 2. The injectivity region $\mathcal{I}(\Gamma, u)$ is the light grey region that contains the stability region $\mathcal{S}(\Gamma, u)$ in dark grey.

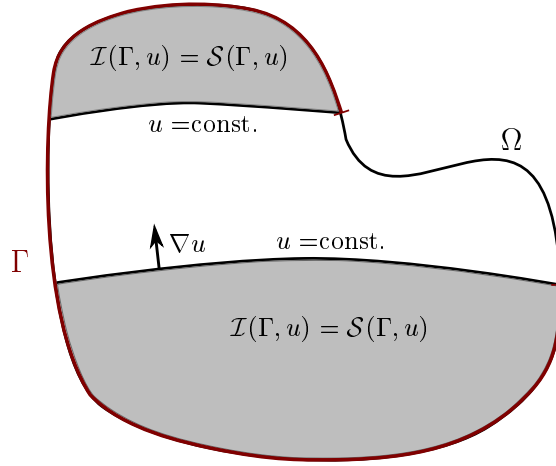


Figure 3. When Γ is connected the visible region and the trajectory region can be the same.

Let \tilde{u} be the voltage potential corresponding to some perturbed conductivity $\tilde{\sigma} \in C^{1,\alpha}(\overline{\Omega})$,

$$(5) \quad \nabla \cdot \tilde{\sigma} \nabla \tilde{u} = 0 \quad \text{in } \Omega, \quad \tilde{u}|_{\partial\Omega} = f,$$

and let $\mathbf{J}(\tilde{\sigma}) := -\tilde{\sigma} \nabla \tilde{u}$ be the corresponding current density field. Let

$$(6) \quad \delta\sigma := \sigma - \tilde{\sigma} \quad \text{and} \quad \delta\mathbf{J} := \mathbf{J} - \tilde{\mathbf{J}}$$

denote the corresponding perturbations.

We prove that $\delta\sigma$ is controlled by the component of $\delta\mathbf{J}$ in the direction $\nabla(u + \tilde{u})$ throughout $\mathcal{S}(\Gamma, u + \tilde{u})$. To the author's knowledge the result below is the first stability result from partial interior data. It is also the first time the recovery is done from just one component of the current density field.

For a vector field \mathbf{w}_0 in Ω we denote the orthogonal projection onto \mathbf{w}_0 by $\Pi_{\mathbf{w}_0}$, and onto the orthogonal complement (both in the Euclidean metric) by $\Pi_{\mathbf{w}_0}^\perp := \text{Id} - \Pi_{\mathbf{w}_0}$ the orthogonal projection; i.e., for an arbitrary vector field \mathbf{w} ,

$$(7) \quad \Pi_{\mathbf{w}_0} \mathbf{w} = \frac{\mathbf{w}_0 \cdot \mathbf{w}}{|\mathbf{w}_0|^2} \mathbf{w}_0 \quad \text{and} \quad \Pi_{\mathbf{w}_0}^\perp \mathbf{w} = \mathbf{w} - \Pi_{\mathbf{w}_0} \mathbf{w}.$$

Theorem 2.1 (Stability with Partial Data). *Let $\Omega \subset \mathbb{R}^n$ be $C^{2,\alpha}$ -diffeomorphic to the unit ball, and $\sigma, \tilde{\sigma} \in C^{1,\alpha}(\overline{\Omega})$ be positive on $\overline{\Omega}$, for some $0 < \alpha < 1$. Let u (respectively \tilde{u}) in $C^{2,\alpha}(\overline{\Omega})$ be σ (respectively $\tilde{\sigma}$)-harmonic. Let $\Gamma \subseteq \partial\Omega$ be the union of finitely many open connected components, and $\Gamma' \subset\subset \Gamma$ be a open subset compactly contained in Γ .*

Assume that

$$(8) \quad \sigma|_{\Gamma} = \tilde{\sigma}|_{\Gamma}, \quad u|_{\Gamma} = \tilde{u}|_{\Gamma},$$

and

$$(9) \quad |\nabla(u + \tilde{u})| > 0, \quad \text{in } \overline{\mathcal{I}(\Gamma', u + \tilde{u})}.$$

Then there exist $C > 0$ dependent on a lower bound of $|\nabla(u + \tilde{u})|$ in $\overline{\mathcal{I}(\Gamma', u + \tilde{u})}$, the domain Ω , and the $C^1(\overline{\Omega})$ -norm of σ and $\tilde{\sigma}$, such that

$$(10) \quad \|\sigma - \tilde{\sigma}\|_{L^2(\mathcal{S}(\Gamma', u + \tilde{u}))} \leq C \left\| \nabla \cdot (\Pi_{\nabla(u + \tilde{u})}(\mathbf{J}(\sigma) - \mathbf{J}(\tilde{\sigma}))) \right\|_{L^2(\mathcal{I}(\Gamma', u + \tilde{u}))}^{\frac{\alpha}{2+\alpha}}.$$

In particular,

$$(11) \quad \|\sigma - \tilde{\sigma}\|_{L^2(\mathcal{S}(\Gamma', u + \tilde{u}))} \leq C \left\| \Pi_{\nabla(u + \tilde{u})}(\mathbf{J}(\sigma) - \mathbf{J}(\tilde{\sigma})) \right\|_{H^1(\mathcal{I}(\Gamma', u + \tilde{u}))}^{\frac{\alpha}{2+\alpha}}.$$

The proof of the Theorem 2.1 is presented in Section 4.

