Recovering the conductivity from a single measurement of interior data

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Abstract. We consider the problem of recovering the conductivity of an object from knowledge of the magnitude of one current density field in its interior. A known voltage potential is assumed imposed at the boundary. We prove identifiability and propose an iterative reconstruction procedure. The computational feasibility of this procedure is demonstrated in some numerical experiments.

1. Introduction

In this paper we consider the problem of recovering the isotropic conductivity $\sigma > 0$ of an object from knowledge of the magnitude of one current density $|J|$ in the interior. The interior data can be obtained from magnetic resonance imaging measurements (MRI) as shown in [16]. We note that this methodology requires determination of all three components of the current density vector field. We hope that the discovery presented here- that it suffices to measure just the magnitude of only one current- may lead to novel physical approaches to obtain this data directly. In [13] the authors proposed a new method of conductivity imaging from the same interior data combined with the Cauchy data (voltage-current) on a part of the boundary. In this paper we only require the interior measurements: we study the corresponding Dirichlet problem, with a given voltage potential on the entire boundary of the object. It is well known that the boundary data has low sensitivity to the variation of σ (see, e.g., [2], [12]) yielding images of low resolution. Knowledge of the interior data $|J|$ restores the image resolution and accuracy, as shown for planar conductivities in [13].

This paper starts out from the result in [13] that the equipotential surfaces are minimal surfaces in a conformal Riemannian metric determined by the given magnitude of the current density. To determine the voltage potential inside the object, one is to study the Plateau problem in this conformal metric for all equipotential surfaces at once; this corresponds to studying the Dirichlet problem for the degenerate elliptic equation

$$\nabla \cdot \left(\frac{|J|}{|\nabla u|} \nabla u \right) = 0 \tag{1}$$

with prescribed boundary data.

In this paper we show that the voltage potential corresponding to the current whose magnitude is measured is the unique minimizer of the functional

$$F[u] = \int_\Omega |J(x)| \cdot |\nabla u(x)| dx, \tag{2}$$

with given Dirichlet data, over the continuous maps in $W^{1,1}(\Omega)$ with non-vanishing gradient almost everywhere. We observe that the Euler-Lagrange equation of this functional is, formally, the degenerate elliptic equation (1). We indicate below why we found it crucial to work directly with the the variational problem rather than the corresponding differential equation.

The degenerate elliptic equation was first introduced in conductivity imaging in [10], where it was observed to follow from Ohm's law $J = -\sigma\nabla u$ combined with the charge conservation law $\nabla \cdot J = 0$. The examples of non-uniqueness and non-existence for the solution to the Neumann problem associated with (1) given in [10] show that knowledge of the applied current at the boundary together with the magnitude of current density field inside is insufficient data to determine the conductivity. In [13] the authors studied the two dimensional Cauchy problem for (1) with data given on a part of the boundary only. A sufficient condition on the boundary voltage was shown to yield identifiability of the conductivity in the whole domain. In this paper, we consider the Dirichlet problem.

Due to the degeneracy of (1) at the points where the gradient vanishes, the notion of a solution needs to be defined carefully. The following example from [17] shows that if one considers solutions in the viscosity sense (see, e.g., [3]) then there is non-uniqueness in the Dirichlet problem for (1) with $|J| \equiv 1$. Let $D = \{x \in R^2 : \ (x_1)^2 + (x_2)^2 < 1\}$ be the unit disk; consider the problem

$$\begin{aligned} &\nabla \cdot \Big(|\nabla u(x)|^{-1} \nabla u(x) \Big) = 0, \ x \in D, \\ &u(x) = (x_1)^2 - (x_2)^2, \ x \in \partial D. \end{aligned} \tag{3}$$

It turns out that for each $\lambda \in [-1, 1]$ the corresponding function

$$u^\lambda(x) = \begin{cases} 2(x_1)^2 - 1, & if \quad |x_1| \geq \sqrt{\frac{1+\lambda}{2}}, \ |x_2| \leq \sqrt{\frac{1-\lambda}{2}}, \\ \lambda, & if \quad |x_1| \leq \sqrt{\frac{1+\lambda}{2}}, \ |x_2| \leq \sqrt{\frac{1-\lambda}{2}}, \\ 1 - 2(x_2)^2, & if \quad |x_1| \leq \sqrt{\frac{1+\lambda}{2}}, \ |x_2| \geq \sqrt{\frac{1-\lambda}{2}} \end{cases} \tag{4}$$

is a viscosity solution to the above boundary value problem. On the other hand, it is only the solution corresponding to $\lambda = 0$ that minimizes the functional $\int_\Omega |\nabla u(x)| dx$ over the space of maps with bounded variation; see [17] for details.

The example above also shows that, in general, the minimization of the functional (2) may lead to solutions which do not represent a voltage potential, since their gradients vanish in open sets. These considerations motivate the following definition.

Definition 1.1 *A pair of functions* $(f, a) \in H^{1/2}(\partial\Omega) \times L^2(\Omega)$ *is called* admissible *if there exists a positive map* $\sigma \in L^\infty(\Omega)$ *bounded away from zero such that, if* $u \in H^1(\Omega)$ *is the weak solution to*

$$\nabla \cdot \sigma \nabla u = 0, \quad u|_{\partial\Omega} = f, \tag{5}$$

then

$$|\sigma \nabla u| = a. \tag{6}$$

The function σ *is called a* generating conductivity *for the pair* (f, a) *and the function* u *is called the* corresponding potential.

Note that the example (3) above shows that the C^∞-smooth pair $((x_1)^2 - (x_2)^2, 1)$ is not admissible. Indeed, if $\tilde{u}$ were the potential corresponding to some generating conductivity, then $\tilde{u}$ would be a minimizer in $W^{1,1}(\Omega)$, see Proposition 1.2 below. But the functional $\int_\Omega |\nabla u| dx$ has a unique minimizer in $BV(\Omega)$ with $u|_{\partial\Omega} = f$, as shown in [18]. Therefore $\tilde{u}$ must be the minimizer u^0 for $\lambda = 0$ in (4). Since u^0 is constant in an open set and $|J|$ is assumed equal to 1 throughout D, the relation (6) cannot be satisfied by a bounded conductivity.

Our first result shows that for an admissible pair, the corresponding voltage potential minimizes F in the entire affine subspace of functions $u \in H^1(\Omega)$ with trace f on the boundary. Throughout the paper we say that $f \in C^{1,\alpha}(\partial\Omega)$ for some $0 < \alpha < 1$ if f is the trace on the boundary $\partial\Omega$ of a map in $C^{1,\alpha}(\overline{\Omega})$. As well, we denote by $L^\infty_+(\Omega)$ the set of functions $\sigma \in L^\infty(\Omega)$ with $\sigma \geq c$ a.e. for some $c > 0$.

