ON A NEW APPROACH TO FREQUENCY SOUNDING OF LAYERED MEDIA

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Abstract.
Frequency sounding of layered media is modelled by a hyperbolic problem. Within the framework of this model, we formulate an inverse problem. Applying the Laplace transform and introducing the impedance function, the latter is first reduced to the inverse boundary value problem for the Riccati equation and then to the Cauchy problem for a first order quadratic equation. The advantage of such transformations is that the quadratic equation does not contain an unknown coefficient. For a specific class of data, it is shown that the Cauchy problem is uniquely solvable. Based on the asymptotic behavior of solutions to both the Riccati and quadratic equations, a stable reconstruction algorithm is constructed. Its feasibility is demonstrated in computational experiments.

Key words. frequency sounding, Riccati equation, inverse problem, asymptotic expansions, Cauchy problem

AMS subject classifications. 35J05, 35J25, 35J45, 45E10, 45E99

1. Introduction. The method of frequency sounding was first introduced and developed by Tikhonov [19, 20] and Cagniard [7] in geophysical prospecting of the Earth's crust using electromagnetic waves emitted by natural sources and measuring the surface impedance or admittance. Later on, the method was generalized and applied to a wide variety of problems in areas that include remote sensing, medical imaging, nondestructive testing and evaluation, marine acoustics and electromagnetics, etc. We refer to [10] and the references therein for a survey on the applications of the method.

In this paper, we consider the problem of acoustic frequency sounding of layered media. An acoustic pulse is emitted at $z = 0$ and propagated through the Epstein layer [8], i.e., the half-space $z > 0$ is filled with an inhomogeneous medium whose sound speed is variable in the interval $(0, L)$ and it is constant $c(z) = c_0$ for $z \geq L$.

The propagation and scattering of the pulsed acoustic wave in such a medium can be modelled by the hyperbolic problem

$$c^{-2}(z)U_{tt}(z,t) - U_{zz}(z,t) = 0, \quad z > 0, \quad t > 0, \tag{1.1}$$

$$U(z,0) = U_z(z,0) = 0, \tag{1.2}$$

$$U(0,t) = \delta(t), \tag{1.3}$$

$$|U(z,t)| \leq Ce^{\sigma_0 t}, \quad \forall t > 0, \tag{1.4}$$

where $C = const > 0$ and $\sigma_0 > 0$ is a known frequency. We formulate the following

Inverse Problem I. *Let the function $U(z,t)$ satisfies the hyperbolic problem (1.1)-(1.4). Given the function $U_z(0,t) = \Psi(t), \quad t > 0$, c_0 and L, find the variable sound speed $c(z)$ in $(0, L)$.*

Traditionally, via the Fourier transform, one first reduces the wave equation (1.1) to the Helmholtz equation containing the refraction coefficient $n(z) = c_0/c(z)$ and then study the inverse problem for this equation. In the mathematics literature there

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are several methods available for solving the inverse problem for the Helmholtz equation, which may be subdivided into three groups as follows.

The methods from the first group (see, e.g., [4], [5], [6], [18]) utilize the trace (asymptotic) formulae. For instance, in [5] the combination of the Riccati equation and trace formulae resulted in a system of integro-differential equations with respect to both the unknown impedance function and scattering potential inside the layer. This system was numerically solved by the forth order Runge-Kutta scheme, and the global convergence result was established for a sufficiently smooth scattering potential. In the second group of methods (see, e.g., [17]), the reconstruction was based on reducing the Riccati equation to an equivalent Volterra integral equation via the nonlinear Riesz transform. This ensured the global convergence. In the third group (see, e.g. [11]), the integral transformation

$$u(x,\omega) = \int_0^\infty e^{-\sqrt{i\omega t}} U(z,t)dt$$

was used to formulate an inverse problem for a family of the second order differential equations and to develop the convexification method. This method was shown to have the global convergence to a minimum norm solution only if the ultra-wide band data is used. However, such a condition is more exceptional than typical in practice.

Unlike the methods indicated above, in the proposed approach, we reduce the Inverse Problem I to an inverse problem for a family of Riccati equations. The latter problems is solved via reducing to the Cauchy problem for the first order quadratic differential equation for the sequence valued maps. This reduction is advantageous, because the latter equations do not contain the unknown refraction coefficient. Although it is similar to the techniques indicated in [5] and [11], the proposed approach is distinctive. Unlike in [5], the asymptotic behavior of solutions to the Cauchy problem is determined from the solution of a first order differential equation for sequence-valued maps. This allow for avoiding computations at large frequencies. Unlike in [11], we are not concerned with a minimum norm solution resulted from solving a constrained minimization problem, but rather solve for the exact solution.