A few remarks are in order. Note first, that the estimates (10) and (11) concern norms over (possibly) different domains. However, in many interesting situation $\mathcal{S}(\Gamma', u + \tilde{u}) = \mathcal{I}(\Gamma', u + \tilde{u})$ as illustrated in Figure 3. In particular, in the full data case with $\Gamma = \partial\Omega$, we have $\mathcal{S}(\Gamma', u + \tilde{u}) = \mathcal{I}(\Gamma', u + \tilde{u}) = \Omega$, and the estimate (11) yields

$$(12) \quad \|\sigma - \tilde{\sigma}\|_{L^2(\Omega)} \leq C \|\mathbf{J}(\sigma) - \mathbf{J}(\tilde{\sigma})\|_{H^1(\Omega)}^{\frac{\alpha}{2+\alpha}}.$$

The stability estimate requires knowledge of a component of \mathbf{J} in a direction $\nabla(u + \tilde{u})$ that is well defined by the exact and perturbed data due to the uniqueness results in [23]. However, if $\tilde{\sigma}$ is a priori close to σ then the level curves of \tilde{u} will be close to the level curves u , and it will suffice to project $\delta\mathbf{J}$ onto the ∇u with a penalty term that will depend on the a priori closeness assumption, as illustrated in Figure ???. More precisely, we have the following local stability result.

Theorem 2.2. *Let $\sigma \in C^{1,\alpha}(\overline{\Omega})$, $0 < \alpha < 1$, be positive in $\overline{\Omega}$ and u be σ -harmonic with $|\nabla u| > 0$ in $\overline{\Omega}$. There exists an $\epsilon > 0$ depending on Ω and some $C > 0$ depending on ϵ , such that the following holds: If $\tilde{\sigma} \in C^{1,\alpha}(\overline{\Omega})$ with*

$$(13) \quad \|\sigma - \tilde{\sigma}\|_{C^{1,\alpha}(\overline{\Omega})} < \epsilon \quad \text{and} \quad \sigma|_{\partial\Omega} = \tilde{\sigma}|_{\partial\Omega},$$

and \tilde{u} is the $\tilde{\sigma}$ -harmonic map with

$$(14) \quad \tilde{u}|_{\partial\Omega} = u|_{\partial\Omega},$$

then

$$(15) \quad \|\sigma - \tilde{\sigma}\|_{L^2(\Omega)} \leq C \left\| \nabla \cdot (\Pi_{\nabla u}(\mathbf{J}(\sigma) - \mathbf{J}(\tilde{\sigma}))) \right\|_{L^2(\Omega)}^{\frac{\alpha}{2+\alpha}}.$$

In particular,

$$\|\sigma - \tilde{\sigma}\|_{L^2(\Omega)} \leq C \|\Pi_{\nabla u}(\mathbf{J}(\sigma) - \mathbf{J}(\tilde{\sigma}))\|_{H^1(\Omega)}^{\frac{\alpha}{2+\alpha}}.$$

As a consequence of Theorem 2.2 let us consider the problem of determining a perturbation from a constant conductivity: If the boundary voltage is uniform in the z -direction, the equipotential surfaces are planes perpendicular to the z -direction and we only need two components of the magnetic field $\mathbf{B} = (B^x, B^y, B^z)$ to stably recover the conductivity, as exemplified below.

Example 1. For simplicity assume that $\sigma \equiv 1$ and $f(x, y, z) = z$. Then $u(x, y, z) = z$ and, by Ampère's law,

$$\delta J^z = \nabla u \cdot \delta \mathbf{J} = \frac{1}{\mu_0} \nabla u \cdot (\nabla \times \delta \mathbf{B}) = \frac{1}{\mu_0} \left(\frac{\partial}{\partial x} \delta B^y - \frac{\partial}{\partial y} \delta B^x \right).$$

By Theorem 2.2, we can stably recover a sufficiently small perturbation of the conductivity from δB^x and δB^y .

3. PRELIMINARIES

The stability result relies on the following decomposition of the perturbation $\delta \mathbf{J}$ in the data.

Proposition 3.1. *Let u and \tilde{u} be σ -harmonic and $\tilde{\sigma}$ -harmonic, respectively. If $\nabla(u + \tilde{u}) \neq 0$ in \bar{V} , for some open subset V of Ω , then*

$$(16) \quad 2\nabla \cdot \Pi_{\nabla(u+\tilde{u})}(\mathbf{J}(\sigma) - \mathbf{J}(\tilde{\sigma})) = L(u - \tilde{u}) \quad \text{in } V.$$

where L is the differential operator defined by

$$(17) \quad Lv := -\nabla \cdot (\sigma + \tilde{\sigma}) \nabla v + \nabla \cdot \left((\sigma + \tilde{\sigma}) \frac{\nabla(u + \tilde{u}) \cdot \nabla v}{|\nabla(u + \tilde{u})|^2} \nabla(u + \tilde{u}) \right).$$

Proof. Consider the difference of two internal measurements

$$(18) \quad 2(\mathbf{J}(\sigma) - \mathbf{J}(\tilde{\sigma})) = (\sigma - \tilde{\sigma}) \nabla(u + \tilde{u}) + (\sigma + \tilde{\sigma}) \nabla(u - \tilde{u}).$$

If we dot product with $\nabla(u + \tilde{u})$ we obtain

$$(19) \quad 2(\mathbf{J}(\sigma) - \mathbf{J}(\tilde{\sigma})) \cdot \nabla(u + \tilde{u}) = (\sigma - \tilde{\sigma}) |\nabla(u + \tilde{u})|^2 + (\sigma + \tilde{\sigma}) \nabla(u - \tilde{u}) \cdot \nabla(u + \tilde{u}).$$

Solving for $\sigma - \tilde{\sigma}$ we have

$$(20) \quad \sigma - \tilde{\sigma} = \frac{2(\mathbf{J}(\sigma) - \mathbf{J}(\tilde{\sigma})) \cdot \nabla(u + \tilde{u})}{|\nabla(u + \tilde{u})|^2} - (\sigma + \tilde{\sigma}) \frac{\nabla(u + \tilde{u}) \cdot \nabla(u - \tilde{u})}{|\nabla(u + \tilde{u})|^2}$$