Proposition 1.2 *Let $\Omega \subset R^n$ be a domain and $(f, |J|) \in H^{1/2}(\partial\Omega) \times L^2(\Omega)$.*

(i) If $(f, |J|)$ is admissible, say generated by some conductivity $\sigma_0 \in L^\infty_+(\Omega)$, and $u_0 \in H^1(\Omega)$ is the corresponding voltage potential, then u_0 is a minimizer for $F[u]$ in (2) over all $u \in H^1(\Omega)$ with $u|_{\partial\Omega} = f$. Moreover, if $f \in C^{1,\alpha}(\partial\Omega)$ and if the generating conductivity $\sigma_0 \in C^\alpha(\overline{\Omega})$, then the corresponding potential $u_0 \in C^{1,\alpha}(\overline{\Omega})$ is a minimizer of F over all $u \in W^{1,1}(\Omega)$ with $u|_{\partial\Omega} = f$.

(ii) If u_0 is a minimizer for $F[u]$ in (2) over all $u \in H^1(\Omega)$ with $u|_{\partial\Omega} = f$ such that $|J|/|\nabla u_0| \in L^\infty_+(\Omega)$, then $(f, |J|)$ is admissible.

Proof. For any $u \in H^1(\Omega)$ with $u|_{\partial\Omega} = f$ we have

$$\begin{aligned} F[u] &= \int_\Omega \sigma_0 |\nabla u_0| \cdot |\nabla u| dx \geq \int_\Omega \sigma_0 |\nabla u_0 \cdot \nabla u| dx \\ &\geq \int_\Omega \sigma_0 \nabla u_0 \cdot \nabla u dx = \int_{\partial\Omega} \sigma_0 \frac{\partial u_0}{\partial \nu} u ds = \langle \Lambda_{\sigma_0} f, f \rangle, \end{aligned} \tag{7}$$

where ν is the outer normal to the boundary and Λ_{σ_0} denotes the Dirichlet-to-Neumann map. The lower bound is achieved at u_0. By elliptic regularity for Hölder-continuous coefficients, if $\sigma_0 \in C^\alpha(\overline{\Omega})$ and $f \in C^{1,\alpha}(\partial\Omega)$, then $u \in C^{1,\alpha}(\overline{\Omega})$ (see, e.g., Theorem 8.34 in [7]), hence $|J| \in C^\alpha(\overline{\Omega})$ and the argument above extends to $u \in W^{1,1}(\Omega)$.

To show (ii), we note first that, by Lebesgue dominated convergence, the functional F is Gateaux-differentiable at points $u \in H^1(\Omega)$ with $|J|/|\nabla u| \in L^\infty_+(\Omega)$. In particular, at a minimizer u_0 we have

$$F'[u_0](\varphi) = \int_\Omega \frac{|J|}{|\nabla u_0|} \nabla u_0 \cdot \nabla \varphi dx = 0, \tag{8}$$

for all $\varphi \in H^1_0(\Omega)$. This means that u_0 is a weak solution to the Euler-Lagrange equation (1) and, therefore, $|J|/|\nabla u_0|$ is a generating conductivity for the pair $(f, |J|)$.

□

Throughout the paper $W^{1,1}_+(\Omega)$ denotes the space of $L^1(\Omega)$ maps with gradient in $L^1(\Omega)$, for which the set of singular points (where the gradient vanishes) has at most Lebesgue measure zero. We prove the following uniqueness result.

Theorem 1.3 (Unique Determination) *Let $\Omega \subset R^n$ be a domain with connected, $C^{1,\alpha}$- boundary and let $(f, |J|) \in C^{1,\alpha}(\partial\Omega) \times C^\alpha(\overline{\Omega})$ be an admissible pair generated by some unknown $C^\alpha(\overline{\Omega})$ -conductivity. Assume that $|J| > 0$ a.e. in Ω. Then the minimization problem*

$$argmin \left\{ F[u] : \quad u \in W^{1,1}_+(\Omega) \bigcap C(\overline{\Omega}),\ u|_{\partial\Omega} = f \right\}, \tag{9}$$

has a unique solution u_0. Moreover, $\sigma_0 = |J|/|\nabla u_0|$ is the unique conductivity in $C^\alpha(\overline{\Omega})$ for which $|J|$ is the magnitude of the current density while maintaining the voltage f at the boundary.

Based on the results in [1], for simply connected planar domains there is a simple sufficient condition to ensure a non-vanishing current density field. As in [13] we say that a map on the connected boundary is *almost two-to-one* if the set of local maxima is either one point or one connected arc. The uniqueness result above then simplifies as follows.

Corollary 1.4 *Let $\Omega \subset R^2$ be a simple connected domain with $C^{1,\alpha}$- boundary, $(f,|J|) \in C^{1,\alpha}(\partial\Omega) \times C^{\alpha}(\overline{\Omega})$ be an admissible pair with f almost two-to-one. Then there is a unique positive conductivity in $C^{\alpha}(\overline{\Omega})$ for which $|J|$ is the magnitude of the current density while maintaining the voltage f on the boundary. Moreover, the corresponding potential u_0 is the unique solution to the minimization problem (9).*

The paper is organized as follows. In the next section, we prove some preliminary results needed in the proof of the uniqueness theorem, presented in the third section. In the fourth section, we describe an iterative procedure which constructs a minimizing sequence. The fifth section presents some numerical experiments based on the iterative procedure. Several remarks conclude the paper.

2. Preliminaries

This section concerns the geometry of the level sets of maps in $W^{1,1}(\Omega)$. The results are based on the regularity result of De Giorgi (see, e.g., [8]) for boundaries of sets of locally finite perimeter (or Caccioppoli sets). For our purposes it suffices to work with nonnegative maps $u \in W^{1,1}(\Omega)$ with $u \geq 0$. For any $t \geq 0$ let E_t denote the super-level set $E_t = \{x \in \Omega : \ u(x) > t\}$ and χ_{E_t} be its characteristic function. Since $u \in W^{1,1}(\Omega)$, from the co-area formula [4],[6] (see also [19])

$$\int_{\Omega} |\nabla u(x)| dx = \int_0^{\infty} |\nabla \chi_{E_t}|(\Omega) dt, \tag{10}$$

we have that $|\nabla \chi_{E_t}|(\Omega) < \infty$ for almost all $t \geq 0$, so that E_t is a Caccioppoli set for such t. In the formula above $\nabla \chi_E$ is the vector valued Radon measure defined by $\partial_{x_i}\chi_E(\phi) = -\int_E \partial_{x_i}\phi$, for all $\phi \in C_0^1(R^n)$. We recall the notions of the *reduced boundary* $\partial^* E$ of a Caccioppoli set E and that of the *measure theoretic outer unit normal*.

Definition 2.1 *Let E be a Caccioppoli set. The reduced boundary $\partial^* E$ consists of all points $x \in R^n$ for which the following conditions hold:*

(i) for all $r > 0$, we have $\int_{B(x,r)} |\nabla \chi_E| > 0$,

(ii) the limit below exists:

$$\nu(x) := -\lim_{r \to 0} \frac{\int_{B(x,r)} \nabla \chi_E}{\int_{B(x,r)} |\nabla \chi_E|},$$

and

(iii) $|\nu(x)| = 1$.

For $x \in \partial^* E$, $\nu(x)$ is called the measure theoretic unit outer normal.