It should be mentioned that, assuming $c \in C^2(0,\infty)$ and introducing a new variable $v(x,t) = U(z(x),t)/S(x)$, where $S(x) = \sqrt{c(z(x))/c_0}$ and

$$x(z) = \int_0^z c^{-1}(y)dy,$$

the hyperbolic problem (1.1)-(1.3) can be transformed to the problem

$$\begin{aligned} &v_{tt}(x,t) - v_{xx}(x,t) - a(x)v(x,t) = 0, \quad x > 0, \quad t > 0, \\ &v(x,0) = v_x(x,0) = 0, \\ &v(0,t) = \delta(t), \end{aligned}$$

with $a(x) = S''(x)/S(x) - 2(S'(x)/S(x))^2$. In [16], Romanov studied the problem of recovering the coefficient $a(x)$ from the given function $v_x(0,t) = \Psi(t)c(0) - \delta(t)c'(0)/2$ for $t \in [0, 2l]$ under the assumption

$$v_x(l,t) + Hv(l,t) = 0,$$

where H is a constant. Our formulation of the problem is different, because the presence of the Epstein layer requires the transmission condition

$$v_x(l,t) + c_0^{-1} v_t(l,t) = 0.$$

Moreover, in the proposed approach, knowledge of $c(0)$ and $c'(0)$ is not required.

It should also be mentioned that there is a relationship between the inverse problem studied in [16] and the inverse Sturm-Liouville problem that was extensively studied during the last five decades (see, e.g., [9], [12], [13], [14], [15],[1]).

In this paper we identify a class of data for which the sound speed $c(z)$ can be recovered in the Epstein layer. Based on the asymptotic behavior of the solutions to the Cauchy problem, we develop a stable reconstruction algorithm.

The paper is outlined as follows. In the section 2, we reduce both the forward problem of acoustic frequency sounding of layered media to an initial value problem for a parametric family of the Riccati equations and formulate the inverse problem for the latter. In the section 3, we derive an auxiliary Cauchy and conduct the asymptotic analysis of the solutions to both the Riccati and Cauchy problems. In the section 4, we establish the existence and uniqueness results for the Cauchy problem. The section 5 concerns the numerical experiments demonstrating the computational feasibility of the proposed approach. Finally, the section 6 concludes the paper.

2. The Riccati formalism. Since the function $U(z,t)$ is of the exponential order, we apply the Laplace transform

$$\tilde{u}(z,\sigma) = \int_{\infty}^{0} e^{-\sigma t} U(z,t)dt, \quad z > 0, \quad \sigma > \sigma_0$$

to the hyperbolic problem (1.1)-(1.4) and obtain the parametric family of elliptic boundary value problems

$$\tilde{u}_{zz}(z,\sigma) - \sigma^2 c^{-2}(z) u(z,\sigma) = 0, \quad z > 0, \quad \sigma > \sigma_0, \tag{2.1}$$

$$\tilde{u}(0,\sigma) = 1, \tag{2.2}$$

$$|\tilde{u}(x,\sigma)| \le \frac{C}{\sigma - \sigma_0}. \tag{2.3}$$

Since $c(z) = c_0$ for $z > L$, the traces of its solutions on the interval $(0, L)$ satisfy the transmission problem

$$\tilde{u}_{zz}(z,\sigma) - \sigma^2 c^{-2}(z) u(z,\sigma) = 0, \quad 0 < z < L, \quad \sigma > \sigma_0, \tag{2.4}$$

$$\tilde{u}(0,\sigma) = 1, \tag{2.5}$$

$$\tilde{u}_x(L,\sigma) + \sigma c_0^{-1} \tilde{u}(L,\sigma) = 0. \tag{2.6}$$

Introducing the dimensionless variables $x = z/L,\ s = \sigma L/c_0,\ n(x) = c_0/c(Lx)$ and denoting

$$u(x,s) = \tilde{u}(Lx, c_0 s/L),$$

we arrive to the dimensionless problem

$$u_{xx}(x,s) - s^2 n^2(x) u(x,s) = 0, \quad 0 < x < 1, \quad s > s_0, \tag{2.7}$$

$$u(0,s) = 1, \tag{2.8}$$

$$u_x(1,s) + s u(1,s) = 0, \tag{2.9}$$

where $s_0 = \sigma_0 L/c_0$.

The existence of a unique solution to the above problem follows from classical arguments. Also, the positivity of $u(x,s)$ follows from the fact that a C^2-function has no negative minima at the points where it is concave down. In frequency sounding, it is not u but rather the surface admittance $Y(s) = -u_x(0,s)/u(0,s)$ that is observed. This motivates introducing the impedance function

$$w(x,s) = -\frac{u_x(x,s)}{u(x,s)}, \quad 0 \le x \le 1, \quad s > s_0. \tag{2.10}$$

Then, the problem (2.7)-(2.9) can be reduced to a parametric family of final value problems for the Riccati equation

$$w_x(x,s) = w^2(x,s) - s^2 n^2(x), \quad 0 < x < 1, \quad s > s_0, \tag{2.11}$$

$$w(1,s) = s. \tag{2.12}$$

Also, we have

$$w(0,s) = L\varphi(s),$$

where $\varphi(s)$ is the Laplace transform of the data $U_z(0.t) = \Psi(t)$. Thus, the Inverse Problem I can be reformulated as follows.