On the other hand it follows easily from (1)

$$(21) \quad \nabla \cdot (\sigma + \tilde{\sigma}) \nabla(u - \tilde{u}) = -\nabla \cdot (\sigma - \tilde{\sigma}) \nabla(u + \tilde{u}) \quad \text{in } \Omega.$$

Using (20) we substitute $\sigma - \tilde{\sigma}$ in (21) and we get

$$\begin{aligned}
L(u - \tilde{u}) &= -\nabla \cdot (\sigma + \tilde{\sigma}) \nabla(u - \tilde{u}) \\
&\quad + \nabla \cdot (\sigma + \tilde{\sigma}) \frac{\nabla(u + \tilde{u}) \cdot \nabla(u - \tilde{u})}{|\nabla(u + \tilde{u})|^2} \nabla(u + \tilde{u}) \\
&= \nabla \cdot \left(\frac{2(\mathbf{J}(\sigma) - \mathbf{J}(\tilde{\sigma})) \cdot \nabla(u + \tilde{u})}{|\nabla(u + \tilde{u})|^2} \nabla(u + \tilde{u}) \right) \\
&= 2\nabla \cdot \Pi_{\nabla(u + \tilde{u})}(\mathbf{J}(\sigma) - \mathbf{J}(\tilde{\sigma}))
\end{aligned}$$

□

We remark that (16) is the non-linear version of the decomposition obtained in the linearized case in [20]. The representation of the operator L in local coordinates can then be obtained similarly as in [20]:

Proposition 3.2. *Let $u, \tilde{u} \in C^2(\bar{\Omega})$, with $\nabla(u + \tilde{u})(x_0) \neq 0$ for $x_0 \in \Omega$. Then locally near Σ_{x_0} , the operator L in (17) is the restriction of $\nabla(\sigma + \tilde{\sigma})\nabla$ onto the level surfaces $(u + \tilde{u}) = \text{const}$. Moreover, if $y^n = u + \tilde{u}$, and $y' = (y^1, \dots, y^{n-1})$ are any local coordinates of the level set Σ_{x_0} , then in a neighborhood of Σ_{x_0} the Euclidean line element ds is given by*

$$(22) \quad ds^2 = c^2(dy^n)^2 + g_{\alpha\beta}dy^\alpha dy^\beta, \quad g_{\alpha,\beta} := \sum_i^{n-1} \frac{\partial x^i}{\partial y^\alpha} \frac{\partial x^i}{\partial y^\beta},$$

where $c = |\nabla(u + \tilde{u})|^{-1}$. In this coordinates the operator L becomes

$$(23) \quad L = - \sum_{\alpha,\beta} \frac{1}{\sqrt{\det g}} \frac{\partial}{\partial y^\beta} (\sigma + \tilde{\sigma}) g^{\alpha\beta} \sqrt{\det g} \frac{\partial}{\partial y^\alpha},$$

where the Greek super/subscripts run from 1 to $n - 1$.

Proof. Let Σ_{x_0} be the level set of $u + \tilde{u}$ passing through x_0 . On Σ_{x_0} choose local coordinates $y' = (y^1, \dots, y^{n-1})$ and set $y^n = (u + \tilde{u})(x)$. Then $y = (y', y^n) = \varphi(x)$ define local coordinates in a neighborhood of Σ_{x_0} . In the new coordinates, the Euclidean metric becomes

$$(24) \quad ds^2 = c^2(dy^n)^2 + g_{\alpha\beta}dy^\alpha dy^\beta, \quad \text{where } g_{\alpha\beta} := \sum_{i=1}^{n-1} \frac{\partial x^i}{\partial y^\alpha} \frac{\partial x^i}{\partial y^\beta},$$

where $c = |\nabla(u + \tilde{u})|^{-1}$. This follows easily by noticing that $u + \tilde{u}$ trivially solves the eikonal equation $c^2|\nabla\phi|^2 = 1$ for the speed $c = |\nabla(u + \tilde{u})|^{-1}$. Then, near x_0 , $u + \tilde{u}$ is the signed distance from the x

to the level surface Σ_{x_0} and the Euclidean metric has block structure given by (24). Denote by g be metric after the change of variables, i.e.,

$$g = \begin{bmatrix} [g_{\alpha,\beta}] & 0 \\ 0 & c^2 \end{bmatrix}$$

Let $\phi \in C_0^\infty(\Omega)$ be a test function supported near some $\hat{x}_0 \in \Sigma_{x_0}$. We have

$$(25) \quad \begin{aligned} \langle Lv, \phi \rangle &= \langle (\sigma + \tilde{\sigma}) \nabla v, \nabla \phi \rangle - \langle (\sigma + \tilde{\sigma}) \Pi_{\nabla(u+\tilde{u})} \nabla v, \nabla \phi \rangle, \\ &= \langle (\sigma + \tilde{\sigma}) \Pi_{\nabla(u+\tilde{u})}^\perp \nabla v, \Pi_{\nabla(u+\tilde{u})}^\perp \nabla \phi \rangle, \end{aligned}$$

where $\langle \cdot, \cdot \rangle$ denotes the scalar product in $L^2(\Omega)$.