Since the super-level set E_t is Caccioppoli, by the Besicovitch's theorem on differentiations of measures (see, e.g., 2.9 of [5]) it follows that $\nu_t(x)$ exists H^{n-1}- a.e. $x \in \partial E_t$, where H^{n-1} is the $(n-1)$-dimensional Hausdorff measure in R^n.

Lemma 2.2 *Suppose $u \in W^{1,1}(\Omega)$. Let $E_t = \{x \in \Omega : u(x) > t\}$ and let $\nu_t(x)$ be the measure theoretic outer unit normal to $\partial^* E_t$. For (Lebesgue-) a.e. t and at H^{n-1}-a.e. $x \in \partial^* E_t$, we have that*

$$\nu_t(x) = -\frac{\nabla u(x)}{|\nabla u(x)|}. \tag{11}$$

Proof. Without loss of generality, we may assume that $u \geq 0$. Fix a vector valued function $f \in C_0^1(\Omega; R^n)$ and a smooth increasing function on the real line $g(t)$ with $|g(t)| \leq c|t|$ for some constant c. On the one hand

$$\begin{aligned}
\int_\Omega f(x) \cdot \nabla g(u(x))dx &= -\int_\Omega (\nabla \cdot f(x))g(u(x))dx \\
&= -\int_\Omega \int_0^\infty (\nabla \cdot f(x))\chi_{\{g(u)>s\}}(x)dsdx \\
&= -\int_\Omega \int_0^\infty (\nabla \cdot f(x))\chi_{E_t}(x)g'(t)dtdx \\
&= -\int_0^\infty g'(t) \int_{E_t} \nabla \cdot f(x)dxdt \\
&= -\int_0^\infty g'(t) \int_{\partial^* E_t} f(x) \cdot \nu_t(x)dH^{n-1}(x)dt, \qquad (12)
\end{aligned}$$

where the last equality is the Gauss-Green formula for Caccioppoli sets. On the other hand

$$\begin{aligned}
\int_\Omega f(x) \cdot \nabla g(u(x))dx &= \int_\Omega g'(u(x))f(x) \cdot \frac{\nabla u}{|\nabla u|}|\nabla u|dx \\
&= \int_0^\infty g'(t) \int_{\partial^* E_t} f(x) \cdot \frac{\nabla u(x)}{|\nabla u(x)|}dH^{n-1}(x)dt, \qquad (13)
\end{aligned}$$

where the last equality follows from the co-area formula for bounded variation functions. From (12) and (13) we find

$$\int_0^\infty g'(t) \int_{\partial^* E_t} f(x) \cdot \left(\frac{\nabla u(x)}{|\nabla u(x)|} + \nu(x)\right) dH^{n-1}(x)dt = 0. \tag{14}$$

Since the equation (14) is valid for any f and g as described above, the equality (11) follows.

□

3. Unique determination

In this section we prove Theorem 1.3. The existence of a minimizer u_0 of the functional (2) comes from the assumption of admissibility. To simplify notation, let $a = |J|$. As shown in the proof of Proposition 1.2, we have $a \in C^\alpha(\overline{\Omega})$ and the functional

$$F[u] = \int_\Omega a|\nabla u|dx \tag{15}$$

is well defined over $W^{1,1}(\Omega)$.

Since $|J| > 0$ a.e. in Ω (by assumption in Theorem 1.3) the equality (6) yields $|\nabla u_0| = 0$ at most on a set of measure zero, which makes $u_0 \in W_+^{1,1}(\Omega) \bigcap C(\overline{\Omega})$. We show in this section that u_0 is the unique minimizer among the maps in $W^{1,1}_+(\Omega)\bigcap C(\overline{\Omega})$ with trace f on $\partial\Omega$.

Assume that $u_1 \in W^{1,1}(\Omega)\bigcap C(\overline{\Omega})$ is another minimizer with $u_1|_{\partial\Omega} = f$ and $|\nabla u_1| > 0$ a.e. in Ω. By possibly adding a constant (and then working with $f + const.$), without loss of generality we may assume that $u_1 > 0$ in $\overline{\Omega}$.

Since u_0 minimizes the functional in the whole space $W^{1,1}(\Omega)$ so does u_1. Equality holds in (7) for $u = u_1$. Since the equality in the Cauchy's inequality can only hold for parallel vectors, we have that

$$\nabla u_1(x) = \lambda(x)\nabla u_0(x), \quad a.e.\ x \in \Omega, \tag{16}$$

for some (Lebesgue-) measurable $\lambda(x)$ nonnegative a.e. In particular, for a.e. $x \in \Omega$ we must have

$$\frac{\nabla u_1(x)}{|\nabla u_1(x)|} = \frac{\nabla u_0(x)}{|\nabla u_0(x)|}. \tag{17}$$

Let $E_t = \{x \in \Omega :\ u_1(x) > t\}$. We claim that the sets $\partial E_t \bigcap \Omega$ are smooth C^1 manifolds in Ω for almost all $t > 0$. Since $u_0 \in C^1(\Omega)$, from the equalities (17) and (11) we have that the measure theoretical normal $\nu_t(x)$ extends continuously from $\partial^* E_t \bigcap \Omega$ to the topological boundary $\partial E_t \bigcap \Omega$. By applying the regularity result of De Giorgi (see, e.g, Theorem 4.11 in [8]), we conclude that $\partial E_t \bigcap \Omega$ is a C^1-hypersurface for almost all $t > 0$ and ν_t is its unit normal field.

As a consequence of (16) and according to the C^1 regularity of ∂E_t, the function u_0 is constant on each of the connected components of ∂E_t, for almost all t. Indeed, let $\gamma : (-\epsilon, \epsilon) \to \partial E_t$ be an arbitrary C^1 curve in ∂E_t. Since $\gamma'(s)$ is orthogonal to $\nu(\gamma(s))$, we have

$$\frac{d}{ds}u_0(\gamma(s)) = \nabla u_0(\gamma(s)) \cdot \gamma'(s) = |\nabla u_0(\gamma(s))|\nu(\gamma(s)) \cdot \gamma'(s) = 0,$$

so that u_0 is constant along γ.

Let t be one of the values for which ∂E_t is a hypersurface (which is the case for a.e. $t > 0$). We show next that each connected component of ∂E_t intersects the boundary $\partial\Omega$.

Arguing by contradiction, assume that $\Sigma_t \subset \Omega$ is a connected component of ∂E_t such that $\Sigma_t \bigcap \partial\Omega = \emptyset$. Then $\partial\Omega \bigcup \Sigma_t$ is a compact manifold with two connected components. Using the Alexander duality theorem in algebraic topology for $\partial\Omega \bigcup \Sigma_t$ (see, e.g. Theorem 27.10 in [9],) we have that $R^n \setminus (\partial\Omega \bigcup \Sigma_t)$ is partitioned into three open connected components: $(R^n \setminus \overline{\Omega}) \bigcup O_1 \bigcup O_2$. Since $\Sigma_t \subset \Omega$ we have $O_1 \bigcup O_2 = \Omega \setminus \Sigma_t$ and then $\partial O_i \subset \partial\Omega \bigcup \Sigma_t$ for $i = 1, 2$.