Inverse Problem 2. *Let the function $w(x,s)$ be a solution of the problem (2.11), (2.12). Given the function $w(0,s)$ for $s > s_0$, find the refraction coefficient $n(x)$.*

3. Asymptotic analysis. In this section we conduct the asymptotic analysis of problems for the Riccati equation (2.11), (2.12), as well as the auxiliary Cauchy problems derived from the previous ones.

To establish the analyticity of the function $w(x,s)$ in the s variable, assume that $s = s_1 + is_2$ is a complex parameter. For some $\alpha \ge 0$ we denote by H_α the closed half-plane,

$$H_\alpha = \{s \in C : \Re(s) \ge \alpha\}.$$

For a C^2-smooth refraction coefficient we show that the Riccati problem has a unique solution at a sufficiently large frequency. Moreover, this solution is analytic in $s \in H_{s_0}$ uniformly in $x \in [0,1]$ for some $s_0 > 0$. We denote by $\|\cdot\|_\infty$ the sup-norm of bounded functions defined on $[0,1]$.

THEOREM 3.1. *For $n \in C^2[0,1]$, $n(x) \ge n_0 > 0$ and $n(1) = 1$, consider the problem (2.11), (2.12) for each $s \in C$. Let*

$$M = \left\|\frac{n'}{n}\right\|_\infty + \frac{1}{2}\left\|\left(\frac{n'}{n}\right)'\right\|_\infty \quad \text{and } s_0 = \frac{2M}{n_0}. \tag{3.1}$$

Then for any $s \in H_{s_0}$ the problem (2.11), (2.12) has a unique solution $w(\cdot,s) \in C^3[0,1]$ of the form

$$w(x,s) = sn(x) + h(x,s), \tag{3.2}$$

where $h(x,s)$ is a bounded continuous function defined on $[0,1] \times H_{s_0}$, such that

$$\sup_{[0,1]\times H_{s_0}} |h(x,s)| \le 2M, \tag{3.3}$$

and its series representation

$$h(x,s) = \sum_{n=1}^{\infty} h_n(x) \left(\frac{1}{s} - \frac{1}{2s_0}\right)^n \tag{3.4}$$

converges absolutely and uniformly in $x \in [0,1]$.

Proof. By introducing the functions

$$\tilde{w}(x,s) = w(1-x,s), \quad g(x,s) = h(1-x,s), \quad b(x) = n(1-x), \quad x \in [0,1],$$

we transform the final value problems (2.11), (2.12) to the initial value problems

$$\left\{ \begin{array}{l} -\tilde{w}_x(x,s) = \tilde{w}^2(x,s) - s^2 b^2(x), \quad 0 < x < 1 \\ \tilde{w}(0,s) = s. \end{array} \right. \tag{3.5}$$

Then, the ansatz (3.2) can be rewritten as

$$\tilde{w}(x,s) = sb(x) + g(x,s). \tag{3.6}$$

Clearly, the function $\tilde{w}$ is a solution of (3.5) if and only if g solves the family of Riccati problems

$$g_x(x,s) + 2sb(x)g(x,s) + g^2(x,s) = -sb'(x), \quad 0 < x < 1, \tag{3.7}$$

$$g(0,s) = 0. \tag{3.8}$$

Solutions to the Cauchy problem (3.7), (3.8) are the fixed points of the Volterra operator

$$F[g](x,s) = g_0(x,s) - \int_0^x g^2(t,s) \exp\left(-2s \int_t^x b(\tau) d\tau\right) dt \tag{3.9}$$

with

$$g_0(x,s) = -s \int_0^x b'(t) \exp\left(-2s \int_t^x b(\tau) d\tau\right) dt.$$

Integrating by parts, we obtain

$$\begin{aligned} g_0(x,s) =& -\frac{1}{2}\left[\frac{d}{dx}(\ln b)(x) - \frac{d}{dx}(\ln b)(0) \exp\left(-2s \int_0^x b(\tau) d\tau\right)\right] \\ &+ \frac{1}{2} \int_0^x \left(\frac{d^2}{dt^2}(\ln b)(t)\right) \exp\left(-2s \int_t^x b(\tau) d\tau\right) dt. \end{aligned}$$

Since $\Re(s), b(x) > 0$, the exponentials indicated above lie in the unit disk and

$$\sup_{[0,1] \times H_{s_0}} |g_0(x,s)| \leq M,$$

where M is defined in (3.1).

In the space of bounded continuous functions on $[0,1] \times H_{s_0}$ with the sup-norm, consider a closed ball $\overline{B(g_0; M)}$ with the radius M at the center g_0. Now we show that the map $F : \overline{B(g_0; M)} \to \overline{B(g_0; M)}$ is contractive with respect to the Bielecki norm

$$\|g\|_\lambda = \sup_{[0,1] \times [s_0, \infty)} |g(x,s)| e^{-\lambda x}, \tag{3.10}$$

for $\lambda > 2M$.