From (24), we get that for any function $v \in C^1(\bar{\Omega})$

$$\Pi_{\nabla(u+\tilde{u})} \nabla_x v = \left(0, \dots, 0, c^2 \frac{\partial v}{\partial y^n} \right).$$

and

$$\Pi_{\nabla(u+\tilde{u})}^\perp \nabla_x v = \left(\frac{\partial v}{\partial y^1}, \dots, \frac{\partial v}{\partial y^{n-1}}, 0 \right).$$

thus in (y', y^n) coordinates (25) becomes

$$(26) \quad \begin{aligned} \langle Lv, \phi \rangle &= \sum_{\alpha,\beta} \int_{\varphi^{-1}(\Omega)} (\sigma + \tilde{\sigma}) \left(g^{\alpha\beta} \frac{\partial v}{\partial y^\alpha} \frac{\partial \phi}{\partial y^\beta} \right) \sqrt{\det g} \, dy \\ &= - \sum_{\alpha,\beta} \int_{\varphi^{-1}(\Omega)} \frac{1}{\sqrt{\det g}} \left(\frac{\partial}{\partial y^\beta} (\sigma + \tilde{\sigma}) g^{\alpha\beta} \sqrt{\det g} \frac{\partial}{\partial y^\alpha} \right) \phi(y) \, dy \end{aligned}$$

Since ϕ and $\hat{x}_0 \in \Sigma_{x_0}$ were arbitrarily we conclude that

$$(27) \quad L = - \sum_{\alpha,\beta} \frac{1}{\sqrt{\det g}} \left(\frac{\partial}{\partial y^\beta} (\sigma + \tilde{\sigma}) g^{\alpha\beta} \sqrt{\det g} \frac{\partial}{\partial y^\alpha} \right).$$

near Σ_0 . □

We note that Proposition 3.2 shows L to be an elliptic differential operator acting onto the level sets of $u + \tilde{u}$, for which the normal component becomes a parameter.

4. STABILITY ESTIMATES FOR PARTIAL DATA

In this section we prove Theorem 2.1 and its corollary. The proof is based on establishing separate estimates for $u - \tilde{u}$ and for the operator L in (16). For points in $\mathcal{S}(\Gamma', u + \tilde{u})$ the visibility from Γ' parallel to $\nabla(u + \tilde{u})$ will allow us to control $u - \tilde{u}$, while the the visibility along the equipotential sets will be used for estimates on L .

Recall that $\mathcal{S}(\Gamma', u + \tilde{u}) = \mathcal{S}^+(\Gamma', u + \tilde{u}) \cup \mathcal{S}^-(\Gamma', u + \tilde{u})$, where

$$\mathcal{S}^\pm(\Gamma', u) = \{p \in \overline{\Omega} : \Sigma_p \cap \partial\Omega \subset \Gamma' \text{ and } l_p^\pm \subset \mathcal{I}(\Gamma, u)\}.$$

We will show that

(28)

$$\|\sigma - \tilde{\sigma}\|_{L^2(\mathcal{S}^\pm(\Gamma', u + \tilde{u}))} \leq C \left\| \nabla \cdot (\Pi_{\nabla(u + \tilde{u})}(\mathbf{J}(\sigma) - \mathbf{J}(\tilde{\sigma}))) \right\|_{L^2(\mathcal{I}(\Gamma', u + \tilde{u}))}^{\frac{\alpha}{2+\alpha}}.$$

We prove (28) for $\mathcal{S}^+(\Gamma', u + \tilde{u})$. The corresponding inequality for $\mathcal{S}^-(\Gamma', u + \tilde{u})$ follows similarly.

To simplify notation, let

$$(29) \quad \mathcal{S} := \mathcal{S}^+(\Gamma, u + \tilde{u}), \quad \mathcal{S}' := \mathcal{S}^+(\Gamma', u + \tilde{u}), \text{ and } \mathcal{I}' := \mathcal{I}(\Gamma', u + \tilde{u}).$$

We assume w.l.o.g. that \mathcal{S}' and \mathcal{S} are connected. Notice that, since there are only finitely many components, we can add the estimates that we obtain for the single-component case, to the more general case.

We will make use of the following two technical estimates in the Lemmas below.

Lemma 4.1. *There exist $C > 0$ dependent on the domain Ω such that*

$$(30) \quad \|\sigma - \tilde{\sigma}\|_{L^2(\mathcal{S}')} \leq C \|u - \tilde{u}\|_{H^2(\mathcal{S}')},$$

where for simplicity we denote $\mathcal{S}' := \mathcal{S}^+(\Gamma', u + \tilde{u})$.

Proof. Since $\nabla(u + \tilde{u}) \neq 0$ in $\overline{\Omega}$, without loss of generality (otherwise work with a diffeomorphic image of Ω , and, possibly, with $-(u + \tilde{u})$), we may assume that

$$(31) \quad \min_{\overline{\Omega}} \frac{\partial(u + \tilde{u})}{\partial x^n} > 0.$$

Since the n -th coordinate now plays a special role, to simplify notation, let $x' := (x^1, \dots, x^{n-1})$, so that $x = (x', x^n)$.

Let Z be the set (possibly empty) of boundary points, where $\nabla(u + \tilde{u})$ is tangential to the boundary,

$$(32) \quad Z := \{x \in \partial\Omega : n(x) \cdot \nabla(u + \tilde{u})(x) = 0\}.$$

The regularity assumptions on the boundary of the domain, and on u, \tilde{u} yield that Z is closed and confined to a co-dimension 1 variety of the boundary, in particular, Z is negligible with respect to the induced area measure on the boundary.

Let $\Gamma' \subset\subset \Gamma \setminus Z$, $x_0 \in \mathcal{S}' = \mathcal{S}^+(\Gamma', u + \tilde{u})$, and $\Sigma_0 := \{x \in \Omega : (u + \tilde{u})(x) = (u + \tilde{u})(x_0)\}$ be the level set passing through x_0 .