We claim that at least one of the ∂O_1 or ∂O_2 is in Σ_t. Assume not, i.e. for each $i = 1, 2$, $\partial O_i \bigcap \partial\Omega \neq \emptyset$. Since $\partial\Omega$ is connected (by assumption), we have that $O_1 \bigcup O_2 \bigcup \partial\Omega$ is connected which implies $O_1 \bigcup O_2 \bigcup (R^n \setminus \Omega)$ is also connected. By applying once again Alexander's duality theorem for $\Sigma_t \subset R^n$, we have that $R^n \setminus \Sigma_t$ has exactly two open connected components, one of which is unbounded: $R^n \setminus \Sigma_t = O_\infty \bigcup O_0$. Since $O_1 \bigcup O_2 \bigcup (R^n \setminus \Omega)$ is connected and unbounded, we have $O_1 \bigcup O_2 \bigcup (R^n \setminus \Omega) \subset O_\infty$, which leaves $O_0 \subset R^n \setminus (O_1 \bigcup O_2 \bigcup (R^n \setminus \Omega)) \subset \Sigma_t$. This is impossible since O_0 is open and Σ_t is a hypersurface. Therefore either O_1 or O_2 or both has the boundary in Σ_t.

To fix ideas, consider $\partial O_1 \subset \Sigma_t$. If this were the case, then we claim that $u_1 \equiv t$ in O_1. Indeed, since O_1 is an extension domain (∂O_1 has a unit normal everywhere) the new map $\tilde{u}_1$ defined by

$$\tilde{u}_1(x) = \begin{cases} u_1(x), & x \in \Omega \setminus O_1, \\ t, & x \in \overline{O_1}, \end{cases} \tag{18}$$

is in $W^{1,1}(\Omega) \bigcap C(\overline{\Omega})$ and decreases the functional in (2), thus contradicting the minimality of u_1. Therefore $u_1 \equiv t$ in O_1, which makes $|\nabla u_1| \equiv 0$ in O_1. Again we reach a contradiction since the set of critical points of u_1 has measure zero.

These contradictions followed from the assumption that $\Sigma_t \bigcap \partial\Omega = \emptyset$. We conclude that each connected component of ∂E_t reaches the boundary $\partial\Omega$.

Since u_0 and u_1 coincide on the boundary, we showed that $u_0|_{\partial E_t} = u_1|_{\partial E_t} = t$ for almost all t. Let G denote the set of values $\{t : \ u_0|_{\partial E_t} = u_1|_{\partial E_t} = t\}$. We claim that the set spanned by the level curves on which $u_0 = u_1$ is dense in $\overline{\Omega}$. By the continuity of u_1, it then follows that $u_0 = u_1$ in $\overline{\Omega}$. Indeed, assume that there is a ball $\overline{B} \subset \Omega$ with $\overline{B} \bigcap \{x : \ u_1(x) \in G\} = \emptyset$. Since u_1 is continuous and $|\nabla u_1| > 0$, $u_1(\overline{B}) = [\alpha, \beta]$ for $\alpha < \beta$. Hence $[\alpha, \beta] \subset Range(u_1) \setminus G$. This is impossible since the latter set has measure zero.

4. On constructing minimizing sequences

In this section we present an algorithm that produces a minimizing sequence for the functional in (2), under certain conditions specified below.

For a given admissible pair (f, a) we consider the following iterative algorithm. For $u_{n-1} \in H^1(\Omega)$ given such that $\frac{a}{|\nabla u_{n-1}|} \in L^\infty_+(\Omega)$, we define

$$\sigma_n = \frac{a}{|\nabla u_{n-1}|} \tag{19}$$

and construct u_n as the unique solution to

$$\begin{cases} \nabla \cdot \sigma_n \nabla u_n = 0, \\ u_n|_{\partial\Omega} = f. \end{cases} \tag{20}$$

The results below specify sufficient conditions, under which the algorithm is well defined and the sequence $\{u_n\}$ is minimizing for the functional in (2). In the numerical experiments considered in the next section, the iteration starts with the harmonic function with trace f on the boundary.

We make use of the following lemma which is not restricted to planar domains.

Lemma 4.1 *Assume that $v \in H^1(\Omega)$ is such that $\frac{a}{|\nabla v|} \in L^\infty_+(\Omega)$ and let $u \in H^1(\Omega)$ be the weak solution of*

$$\begin{cases} \nabla \cdot \frac{a}{|\nabla v|} \nabla u = 0 \ \textit{in} \ \Omega, \\ u|_{\partial\Omega} = v|_{\partial\Omega}. \end{cases} \tag{21}$$

Then the following inequalities hold:

$$\int_\Omega a|\nabla u|dx \le \int_\Omega a|\nabla v|dx; \tag{22}$$

$$\int_\Omega a|\nabla u|dx \ge \int_\Omega \frac{a}{|\nabla v|}|\nabla u|^2 dx; \tag{23}$$

$$\begin{aligned}\frac{1}{2}\int_\Omega \left(a|\nabla v| - \frac{a}{|\nabla v|}|\nabla u|^2\right) dx &\le \int_\Omega (a|\nabla v| - a|\nabla u|)\, dx \\ &\le \int_\Omega \left(a|\nabla v| - \frac{a}{|\nabla v|}|\nabla u|^2\right) dx. \end{aligned}\tag{24}$$

We also have the identity:

$$\int_\Omega \left(a|\nabla v| - \frac{a}{|\nabla v|}|\nabla u|^2\right) dx = \int_\Omega \frac{a}{|\nabla v|}|\nabla v - \nabla u|^2 dx. \tag{25}$$

Moreover, equality in either of (22) or (23) holds if and only if $u = v$.

Proof. From the Dirichlet principle we know that

$$\int_\Omega \frac{a}{|\nabla v|}|\nabla u|^2 dx \le \int_\Omega \frac{a}{|\nabla v|}|\nabla v|^2 dx, \tag{26}$$

with equality if and only if $u = v$. Now

$$\begin{aligned}\int_\Omega a|\nabla u|dx &= \int_\Omega \frac{a^{1/2}}{|\nabla v|^{1/2}}|\nabla u| a^{1/2}|\nabla v|^{1/2} dx \\ &\le \left(\int_\Omega \frac{a}{|\nabla v|}|\nabla u|^2 dx\right)^{1/2} \left(\int_\Omega a|\nabla v|dx\right)^{1/2} \\ &\le \left(\int_\Omega \frac{a}{|\nabla v|}|\nabla v|^2 dx\right)^{1/2} \left(\int_\Omega a|\nabla v|dx\right)^{1/2} = \int_\Omega a|\nabla v|dx,\end{aligned}$$

where the second inequality uses (26). If equality holds in (22) then we have equality in (26), which can happen only if $u = v$.