Indeed, for any $s \in H_{s_0}$, we have

$$\begin{aligned} |Fg(x,s) - g_0(x,s)| &\leq \int_0^x |g(t,s)|^2 \left| \exp\left(-2s\int_t^x b(\tau)d\tau\right)\right| dt \\ &\leq 4M^2 \int_0^x \exp\left(-2\Re(s)\int_t^x b(\tau)d\tau\right) dt \\ &= \frac{2M^2}{\Re(s)b(x)}\left(1-\exp\left(-2\Re(s)\int_0^x b(\tau)d\tau\right)\right) \\ &\leq \frac{2M^2}{\Re(s)n_0} \leq \frac{2M^2}{s_0 n_0} = M. \end{aligned}$$

For $g, \tilde{g} \in \overline{B(g_0; M)}$, we also have

$$\begin{aligned} |Fg(x,s) - F\tilde{g}(x,s)| &\leq \int_0^x |g^2(t,s) - \tilde{g}^2(t,s)| \exp\left(-2\Re(s)\int_t^x b(\tau)d\tau\right) dt \\ &\leq 2M \int_0^x |g(t,s) - \tilde{g}(t,s)| e^{-\lambda t} e^{\lambda t} dt \leq \frac{2M}{\lambda} \|g - \tilde{g}\|_\lambda e^{\lambda x}, \end{aligned}$$

from where we obtain

$$\|Fg - F\tilde{g}\|_\lambda \leq (2M/\lambda)\|g - \tilde{g}\|_\lambda.$$

An application of the contraction mapping principle proves the existence and uniqueness of the solution of (2.11) and (2.12). Because of the uniform bound in (3.3), the asymptotic behavior at large frequencies is proven as well.

To prove the analyticity in $s \in H_{s_0}$, let us consider the first order system

$$\begin{aligned} &\frac{\partial g}{\partial x}(x,s) + 2sb(x)g(x,s) + g^2(x,s) = -sb'(x), \\ &\frac{\partial l}{\partial x}(x,s) + 2[sb(x) + g(x,s)]l(x,s) + 2b(x)g(x,s) = -b'(x), \end{aligned}$$

subject to the initial conditions $g(0,s) = 0$ and $l(0,s) = 0$. The second equation of the system comes from the formal application of the operator $\partial_s = (\partial_{s_1} - i\partial_{s_2})/2$ to (3.7). This system has a unique solution in $[0,1]$ for all $s \in H_{s_0}$. More precisely, g is the unique solution of (3.7) and (3.8), and the function $l(x,s)$ is given by

$$l(x,s) = -\int_0^x [b'(t) + 2b(t)g(t,s)] \exp\left(-2\int_t^x [sb(\tau) + g(\tau,s)]d\tau\right) dt. \tag{3.11}$$

Therefore g is holomorphic in $s \in H_{s_0}$ and $\partial_s g(x,s) = l(x,s)$.

To prove the absolute convergence of the series (3.4), we notice that $z \to \frac{1}{z}$ maps the closed disk $|z - \frac{1}{2s_0}| \leq \frac{1}{2s_0}$ into H_{s_0}. This fact makes the map $z \to h(x, \frac{1}{z})$ analytic in the closed disk, and the Taylor expansion

$$h(x, \frac{1}{z}) = \sum_{n=0}^{\infty} h_n(x) \left(z - \frac{1}{2s_0}\right)^n$$

converges absolutely and uniformly in $x \in [0,1]$. □

Remark 1. The asymptotic formula (3.2) can also be obtained from the WKB solutions of the equation (2.7) by analogy with the method used in [11].

Below, the parameter s is assumed to be real. Let $w(x,s)$ be a solution of (2.11). We introduce a new function

$$p(x,s) = -s^2 \frac{\partial}{\partial s}\left(\frac{w(x,s)}{s^2}\right). \tag{3.12}$$

Since $\lim_{s\to\infty} w(x,s) = 0$, the quotient $p(x,s)/s^2$ is integrable for each $x \in (0,L)$ and

$$\frac{w(x,s)}{s^2} = \int_s^\infty \frac{p(x,\nu)}{\nu^2} d\nu. \tag{3.13}$$

Differentiating the equation (2.11) and taking into account (3.13), one can show that the function p satisfies the integro-differential equation

$$\frac{\partial p}{\partial x}(x,s) = -2s^2 \left(\int_s^\infty \frac{p(x,\nu)}{\nu^2} d\nu\right)\left[s\left(\int_s^\infty \frac{p(x,\nu)}{\nu^2} d\nu\right) - p(x,s)\right]. \tag{3.14}$$

THEOREM 3.2. *Let s_0 and M be as defined in (3.1). The function $p(x,s)$ is represented as a series*

$$p(x,s) = n(x) + \sum_{n=1}^{\infty} \frac{(n+1)p_n(x)}{s^n}, \tag{3.15}$$

which converges absolutely for $s \geq s_0$ and uniformly in $x \in [0,1]$. Moreover, for $s \geq 2s_0$ the following estimates hold

$$|p(x,s) - n(x)| \leq \frac{9M}{s}, \tag{3.16}$$

$$\left|\frac{\partial p}{\partial s}(x,s)\right| < \frac{73M}{2s^2}. \tag{3.17}$$

Proof. It follows from the asymptotic formula (3.2) that

$$\begin{aligned} p(x,s) &= n(x) - \frac{\partial h}{\partial s}(x,s) + \frac{2h(x,s)}{s}, \\ \frac{\partial p}{\partial s}(x,s) &= \frac{\partial^2 h}{\partial s^2}(x,s) + \frac{2}{s}\frac{\partial h}{\partial s}(x,s) - \frac{2h}{s^2}(x,s). \end{aligned} \tag{3.18}$$

Replacing h in (3.18) with its absolutely convergent series representation (3.4) and combining terms of the same order, we obtain the series (3.15). The factor $(n+1)$ in (3.15) is merely a scaling.