Since $\frac{\partial(u + \tilde{u})}{\partial x^n}(x'_0, x_0^n) \neq 0$, by the implicit function theorem, in a neighborhood $O \subset \mathbb{R}^n$ of x'_0 there is a local representation of Σ_0 : $O \ni x' \mapsto$

$(x', g(x'))$, for some $C^{1,\alpha}$ - smooth coordinate map $g : O \subset \mathbb{R}^{n-1} \rightarrow \mathbb{R}$ with $g(x'_0) = x_0^n$.

We define next a tubular neighborhood T_{x_0} by flowing along $\nabla(u + \tilde{u})$ the points on $O \subset \Sigma_0$ until the boundary is met. More precisely, we consider the family (indexed in $x' \in O$) of the initial value problems

$$(33) \quad \frac{d}{dt}\gamma(t; x') = \frac{\nabla(u + \tilde{u})}{|\nabla(u + \tilde{u})|}(\gamma(t; x')), \quad \gamma(0) = (x', g(x')), \quad x' \in O.$$

Since $x_0 \in \mathcal{S}'$, for the solution $\gamma(\cdot; x'_0)$ of (33) with parameter $x' = x'_0$ meets the boundary transversally, hence there exists a smallest time $t_{x'_0}$ such that $\gamma(t_{x'_0}; x'_0) \in \Gamma'$, and $\gamma(t; x'_0) \in \Omega$ for $0 \leq t < t_{x'_0}$.

The continuous dependence with parameters of solutions of initial value problems for ODE's shows that, by possibly shrinking the neighborhood O , we have that $t \mapsto \gamma(t, x')$ are defined on a maximal interval with $\gamma(\cdot, x') : [0, t_{x'}] \rightarrow \Omega$ and $\gamma(t_{x'}; x') \in \Gamma'$.

We consider the following change of coordinates $(x', x^n) \rightarrow (y', y^n)$, where $y' = x'$ and $x^n = \gamma(y^n; y')$. In the new coordinates, the tubular neighborhood T_{x_0} rewrites

$$T_{x_0} = \{(y', y^n) \in \mathbb{R}^n : y' \in O, 0 \leq y^n \leq t_{y'}\}.$$

Let $J(y', y^n)$ denotes the absolute value of the Jacobian determinant of the change of coordinates above at some $(y', y^n) \in T_{x_0}$, and let $\beta > 0$ denote an upper bound on the quotient

$$(34) \quad \beta := \frac{\max_{T_{x_0}} J}{\min_{T_{x_0}} J} < \infty$$

Recall the equation (1) interpreted as a transport equation for $\delta\sigma = (\sigma - \tilde{\sigma})$:

$$(35) \quad \nabla(u + \tilde{u}) \cdot \nabla(\delta\sigma) + (\delta\sigma)\Delta(u + \tilde{u}) = G \quad \text{in } \Omega,$$

where, for brevity, we let

$$(36) \quad G := -\nabla \cdot (\sigma + \tilde{\sigma})\nabla(u - \tilde{u}).$$

In the local coordinates in the interior of T_{x_0} , the equation (35) becomes

$$(37) \quad \frac{1}{c^2} \frac{\partial(\delta\sigma)}{\partial y^n} + (\delta\sigma)\Delta(u + \tilde{u}) = G,$$

where $c = |\nabla(u + \tilde{u})|^{-1}$. By (9),

$$(38) \quad \max_{\Omega} c < \infty.$$

By hypothesis (13) we also have that

$$(39) \quad (\delta\sigma)(y', t_{y'}) = 0, \quad \forall y' \in O.$$

Using an integral factor we can solve (37) and (39) to get

$$(40) \quad (\delta\sigma)(y', y^n) = \left(\frac{1}{\mu(y', y^n)} \int_{t_{y'}}^{y^n} c^2 G(y', t) \mu(y', t) dt \right),$$

where

$$\mu(y', y^n) = \exp \left(- \int_{y^n}^{t_{y'}} c^2 \Delta(u + \tilde{u})(y', t) dt \right).$$

Let m, M (not necessarily positive) be such that

$$(41) \quad m \leq c^2 \Delta(u + \tilde{u})(x) \leq M, \quad \forall x \in \bar{\Omega}.$$

Then, it is easy to see that, for $y' \in O$ and $0 \leq t \leq y^n \leq t_{y'}$, we have

$$(42) \quad e^{-|M|diam(\Omega)} \leq \frac{\mu(y', t)}{\mu(y', y^n)} \leq e^{|m|diam(\Omega)}.$$

For a constant $\tilde{C} > 0$ depending on the bounds in (42) and (38), and β in (34), we estimate

$$(43) \quad \begin{aligned} \|\delta\sigma\|_{L^2(T_{x_0})}^2 &= \int_{y' \in O} \int_0^{t_{y'}} \left| \int_0^{y^n} c^2 G(y', t) \frac{\mu(y', t)}{\mu(y', y^n)} dt \right|^2 J(y', y^n) dy^n dy' \\ &\leq \tilde{C} \int_{y' \in O} \int_0^{t_{y'}} \int_0^{y^n} |G(y', t)|^2 J(y', y^n) dt dy^n dy' \\ &= \tilde{C} \int_{y' \in O} \int_0^{t_{y'}} \int_t^{t_{y'}} |G(y', t)|^2 \frac{J(y', y^n)}{J(y', t)} dy^n J(y', t) dt dy' \\ &\leq \tilde{C} \beta \int_{y' \in O} \int_0^{t_{y'}} (t_{y'} - t) |G(y', t)|^2 dy^n J(y', t) dt dy' \\ &\leq \tilde{C} \beta diam(\Omega) \int_{y' \in O} \int_0^{t_{y'}} |G(y', t)|^2 dy^n J(y', t) dt dy' \\ &= \tilde{C} \beta diam(\Omega) \|G\|_{L^2(T_{x_0})}^2 \\ &\leq \tilde{C} \beta diam(\Omega) \|\nabla \cdot (\sigma + \tilde{\sigma}) \nabla(u - \tilde{u})\|_{L^2(S')}^2, \end{aligned}$$