To show (23) note that

$$\begin{aligned}\int_\Omega a|\nabla u|dx &= \int_\Omega \frac{a}{|\nabla v|}|\nabla u||\nabla v|dx \\ &\ge \int_\Omega \frac{a}{|\nabla v|}\nabla u \cdot \nabla v dx = \int_\Omega \frac{a}{|\nabla v|}|\nabla u|^2 dx,\end{aligned}$$

where the last equality holds since u is a weak solution of the problem (21) and $u - v \in H_0^1(\Omega)$.

To prove the first inequality in (24) consider the inequality

$$\beta - \frac{\alpha^2}{\beta} \le 2(\beta - \alpha),$$

(valid for all $\alpha \in R$ and $\beta > 0$) with $\alpha = |\nabla u|$ and $\beta = |\nabla v|$. Multiply by $a(x) \ge 0$ and integrate over Ω. The second inequality in (24) is a direct consequence of (23).

From (24), we see that equality in (23) implies equality in (22), which implies $u = v$.

Finally, to show the identity (25) we write

$$\int_\Omega \frac{a}{|\nabla v|}|\nabla v - \nabla u|^2 dx = \int_\Omega \frac{a}{|\nabla v|}(|\nabla v|^2 + |\nabla u|^2 - 2\nabla v \cdot \nabla u) dx$$

and note that, since u is a solution to (21) and $u - v \in H_0^1(\Omega)$, we have

$$\int_\Omega \frac{a}{|\nabla v|}\nabla v \cdot \nabla u dx = \int_\Omega \frac{a}{|\nabla v|}|\nabla u|^2 dx.$$

□

We note that the first inequality in (24) does not require u to be the solution of (21).

Proposition 4.2 *Let Ω be a $C^{1,\alpha}$ simply connected domain in R^2, $a \in C^\alpha(\overline{\Omega})$, $a > 0$, $f \in C^{1,\alpha}(\overline{\Omega})$ with f almost two-to-one on the boundary $\partial\Omega$. Given $u_{n-1} \in C^{1,\alpha}(\overline{\Omega})$ with $\nabla u_{n-1} \neq 0$, construct u_n as the unique solution to (20). Then $u_n \in C^{1,\alpha}(\overline{\Omega})$, $\nabla u_n \neq 0$, $\sigma_n := a/|\nabla u_{n-1}| \in C^\alpha(\overline{\Omega})$ is bounded above and below away from zero and the iteration can proceed.*

Moreover, the sequence $\int_\Omega a|\nabla u_n|dx$ is decreasing and positive and the following limits hold:

$$\lim_{n\to\infty} \langle \Lambda_{\sigma_n} f, f\rangle = \lim_{n\to\infty} \int_\Omega a|\nabla u_n| dx, \tag{27}$$

$$\lim_{n\to\infty} \int_\Omega \sigma_n |\nabla u_{n-1} - \nabla u_n|^2 dx = 0, \tag{28}$$

$$\lim_{n\to\infty} \int_\Omega a|\nabla u_n - \nabla u_{n-1}| dx = 0. \tag{29}$$

Proof. The fact that $u_n \in C^{1,\alpha}(\overline{\Omega})$, is a consequence of the elliptic regularity with Hölder coefficients (see, e.g., Theorem 8.34 in [7]). The choice of boundary data (almost two-to-one maps) ensures that u_n is free of singular points, see [1, 13]. By the inequality (22) we get that the sequence $\int_\Omega a|\nabla u_n|dx$ is decreasing and convergent (being positive). Since

$$\lim_{n\to\infty} \int_\Omega a\left(|\nabla u_{n-1}| - |\nabla u_n|\right) dx = 0,$$

it follows from (24) that

$$\lim_{n\to\infty} \int_\Omega \frac{a}{|\nabla u_{n-1}|}|\nabla u_n|^2 dx = \lim_{n\to\infty} \int_\Omega a|\nabla u_{n-1}| dx. \tag{30}$$

But

$$\int_\Omega \frac{a}{|\nabla u_{n-1}|}|\nabla u_n|^2 dx = \int_{\partial\Omega} u_n \frac{a}{|\nabla u_{n-1}|}\frac{\partial u_n}{\partial \nu} ds = \langle \Lambda_{\sigma_n} f, f\rangle.$$

The limit in (28) now follows from the identity (25) and the equality (30). To prove (29) we estimate using Cauchy's inequality

$$\left(\int_\Omega a|\nabla u_{n-1} - \nabla u_n| dx\right)^2 \leq \int_\Omega a|\nabla u_{n-1}| dx \int_\Omega \frac{a}{|\nabla u_{n-1}|}|\nabla u_n - \nabla u_{n-1}|^2 dx.$$

The first integral is bounded in n, whereas by (28), the second integral converges to zero.

□

The next result provides a sufficient condition under which the algorithm produces a minimizing sequence.

Proposition 4.3 *In addition to the hypotheses of Proposition 4.2, assume that the functions σ_n are uniformly bounded from above; i.e., there exists $\sigma_+ > 0$ such that*

$$\sigma_n \le \sigma_+, \quad \textit{for all} \quad n. \tag{31}$$

Then

$$\lim_{n\to\infty} \int_\Omega a|\nabla u_n| dx = \min\{\int_\Omega a|\nabla u| dx : \ u \in H^1(\Omega), \ u|_{\partial\Omega} = f\}. \tag{32}$$

Proof. Clearly,

$$\lim_{n\to\infty} \int_\Omega a|\nabla u_n| dx \ge \min\{\int_\Omega a|\nabla u| dx : \ u \in H^1(\Omega), \ u|_{\partial\Omega} = f\};$$

we show next the reverse inequality.

Let $u \in H^1(\Omega), \ u|_{\partial\Omega} = f$ be arbitrary. Since for any two vectors $x, y \in R^2$ with $y \ne 0$ we have $|x| - |y| \ge \frac{1}{|y|} y \cdot (x - y)$, it follows that

$$\int_\Omega a(|\nabla u| - |\nabla u_{n-1}|) dx \ge \int_\Omega \frac{a}{|\nabla u_{n-1}|} \nabla u_{n-1} \cdot (\nabla u - \nabla u_{n-1}) dx. \tag{33}$$

We show that the right hand side above converges to zero as $n \to \infty$ and this will complete the proof of (32). Since u_n is a solution to (20) and $u - u_{n-1} \in H_0^1(\Omega)$, we have that

$$\int_\Omega \frac{a}{|\nabla u_{n-1}|} \nabla u_n \cdot (\nabla u - \nabla u_{n-1}) dx = 0. \tag{34}$$

Consider the estimate for the difference between the above and the right side of the inequality (33):

$$\begin{aligned} &\left| \int_\Omega \frac{a}{|\nabla u_{n-1}|} (\nabla u_{n-1} - \nabla u_n) \cdot (\nabla u - \nabla u_{n-1}) dx \right| \\ &\le \left(\int_\Omega \frac{a}{|\nabla u_{n-1}|} |\nabla u_{n-1} - \nabla u_n|^2 dx \right)^{1/2} \left(\int_\Omega \frac{a}{|\nabla u_{n-1}|} |\nabla u - \nabla u_{n-1}|^2 dx \right)^{1/2}. \end{aligned}$$