If $s \geq 2s_0 = 4M/a_0$, then $sb(t) + g(t,s) \geq sa_0 - 2M \geq sa_0/2 > 0$. The bounds

$$\sup_{[0,1]\times[2s_0,\infty)} \left|\frac{\partial h}{\partial s}(x,s)\right| \leq \frac{5M}{s}, \tag{3.19}$$

$$\sup_{[0,1]\times[2s_0,\infty)} \left|\frac{\partial^2 h}{\partial s^2}(x,s)\right| \leq \frac{65M}{2s^2} \tag{3.20}$$

can be obtained from the relations

$$\frac{\partial g}{\partial s}(x,s) = -\int_0^x \frac{b'(t) + 2b(t)g(t,s)}{2sb(t) + 2g(t,s)} \cdot \frac{d}{dt}\left\{\exp\left(-2\int_t^x [sb(\tau) + g(\tau,s)]d\tau\right)\right\} dt,$$

$$\frac{\partial^2 g}{\partial s^2}(x,s) = \int_0^x \frac{2b(t)g_s(t,s)+g_s^2(t,s)}{sb(t)+g(t,s)} \cdot \frac{d}{dt}\left\{\exp\left(-2\int_t^x [sb(\tau)+g(\tau,s)]d\tau\right)\right\} dt$$

and from the fact that

$$\frac{d}{dt}\left\{\exp\left(-2\int_t^x [sb(\tau)+g(\tau,s)]d\tau\right)\right\} > 0.$$

The estimates (3.16) and (3.17) follow directly from (3.3), (3.19) and (3.20).
□

4. Existence and uniqueness of a solution to the Cauchy problem. In the previous section, it was shown that introducing the new variables w and q and performing the transformations $u \to w \to p$, one can reduce the second-order elliptic equation (2.7) to the integro-differential equation (3.14). The most attractive feature of the latter is that it does not contain the unknown refraction coefficient $n(x)$. The entire information about $n(x)$ is contained in the boundary data

$$p(0,s) = -Ls^2 \frac{\partial}{\partial s}\left(\frac{\varphi(s)}{s^2}\right).$$

Therefore, solving efficiently the Cauchy problem for (3.14) is crucial. Motivated by the series representation (3.15), we seek the solution in the form

$$p(x,s) = \sum_{n=0}^{\infty} \frac{(n+1)p_n(x)}{s^n}, \tag{4.1}$$

where

$$p(0,s) = \sum_{n=0}^{\infty} \frac{(n+1)b_n}{s^n} \tag{4.2}$$

More precisely, we look for the solutions in the Banach space X of the sequence valued maps $x \to \mathbf{p}(x) = (p_0(x), p_1(x), p_2(x), ...)$ with the norm

$$\|\mathbf{p}\|_a = \sup_{n\ge 1} \sup_{x\in[0,1]} |p_n(x)| a^{-n}(1+x)^{-n} < \infty,$$

where $a > 0$ is constant to be specified below.

Since

$$\left|\sum_{n=0}^{\infty} \frac{(n+1)p_n(x)}{s^n}\right| \le C \sum_{n=0}^{\infty} (n+1)\left(\frac{2a}{s}\right)^n,$$

the series (4.1) converges uniformly for $s \ge s_0 > 2a$ and $0 \le x \le 1$. Because of the uniform continuity, substituting (4.1) in (3.14), we obtain the equivalent initial value problem for the map $x \to \mathbf{p}(x)$

$$p_n'(x) = \frac{2}{n+1}\sum_{k=0}^{n} (k+1)p_{k+1}(x)p_{n-k}(x), \tag{4.3}$$

$$p_n(0) = b_n, \quad (n = 0, 1, 2, ...)$$

The following theorem establishes the existence and uniqueness results.

THEOREM 4.1. *Let the initial data b_n in the problem (4.3) satisfies the condition*

$$|b_n| \leq Ra^n, \quad n \geq 0, \tag{4.4}$$

for some a and $R > 0$ with

$$16Ra < 1. \tag{4.5}$$

Then the problem (4.3) has a unique solution, which can obtained by Picard successive approximations with the initial approximation $\mathbf{b} = (b_0, b_1, ...)$.

Proof.