Since \bar{S}' can be covered by finitely many tubular neighborhoods, and the norm in the right hand side of (43), we conclude that

$$(44) \quad \|\delta\sigma\|_{L^2(S')}^2 \leq C \|\nabla \cdot (\sigma + \tilde{\sigma}) \nabla(u - \tilde{u})\|_{L^2(S')}^2 \leq C \|u - \tilde{u}\|_{H^2(S')},$$

for a constant C depending on u, \tilde{u} and the domain Ω , and a priori bounds on C^1 -norm of σ and $\tilde{\sigma}$. This finishes the proof of (30). \square

Lemma 4.2. *There exists $C > 0$ dependent on Ω and a lower bound on $\sigma + \tilde{\sigma}$, such that,*

$$(45) \quad \|u - \tilde{u}\|_{L^2(\mathcal{I}')} \leq C \|L(u - \tilde{u})\|_{L^2(\mathcal{I}')} ,$$

where, for simplicity we denote by $\mathcal{I}' := \mathcal{I}(\Gamma', u + \tilde{u})$.

Proof. Let x_0 be arbitrarily fixed in the injectivity region $\overline{\mathcal{I}'}$. Assume first that the level set Σ_0 of $u + \tilde{u}$ passing through x_0 intersects the boundary in Γ' , transversally at any point of intersection. Consider the local coordinates (y', y^n) for the equipotential set Σ_0 , introduced in Proposition 3.2. By continuity of $y^n = u + \tilde{u}$ w.r.t. y' , for sufficiently small ϵ , the neighborhood U_ϵ of Σ_0 defined by

$$U_\epsilon := \{(y', y^n) \in \mathcal{I}' : y' \in \Sigma_0, \text{ and } y^n \in (-\epsilon, \epsilon)\}$$

is visible from Γ' along the equipotential sets of $u + \tilde{u}$.

Using the local representation of L obtained in (26) and local representation the metric g on U_ϵ we estimate $L(\delta u)$. We denote by g' the restriction of the metric to y' , i.e., $g' = [g_{\alpha\beta}]$ for $1 \leq \alpha, \beta \leq n-1$. Let $\delta u = u - \tilde{u}$. Note that in U_ϵ , $(u - \tilde{u})|_{\partial\Omega} = 0$, hence Poincaré's inequality is valid in Σ_0 . We estimate

$$\begin{aligned} \langle L\delta u \cdot \delta u \rangle|_{U_\epsilon} &= \int_{-\epsilon}^{\epsilon} \int_{\Sigma_0} \left(\frac{1}{\sqrt{\det g}} \frac{\partial \delta u}{\partial y^\beta} (\sigma + \tilde{\sigma}) g^{\alpha\beta} \sqrt{\det g} \frac{\partial \delta u}{\partial y^\alpha} \right) \cdot \delta u \sqrt{\det g} \, dy' dy^n \\ &= \int_{-\epsilon}^{\epsilon} \int_{\Sigma_0} (\sigma + \tilde{\sigma}) g^{\alpha\beta} \frac{\partial \delta u}{\partial y^\alpha} \frac{\partial \delta u}{\partial y^\beta} \sqrt{\det g} \, dy' dy^n \\ &\geq \delta \int_{-\epsilon}^{\epsilon} \int_{\Sigma_0} |\nabla_{y'} \delta u|^2 \, dy' dy^n, \\ &\geq \delta \int_{-\epsilon}^{\epsilon} \int_{\Sigma_0} |\delta u|^2 \, dy' dy^n \\ (46) \quad &= \delta \|\delta u\|_{L^2(U_\epsilon)}^2, \end{aligned}$$

where $\delta > 0$ depends on an a priori lower bound of $\sigma + \tilde{\sigma}$ and of $g^{\alpha,\beta}$ on $\overline{\Omega}$ for $1 \leq \alpha, \beta \leq n$. The second inequality is the Poincaré inequality. Since $U_\epsilon \subset \mathcal{I}'$ we obtain from (46)

$$(47) \quad \|\delta u\|_{L^2(U_\epsilon)}^2 \leq C \langle L(\delta u), \delta u \rangle_{L^2(\mathcal{I}')},$$

By compactness, we can get a finite covering of $\overline{\mathcal{I}'}$ with neighborhoods U_ϵ . Summing (47) over the covering of U_ϵ we obtain

$$(48) \quad \|u - \tilde{u}\|_{L^2(\mathcal{I}')}^2 \leq C(L(u - \tilde{u}), u - \tilde{u})_{L^2(\mathcal{I}')},$$

for some constant C which depends on Ω , and the lower bound on $\sigma + \tilde{\sigma}$. Finally from Cauchy-Schwartz inequality (45) follows.

Finally the level set Σ_0 intersects tangentially the boundary of Ω as some point then the analysis presented above could fail depending on the degeneracy of such interseccion. In particular it is not clear how to justify integration by parts on U_ϵ in the intersection is degenerate. To avoid this difficulty we will extend $\sigma, \tilde{\sigma}, u$ and \tilde{u} in two different ways to an open domain Ω_1 containing $\overline{\mathcal{I}'}$. The first extension takes advantage of the boundary information while the second extension uses the regularity of the functions to extend the geometry and the operators in decomposition (16).