The first factor converges to zero, by (28). The second factor can be bounded as follows:

$$\begin{aligned} \int_\Omega \frac{a}{|\nabla u_{n-1}|} |\nabla u - \nabla u_{n-1}|^2 dx &= \int_\Omega \frac{a}{|\nabla u_{n-1}|} |\nabla u|^2 dx \\ &\quad + \int_\Omega a|\nabla u_{n-1}| dx - 2 \int_\Omega \frac{a}{|\nabla u_{n-1}|} \nabla u \cdot \nabla u_{n-1} dx \\ &\le \sigma_+ \int_\Omega |\nabla u|^2 dx + \int_\Omega a|\nabla u_1| dx + 2 \int_\Omega a|\nabla u| dx. \end{aligned}$$

□

Proposition 4.4 *In addition to the hypotheses of the Propositions 4.2 and 4.3, assume that the functions σ_n are uniformly bounded from below; i.e., there exists $\sigma_- > 0$ such that*

$$\sigma_- \leq \sigma_n, \quad \textit{for all} \;\; n. \tag{35}$$

Also assume that the sequence $\{F[u_n]\}$ decreases fast enough so that

$$\sum_{n=1}^{\infty} \left(F[u_n] - F[u_{n+1}]\right)^{1/2} < \infty. \tag{36}$$

Then there exists $u_\infty \in H^1(\Omega)$ with $u_\infty|_{\partial\Omega} = f$ and with a negligible set of singular points $\{x \in \Omega : |\nabla u_\infty| = 0\}$ such that

$$\|u_n - u_\infty\|_{H^1(\Omega)} \to 0, \quad \textit{as} \;\; n \to \infty. \tag{37}$$

Moreover, the data (f, a) is admissible and there is a unique generating conductivity σ determined by

$$\sigma = \frac{a}{|\nabla u_\infty|} \in L^\infty_+(\Omega). \tag{38}$$

Proof. We show that the sequence $\{u_n\}$ is Cauchy in $H^1(\Omega)$. Since all the terms have the same boundary value, to prove that the sequence is Cauchy in $H^1(\Omega)$ is equivalent to proving that it is Cauchy in $H^1_0(\Omega)$. Using (24) and (25) we have

$$\|u_n - u_{n+1}\|^2_{H^1_0} \leq \frac{1}{\sigma_-}\int_\Omega \frac{a}{|\nabla u_n|}|\nabla u_n - \nabla u_{n+1}|^2 dx \tag{39}$$

$$\leq \frac{2}{\sigma_-}\int_\Omega a(|\nabla u_n| - |\nabla u_{n+1}|)dx. \tag{40}$$

Taking the square root and summing we obtain

$$\|u_n - u_{n+p}\|_{H^1_0} \leq \sqrt{\frac{2}{\sigma_-}} \sum_{j=1}^{p} \left(\int_\Omega a|\nabla u_{n+j-1}|dx - \int_\Omega a|\nabla u_{n+j}|dx\right)^{1/2}. \tag{41}$$

The assumption (36) thus ensures that $\{u_n\}$ is a Cauchy sequence in $H^1(\Omega)$. Let

$$u_\infty := \lim_{n\to\infty} u_n.$$

From

$$\left|\int_\Omega a(|\nabla u_n| - |\nabla u_\infty|)dx\right| \leq \int_\Omega a|\nabla u_n - \nabla u_\infty|dx \leq \|a\|_{L^2}\|u_n - u_\infty\|_{H^1_0} \tag{42}$$

we obtain

$$\lim_{n\to\infty}\int_\Omega a|\nabla u_n|dx = \int_\Omega a|\nabla u_\infty|dx. \tag{43}$$

We show last that the set $\{x \in \Omega : \; |\nabla u_\infty| = 0\}$ of singular points of u_∞ is of measure zero. From the definition (19) of σ_n and the uniform bounds (31) we have that $|\nabla u_n| \geq \min_\Omega(a)/\sigma_+$. Now $|\nabla u_n| \to |\nabla u_\infty|$ in L^1 hence a.e. in Ω. By Egoroff's Theorem (see, e.g. [14]) for any given $\epsilon > 0$, there is a set B_ϵ of measure less than ϵ

such that on the complement $\Omega_\epsilon := \Omega \setminus B_\epsilon$ we have $|\nabla u_n| \to |\nabla u_\infty|$ uniformly. Since $|\nabla u_\infty| \geq \frac{\min_\Omega(a)}{\sigma_+} + (|\nabla u_\infty| - |\nabla u_n|)$, it follows that on Ω_ϵ

$$|\nabla u_\infty| \geq \frac{\min_\Omega(a)}{\sigma_+}. \tag{44}$$

Since the lower bound in (44) is independent of ϵ it holds on the union $\bigcup_{\epsilon>0} \Omega_\epsilon$. But the set $\Omega \setminus \bigcup_{\epsilon>0} \Omega_\epsilon$ has measure zero since its contained in the intersection of sets of arbitrary small measure. Therefore the bound (44) holds everywhere but on a negligible set. Similarly, from $|\nabla u_\infty| \leq \frac{max_\Omega(a)}{\sigma_-} + |\nabla u_\infty| - |\nabla u_n|$, it follows that a.e. in Ω

$$|\nabla u_\infty| \leq \frac{\max_\Omega(a)}{\sigma_-}. \tag{45}$$

On the one hand, from (44) and (45) we have that $a/|\nabla u_\infty| \in L^\infty_+(\Omega)$. On the other hand, from (32) and (43) we have that u_∞ is a minimizer of the functional in (15). From part (ii) in Proposition 1.2 we have that (a, f) is admissible and $a/|\nabla u_\infty| =: \sigma$ is a generating conductivity. We show next that σ is the unique generating conductivity. Let σ_* be another generating conductivity and u_* be its corresponding potential. According to the part (i) in Proposition 1.2, u_* is also a minimizer of the functional in (15) over all $u \in H^1(\Omega)$ with $u|_{\partial\Omega} = f$. The uniqueness result in the Corollary 1.4 yields $u_\infty = u_*$. Now

$$\sigma = \frac{a}{|\nabla u_\infty|} = \frac{a}{|\nabla u_*|} = \sigma_*.$$

□

The condition (36) assumes lower bounds on the speed with which the functional decreases on the sequence u_n. This speed depends not only on the pair (f, a) but also on the initial guess. For example, if we have an admissible data (f, a), and start the algorithm with the corresponding potential, then the minimizing sequence is constant and the condition (36) is trivially satisfied.

5. Numerical experiments

To check the computational feasibility of the proposed procedure, we perform some numerical experiments. Let $\Omega = (0, 1) \times (0, 1)$. At the boundary we maintain the almost two-to-one voltage potential f, which equals the trace of the harmonic function $u_h(x, y) = y$.