Let $\overline{B(\mathbf{b}; R)} \subset X$ be the closed ball with the radius R and the center at $\mathbf{b}$, such that any $\mathbf{p} \in \overline{B(\mathbf{b}; R)}$ has $\|\mathbf{p}\|_a \leq \|\mathbf{b}\|_a + R \leq 2R$, or

$$|p_n(x)| \leq 2Ra^n(1+x)^n, \; n \geq 0.$$

Solutions $\mathbf{p}$ of the problem (4.3) are the fixed points of the Volterra operator $T : X \to X$ defined for all $n \geq 0$ by

$$(T\mathbf{p})_n(x) = b_n + \frac{2}{n+1}\sum_{k=0}^{n}(k+1)\int_0^x p_{k+1}(t)p_{n-k}(t)dt.$$

We first show that $T : \overline{B(\mathbf{b}; R)} \to \overline{B(\mathbf{b}; R)}$. Indeed, for any $\mathbf{p} \in \overline{B(\mathbf{b}; R)}$ we have

$$\begin{aligned} |(T\mathbf{p} - \mathbf{b})_n(x)| \leq & \frac{2}{n+1}\sum_{k=0}^{n}(k+1)\int_0^x |p_{k+1}(t)| \cdot |p_{n-k}(t)|dt \\ \leq & \frac{2(2R)^2 a^{n+1}}{n+1}\sum_{k=0}^{n}(k+1)\int_0^x (1+t)^{n+1}dt \\ \leq & \, 4R^2 a^{n+1}(1+x)^{n+2}. \end{aligned}$$

From where

$$\|(T\mathbf{p} - \mathbf{b})\|_a \leq 16R^2 a \leq R.$$

Now we show that the operator T is also a contraction on $\overline{B(\mathbf{b}; R)}$. Indeed, for $\mathbf{p}, \mathbf{q} \in \overline{B(\mathbf{b}; R)}$ and $n \geq 0$ we estimate

$$\begin{aligned} |(T\mathbf{p} - T\mathbf{q})_n(x)| \leq & \frac{2}{n+1}\sum_{k=0}^{n}(k+1)\int_0^x |p_{k+1}(t) - q_{k+1}(t)| \cdot |p_{n-k}(t)|dt \\ & + \frac{2}{n+1}\sum_{k=0}^{n}(k+1)\int_0^x |p_{n-k}(t) - q_{n-k}(t)| \cdot |q_{k+1}(t)|dt \\ \leq & 4Ra^{n+1}(1+x)^{n+2}\|\mathbf{p} - \mathbf{q}\|_a. \end{aligned}$$

Therefore, the operator T is contractive with a Lipschitz constant bounded by $16Ra < 1$. An application of the contraction mapping principle on the ball $\overline{B(\mathbf{b}; R)} \subset X$ finishes the proof. □

Remark 2. The assumptions (4.4) and (4.5) can be replaced by

$$|b_n| < \frac{1}{16}a^{n-1}, n \geq 0, \tag{4.6}$$

for some $a > 0$. In particular, we need $|b_1| < 1/16$ regardless of the value of a. However, different choices of a can relax the requirement on b_0 at the expense of faster decay on b_n's for $n \geq 2$. This property is due to a scaling symmetry in the integro-differential equation (3.14). For $\mu > 0$, let S_μ be the scaling operator defined by

$$[S_\mu p](x, s) = \mu p(x, \mu s).$$

By changing the variables in the integrals in (3.14), we establish the following scaling lemma.

LEMMA 4.2. *If $p(x, s)$ is a solution of the integro-differential equation (3.14), then $S_\mu p(x, s)$ is also a solution for any $\mu > 0$.*

Clearly, if the coefficients $(b_0, b_1, ..., b_n, ...)$ correspond to the initial data for a solution p, then the coefficients $(\mu b_0, b_1, b_2/\mu, ...)$ correspond to the initial data for a solution $S_\mu p$ and the class of initial data for which the system (4.3) has a unique solution is thus enlarged.

5. Numerical experiments. The results indicated in the previous sections serve as a basis for the algorithm development. In this section we present some results of numerical experiments demonstrating the computational feasibility of the reconstruction algorithm.

To simulate the data $w(0, s) = -u_x(0, s)/u(0, s)$, for a given $n(x)$, we solve numerically the forward problem (2.7)-(2.9) by the finite-difference method. The s-parametric family of the three-diagonal systems of linear equations resulted from approximating the differential operator of the second order on a grid was solved by a special form of the Gauss method for the systems with three-diagonal matrices.

We tested both asymptotic formulae (3.2) and (3.15) by performing the transformations (2.10) and (3.12) for three different frequency bands $[s_1, s_2]$ as follows. The parameter $s_1 > s_0$ was first chosen. Then its upper bound was determined as $s_2 = \beta s_1,\ \beta > 1$. The three types of the data are: (1) the narrow band (NB) data (for $\beta = 1.5$), (2) the wide band (WB) data (for $\beta = 15$), and (3) the ultra-wide band (UWB) data (for $\beta = 150$). Figures 5.1 and 5.2 show the graphs of the functions $w(x, s_2)/s_2$ and $p(x, s_2)$ for all types of the data. It can be seen from these figures that in accordance with (3.2) and (3.15), both functions $w(x, s)/s$ and $p(x, s)$ converge to the refraction coefficient $n(x)$ as $s \to \infty$.