Since Γ' is compactly supported in Γ it follows that $\overline{\mathcal{I}'} \subset \mathcal{I}$. Thus there exists an open bounded Ω_1 with smooth boundary, such that $\overline{\mathcal{I}'} \subset \Omega_1$ and $\Omega_1 \cap \Omega \subset \mathcal{I}$. First, using the fact that $(\sigma - \tilde{\sigma})|_\Gamma = 0$ we can find $H^1(\Omega_1)$ extensions σ_1 and $\tilde{\sigma}_1$ of σ and $\tilde{\sigma}$, respectively, such that $(\sigma_1 - \tilde{\sigma}_1) = 0$ in $\Omega_1 \setminus \Omega$; (e.g., extend first σ to some $\sigma_1 \in C^2(\Omega_1)$, and define $\tilde{\sigma}_1 := \sigma_1$ in $\Omega_1 \setminus \Omega$. Since $\tilde{\sigma}_1$ coincide with σ_1 on $\partial\Omega$, the difference $\tilde{\sigma}_1 - \sigma_1$, and thus $\tilde{\sigma}_1$ lie in $H^1(\Omega_1)$. Moreover, since σ and $\tilde{\sigma}$ are positive in $\overline{\Omega}$, we may assume that σ_1 and $\tilde{\sigma}_1$ are positive in Ω_1 . Second, denote by $\sigma_2, \tilde{\sigma}_2 \in C^2(\Omega_1)$ extensions of σ and $\tilde{\sigma}$, respectively. We can take Ω_1 close enough to the boundary of Ω so that σ_2 and $\tilde{\sigma}_2$ are positive in $\overline{\Omega_1}$.

In a similar way we extend u and \tilde{u} as follows. Since $(u - \tilde{u})|_\Gamma = 0$, we can find $H^1(\Omega_1)$ extensions u_1 and \tilde{u}_1 of u and \tilde{u} , respectively, such that $(u_1 - \tilde{u}_1)|_{\Omega_1 \setminus \Omega} = 0$. Also denote by $u_2, \tilde{u}_2 \in C^2(\Omega_1)$ extensions of u and \tilde{u} respectively. We can take Ω_1 close enough to the boundary of Ω so that $\nabla(u_2 + \tilde{u}_2) \neq 0$ on $\overline{\Omega_1}$. Finally we denote by g_1 a $C^1(\Omega_1)$ extension to Ω_1 of the metric g obtained in (22).

We then define U_ϵ in Ω_1 instead of $\overline{\mathcal{I}'}$ by

$$U_\epsilon := \{(y', y^n) \in \Omega_1 : y' \in \Sigma_0, \text{ and } y^n \in (-\epsilon, \epsilon)\}$$

Now, even if the intersection of Σ_0 with $\partial\Omega_1$ is tangential the function that we are integrating is compactly supported, so we can justify integration by parts in (47). \square

Using the estimates in Lemmas 4.1 and 4.2 we proceed to proving (10). Recall from Schauder theory that $u, \tilde{u} \in C^{2,\alpha}(\Omega)$. By using in order the estimate (30), an interpolation estimate in [?, Sec. 4.3.1],

(45), and Proposition 3.1 we obtain

$$\begin{aligned}
(49) \quad & \|\sigma - \tilde{\sigma}\|_{L^2(V')} \leq C \|u - \tilde{u}\|_{H^2(V')} \\
& \leq C \|u - \tilde{u}\|_{L^2(V')}^{\frac{\alpha}{2+\alpha}} \cdot \|u - \tilde{u}\|_{H^{2+\alpha}(V')}^{\frac{2}{2+\alpha}} \leq C \|u - \tilde{u}\|_{L^2(\mathcal{I}')}^{\frac{\alpha}{2+\alpha}} \\
& \leq C \|L(u - \tilde{u})\|_{L^2(\mathcal{I}')}^{\frac{\alpha}{2+\alpha}} \\
& = C \left\| 2\nabla \cdot \Pi_{\nabla(u+\tilde{u})}(\mathbf{J}(\sigma) - \mathbf{J}(\tilde{\sigma})) \right\|_{L^2(\mathcal{I}')}^{\frac{\alpha}{2+\alpha}}
\end{aligned}$$

This finishes the proof of Theorem 2.1.

Proof of Theorem 2.2. Recall $\epsilon > 0$ in (13). The Schauder stability estimates for solutions of elliptic equations with $C^\alpha(\Omega)$ -coefficients and $C^{2,\alpha}$ traces on the boundary, yield

$$(50) \quad \|u - \tilde{u}\|_{C^{2,\alpha}(\bar{\Omega})} \leq M\epsilon,$$

for a constant M dependent only on Ω and a lower bound of σ . In particular, for $\epsilon < 1/M$ we have that

$$(51) \quad \|\tilde{u}\|_{C^2(\bar{\Omega})} \leq \|u\|_{C^2(\bar{\Omega})} + 1.$$

Moreover, since $m := \min_{\bar{\Omega}} |\nabla u| > 0$, for $\epsilon < m/M$, we also have

$$(52) \quad |\nabla(u + \tilde{u})| \geq 2|\nabla u| - |\nabla(u - \tilde{u})| \geq 2m - M\epsilon \geq m > 0.$$

Denote by $\delta\mathbf{J} = \mathbf{J}(\sigma) - \mathbf{J}(\tilde{\sigma})$. From Theorem 2.1 we have that there exist a $C > 0$ independent of ϵ , such that

$$(53) \quad \|\sigma - \tilde{\sigma}\|_{L^2(\Omega)} \leq C \left\| \frac{\delta\mathbf{J} \cdot \nabla(u + \tilde{u})}{|\nabla(u + \tilde{u})|} \right\|_{H^1(\Omega)}.$$

Write

$$(54) \quad \frac{\delta\mathbf{J} \cdot \nabla(u + \tilde{u})}{|\nabla(u + \tilde{u})|} = 2 \frac{|\nabla u|}{|\nabla(u + \tilde{u})|} \left(\frac{\delta\mathbf{J} \cdot \nabla u}{|\nabla u|} \right) + \frac{\delta\mathbf{J} \cdot \nabla(\tilde{u} - u)}{|\nabla(u + \tilde{u})|}.$$

Note that all the terms in (54) are $C^1(\bar{\Omega})$ -regular.