5.1. Simulation of the interior data

We employ the four mode model conductivity distribution

$$\sigma(x, y) = 1 + \sigma_s(x, y),$$

where σ_s is a function with support in Ω, which is given by

$$\sigma_s(x, y) = 0.3 \cdot (\alpha(x, y) - \beta(x, y) - \gamma(x, y)),$$

$$\alpha = 0.3 \cdot (1 - 3(2x-1))^2 \cdot \exp[-9 \cdot (2x-1)^2 - (6y-2)^2],$$

$$\beta = \left(\frac{3(2x-1)}{5} - 27 \cdot (2x-1)^3 - (3 \cdot (2y-1))^5 \right) \\ \times \exp[-(9 \cdot (2x-1)^2 + 9 \cdot (2y-1)^2)],$$

$$\gamma = \exp[-(3 \cdot (2x-1) + 1)^2 - 9 \cdot (2y-1)^2].$$

For σ and f given as above, we solve numerically the problem

$$\nabla \cdot \sigma \nabla u = 0, \quad u|_{\partial\Omega} = f$$

by reducing it to the auxiliary problem for $v = u - u_h$

$$\nabla \cdot \sigma \nabla v = -\frac{\partial \sigma}{\partial y}, \quad v|_{\partial\Omega} = 0.$$

For the Poisson problem above we use the finite element method solver from the Matlab PDE toolbox.

Once the solution u is found, the interior data $a = \sigma|\nabla u|$ is computed, so that the pair (f, a) is admissible. The simulated conductivity distribution computed on a 48×48 grid is shown in the Figure 1 (left). All computations are performed on the Dell workstation T5400 with the quad CPU. Each iteration step in the reconstruction algorithm took about 19.2 seconds.

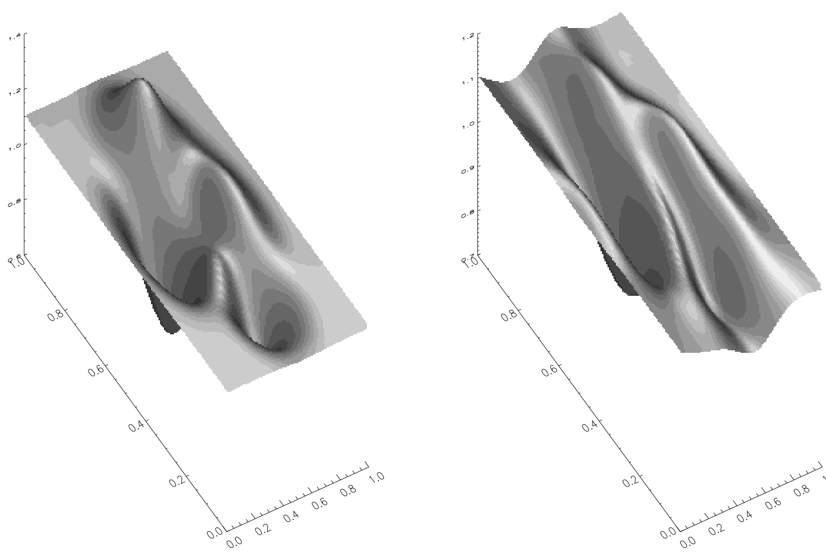

Figure 1. The original conductivity distribution (left) and the initial approximation (right).

5.2. Recovering planar conductivities

To recover the conductivity, we use the iterative procedure of section 4. For $\sigma_0 \equiv 1$ we get $u_0 = u_h$ the harmonic map with boundary values equal to f, and we can start the iteration computations with the initial approximation $\sigma_1 = |J|/|\nabla u_0| = |J|/|\nabla u_h| = |J|$.

Inside each iteration, the Dirichlet problem is solved by the longitudinal-transverse finite difference scheme with an optimal re-ordered set of Chebyshev's parameters (see, e.g., [15]). The numerical differentiation is done by using the three-point Lagrangian interpolation.

In the first experiment, the interior data $|J|$ contains only algorithmic and roundoff errors at levels which do not exceed 10^{-6}. Figure 2 shows the approximations of the conductivity after five and fifty iterates. The relative error as a function of the number of iterates is shown in Figure 6 (bullets). In the second experiment, we use the interior data corrupted by noise. To simulate noisy data, the magnitude of the current density $|J|$ is perturbed by adding a Gaussian pseudo-random matrix with zero mean: we choose the standard deviations to provide the preassigned level of errors of 0.1% and 1%.

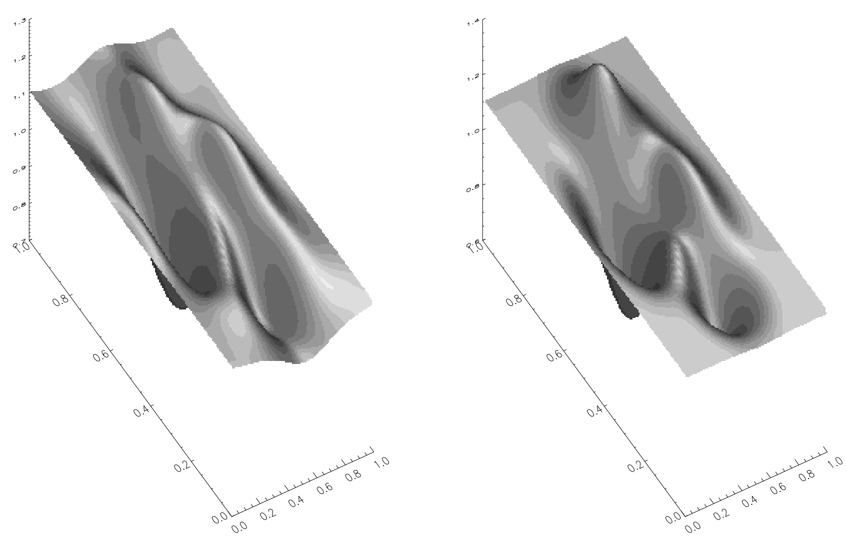

Figure 2. The shaded surfaces of conductivity reconstructed from the noiseless data after 5 (left) and 50 (right) iterates.

To demonstrate quantitatively the numerical convergence, some vertical slices are shown in Figure 3. The results of reconstruction from the perturbed data are shown in Figures 4 and 5. Figure 6 shows the dependence of the relative errors on the number of iterations; the logarithmic scale is used.

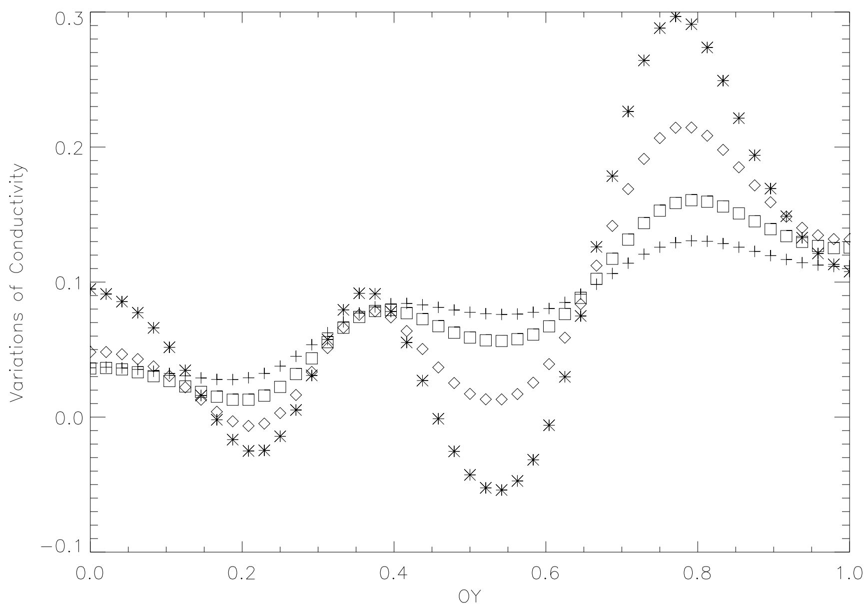

Figure 3. The vertical slices of variations of conductivity over the unit one reconstructed from the noiseless data: crosses - the initial approximation, squares - after 5 iterates, diamonds - after 50 iterates, and stars - after 100 iterates. The latter coincides with the simulated conductivity.