The reconstruction algorithm is based on two facts. On one hand, following (3.13) and (3.15), we know that the data

$$w(0, s) = s^2 \int_s^\infty \frac{p(0, \nu)}{\nu^2} d\nu \tag{5.1}$$

has a converging series representation:

$$w(0, s) = s b_0 + b_1 + \frac{b_2}{s} + \frac{b_3}{s^2} + \frac{b_4}{s^3} + ... + \frac{b_{n+1}}{s^n} + ... \tag{5.2}$$

On the other hand, we have the initial value problem for the system of the first order ODEs (4.3)

$$\begin{aligned} p_0'(x) &= 2p_0 p_1, \\ p_1'(x) &= p_1 p_1 + 2p_0 p_2, \\ p_2'(x) &= \frac{2}{3}(p_1 p_2 + 2p_2 p_1 + 3p_0 p_3), \end{aligned} \tag{5.3}$$

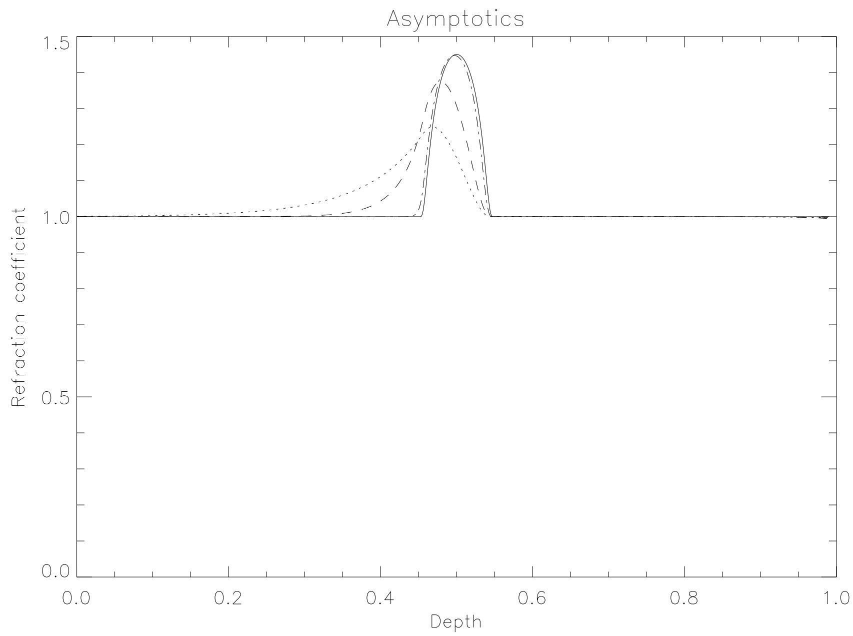

FIG. 5.1. *The function $w(x, s_2)/s_2$ for the different data: (1) the NB data (dotted), (2) the WB data (dashed), (3) the UWB data (dashed/dotted). The graph of the refraction coefficient $n(x)$ is shown by the solid line.*

$$
\begin{aligned}
&p_3'(x) = \frac{1}{2}(p_1p_3 + 2p_2p_2 + 3p_0p_4),\\
&\text{...},\\
&p_n'(x) = \frac{2}{n+1}(p_1p_n + 2p_2p_{n-1} + ... + np_np_1 + (n+1)p_{n+1}p_0),\\
&\text{...},\\
&p_0 = b_0,\ p_1 = b_1, ..., p_n = b_n, ...
\end{aligned}
$$

These facts motivate the following reconstruction procedure. We first fit the given data $w(0, s)$ by truncating the series in the right-hand side in (5.2) up to the $(n+2)$-nd term. As a result, we obtain the coefficients $b_0, b_1, ..., b_{n+1}$. The main difficulty in solving the system (5.3) is that the n-th ODE contains the term p_{n+1} in the right-hand side. To overcome this difficulty, we assign $p_{n+1}(x) \approx b_{n+1}$ and truncate the system (5.3) by the n-th equation. The resulting system is then solved by the Runge-Kutta method of the forth order. Once the coefficients $p_0, p_1, ..., p_n$ are found, an approximate refraction coefficient is determined as

$$n(x) \simeq p_0(x).$$

However, Theorem 4.1 is valid provided that the coefficients b_n satisfy (4.6). Due to (5.2), such an assumption restricts the class of refractions coefficients for which the reconstruction procedure is meaningful. To provide the consistency, we simulated a refraction coefficient n within the range condition (4.6). For a given sequence $\{b_0, b_1, ..., b_{n+1}\}$ satisfying (4.6) we solved the system (5.3) and assign $n(x) = p_0(x)$. The refraction coefficient shown with a solid line in Figure 5.3 is obtained from the finite sequence $b_0 = 1,\ b_1 = 1/2,\ b_2 = 1/32,\ b_3 = 1/128,\ b_4 = 1/512.$