By using the identity (19), the second term in the right hand side of (54) rewrites as

$$(55) \quad \frac{\delta\mathbf{J} \cdot \nabla(\tilde{u} - u)}{|\nabla(u + \tilde{u})|} = \frac{1}{2}(\sigma - \tilde{\sigma})|\nabla(u - \tilde{u})| + \frac{1}{2}(\sigma + \tilde{\sigma}) \frac{|\nabla(u - \tilde{u})|^2}{|\nabla(u + \tilde{u})|}.$$

Using (21), (51), and the Cauchy-Schwartz inequality we obtain

$$\begin{aligned}
\langle (\sigma + \tilde{\sigma})\nabla(\tilde{u} - u), \nabla(\tilde{u} - u) \rangle_{L^2(\Omega)} &= -\langle \nabla \cdot (\sigma + \tilde{\sigma})\nabla(\tilde{u} - u), \tilde{u} - u \rangle_{L^2(\Omega)} \\
&= \langle \nabla \cdot (\tilde{\sigma} - \sigma)\nabla(u + \tilde{u}), \tilde{u} - u \rangle_{L^2(\Omega)} \\
&\leq \|(\tilde{\sigma} - \sigma)\nabla(u + \tilde{u})\|_{L^2(\Omega)} \|\nabla(\tilde{u} - u)\|_{L^2(\Omega)} \\
(56) \qquad \qquad \qquad &\leq C\|\tilde{\sigma} - \sigma\|_{L^2(\Omega)} \|\nabla(\tilde{u} - u)\|_{L^2(\Omega)},
\end{aligned}$$

for C dependent on the C^1 -norm of u but independent of ϵ . From (56), for

$$\delta := \min_{\bar{\Omega}}(\sigma + \tilde{\sigma}) > 0,$$

we obtain

$$(57) \qquad \qquad \qquad \|\nabla(u - \tilde{u})\|_{L^2(\Omega)} \leq \frac{C}{\delta} \|\sigma - \tilde{\sigma}\|_{L^2(\Omega)}.$$

We claim that

$$(58) \qquad \qquad \qquad \left\| \frac{\delta \mathbf{J} \cdot \nabla(u - \tilde{u})}{|\nabla(u + \tilde{u})|} \right\|_{H^1(\Omega)} \leq C\epsilon \|\sigma - \tilde{\sigma}\|_{L^2(\Omega)}.$$

for some C independent of ϵ . Indeed: one differentiation in (55) in any variable, which for simplicity we denote by subscript x , yields four terms. We treat each term individually, the appearing constants are independent of ϵ . To bound the first term we use (13), the Cauchy-Schwartz inequality and (57) to obtain

$$\int_{\Omega} |(\sigma - \tilde{\sigma})_x| |\nabla(u - \tilde{u})| dx \leq C_1\epsilon \|\sigma - \tilde{\sigma}\|_{L^2(\Omega)}.$$

To bound the second term we use the Cauchy-Schwartz inequality and uniform bounds (dependent on u only via (51)) on the second derivatives of u and \tilde{u} and obtain

$$\int_{\Omega} |\sigma - \tilde{\sigma}| (|\nabla(u - \tilde{u})|)_x dx \leq C_2\epsilon \|\sigma - \tilde{\sigma}\|_{L^2(\Omega)}.$$

To bound the third term we use uniform bounds on the second derivatives of u (and implicitly on \tilde{u} via (51)) and on the first derivatives of σ (and implicitly of $\tilde{\sigma}$ via (13)), (52) and (57) and obtain

$$\int_{\Omega} \left(\frac{\sigma + \tilde{\sigma}}{|\nabla(u + \tilde{u})|} \right)_x |\nabla(u - \tilde{u})|^2 dx \leq C_3\epsilon \|\sigma - \tilde{\sigma}\|_{L^2(\Omega)}.$$

To bound the fourth term we use the Cauchy-Schwartz inequality and uniform bounds on $\sigma, \tilde{\sigma}$, and C^2 -bounds on u (and implicitly via on \tilde{u} (51)), (52), and (57) and obtain

$$\int_{\Omega} \frac{\sigma + \tilde{\sigma}}{|\nabla(u + \tilde{u})|} (|\nabla(u - \tilde{u})|)_x |\nabla(u - \tilde{u})| dx \leq C_4\epsilon \|\sigma - \tilde{\sigma}\|_{L^2(\Omega)}.$$

This finishes the proof of (58).

From Equations (53), (54) and (58) we get

$$(59) \quad \|\sigma - \tilde{\sigma}\|_{L^2(\Omega)} \leq C \left\| \frac{\delta \mathbf{J} \cdot \nabla u}{|\nabla u|} \right\|_{H^1(\Omega)} + C\epsilon \|\sigma - \tilde{\sigma}\|_{L^2(\Omega)},$$

for a constant C independent of ϵ .

Finally, using (59) for $\epsilon < 1/C$ we obtain

$$\|\sigma - \tilde{\sigma}\|_{L^2(\Omega)} \leq \frac{C}{1 - C\epsilon} \left\| \frac{\delta \mathbf{J} \cdot \nabla u}{|\nabla u|} \right\|_{H^1(\Omega)}$$

which implies (15) and proves Theorem 2.2. \square

Acknowledgments

We would express our gratitude to Plamen Stefanov for his valuable suggestions during the writing of this paper.

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