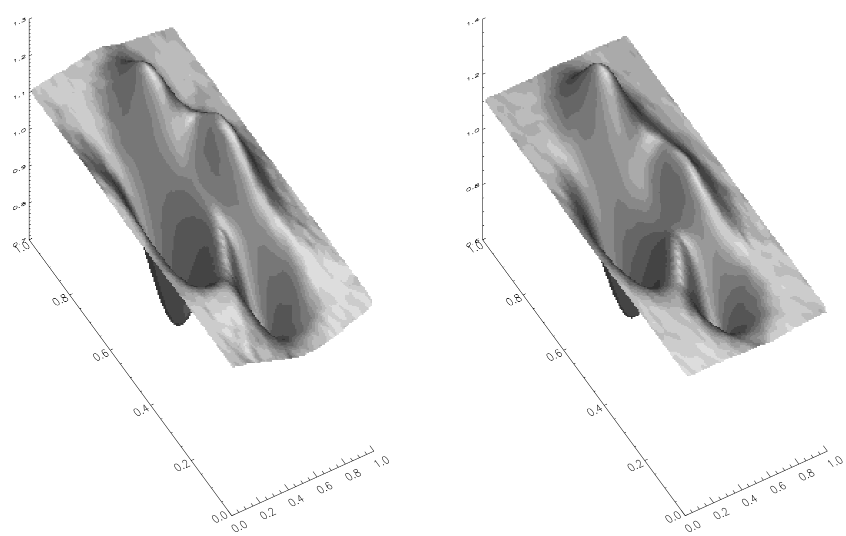

Figure 4. The shaded surfaces of conductivity reconstructed from the perturbed data after 5 (left) and 50 (right) iterates. The noise level was 0.1%.

6. Conclusions

Hölder-continuous conductivities of domains with connected boundary are uniquely determined by the interior measurement of the magnitude of one current density field,

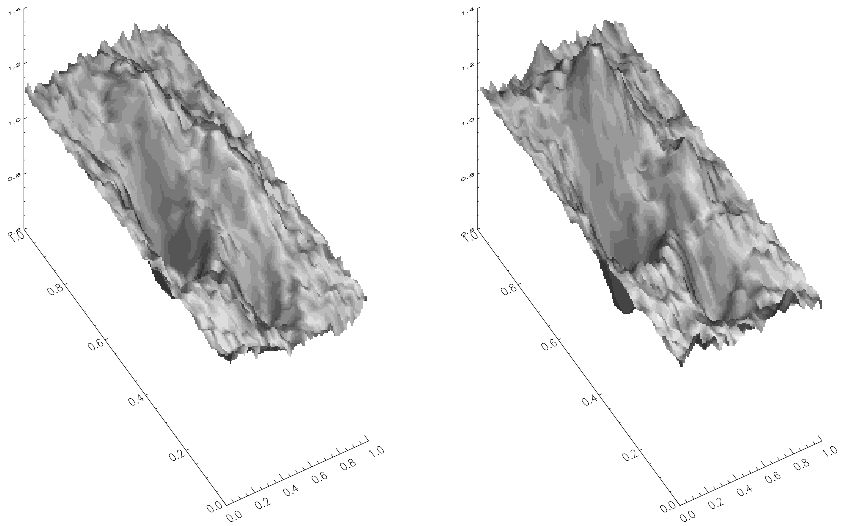

Figure 5. The shaded surfaces of conductivity reconstructed from the perturbed data after 5 (left) and 50 (right) iterates. The noise level is 1%.

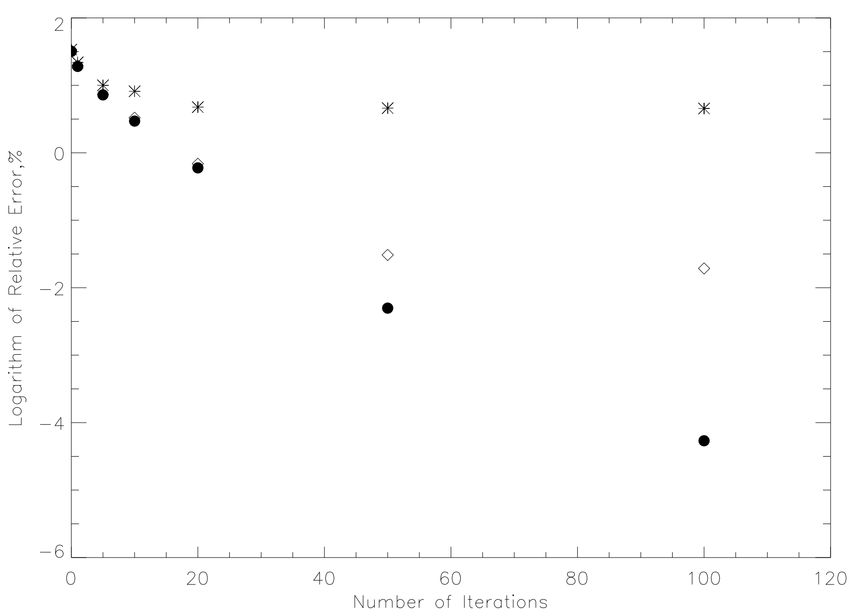

Figure 6. The relative error as a function of the number of iterates for the noiseless (bullets) and perturbed (diamonds = 0.1% and asterisks = 1%) data.

generated while maintaining a given voltage on the boundary. We indicated that such measurements can be obtained from Current Density Imaging, but the discovery that it suffices to measure the magnitude of one current may lead to more direct methodologies to make the physical measurements.

The reconstruction method is reduced to a non-smooth minimization problem. The minimization problem proposed is of independent interest. Note, however, that we only considered the minimization problem (9) for admissible data.

The connectivity of $\partial\Omega$ is essential to our proof of uniqueness. If $\partial\Omega$ is not connected, then it is possible to have level sets of C^1 maps, which do not reach the boundary and such that ∇u never vanishes. As an example, one may consider two different radially symmetric functions in an annulus, which coincide on the two connected components of the boundary.

An algorithm to construct a minimization sequence has been proposed and tested in the two dimensional domains for an almost two-to-one boundary voltage. The algorithm's computational feasibility has been shown in numerical experiments with simulated interior data.

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