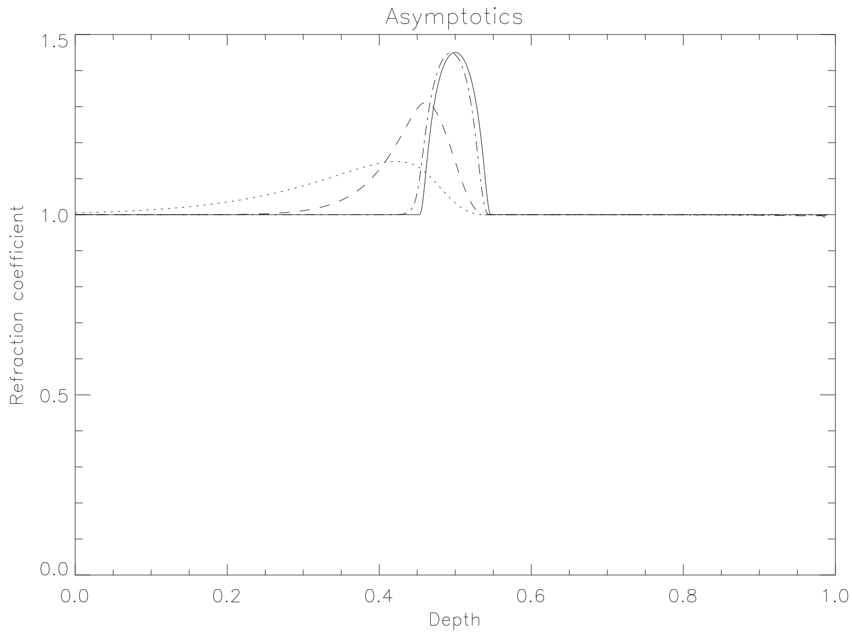

FIG. 5.2. *The function $p(x, s_2)$ for the different data: (1) the NB data (dotted), (2) the WB data (dashed), (3) the UWB data (dashed/dotted). The graph of the refraction coefficient $n(x)$ is shown by the solid line.*

Given the refraction coefficient $n(x)$, we solved the forward problem (2.7)-(2.9) to simulate the data $w(0, s)$. Then, following the reconstruction procedure, we did a non-linear least squares fit to a function indicated in (5.2). In particular, we used the Levenberg-Marquardt algorithm, which combined the steepest descent and inverse-Hessian function fitting method. Figure 5.3 shows the results of the reconstruction for the three types of the frequency data used in the fitting step.

6. Conclusions. We proposed a new approach for solving the inverse problem of frequency sounding of layered media. Unlike the existing methods, we reduce an original inverse problem to a family of the Cauchy problems for the Riccati equation and then to the Cauchy problem for a first order differential equation for the sequence valued maps. The latter equation does not contain the unknown coefficient. We studied the asymptotic behavior of solutions to both the Riccati and quadratic equations and established the existence and uniqueness results for the reduced inverse problem with the specified data. Moreover, we developed a reconstruction algorithms based on the analyticity (in the s variable) of the data. Utilizing the Runge-Kutta method of the forth order, this algorithm is fast and easy to implement. The numerical results have shown that an accurate reconstruction of the refraction coefficient can be obtained for a specific class of the data.

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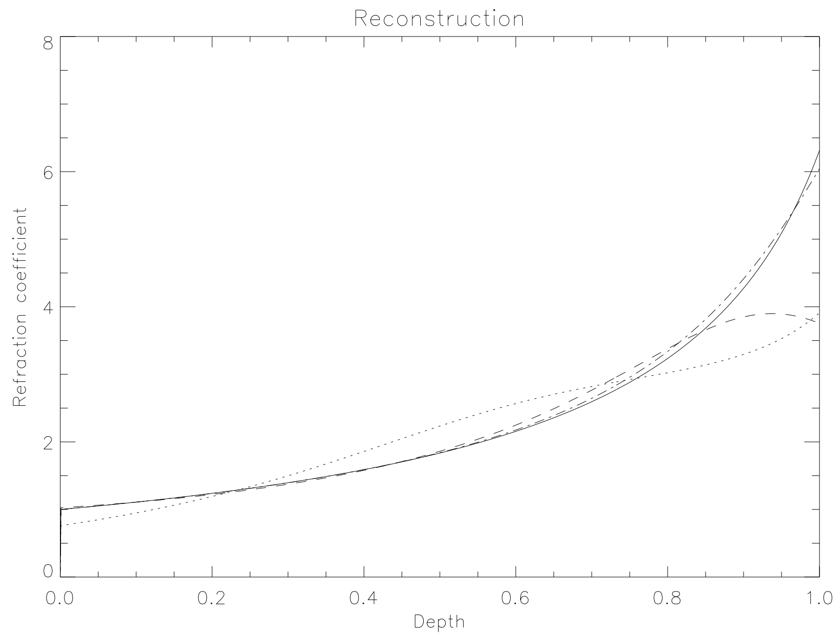

Fig. 5.3. *Reconstruction of the refraction coefficient from: (1) the NB data (dotted), (2) the WB data (dashed), (3) the UWB data (dashed/dotted). The graph of the simulated refraction coefficient $n(x)$ is shown by the solid line.*

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