A CALDERÓN PROBLEM WITH FREQUENCY-DIFFERENTIAL DATA IN DISPERSIVE MEDIA

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ABSTRACT. We consider the problem of identifying a complex valued coefficient $\gamma(x, \omega)$ in the conductivity equation $\nabla \cdot \gamma(\cdot, \omega)\nabla u(\cdot, \omega) = 0$ from knowledge of the frequency differentials of the Dirichlet-to-Neumann map. For a frequency analytic $\gamma(\cdot, \omega) = \sum_{k=0}^{\infty}(\sigma_k + i\epsilon_k)\omega^k$, in three dimensions and higher, we show that $\frac{d^j}{d\omega^j}\Lambda_{\gamma(\cdot,\omega)}\big|_{\omega=0}$ for $j = 0, 1, ..., N$ recovers $\sigma_0, ...\sigma_N$ and $\epsilon_1, , , , \epsilon_N$. This problem arises in frequency differential electrical impedance tomography of dispersive media.

1. INTRODUCTION

Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$, be a bounded domain with Lipschitz boundary Γ. For some $\omega_0 > 0$ and $c > 0$, let $\gamma \in C([0, \omega_0]; L^{\infty}(\Omega))$ be complex valued satisfying

$$(1) \qquad \mathbb{Re}(\gamma(x, \omega)) \geq c > 0, \ (x, \omega) \in \Omega \times [0, \omega_0].$$

For each fixed ω and $f \in H^{1/2}(\Gamma)$ the Dirichlet problem

$$(2) \qquad \nabla \cdot \gamma(\cdot, \omega)\nabla u(\cdot, \omega) = 0 \text{ in } \Omega, \ u(\cdot, \omega)|_{\Gamma} = f,$$

has a unique solution. The Dirichlet-to-Neumann map

$$\Lambda_{\gamma(\cdot, \omega)} : H^{1/2}(\Gamma) \to H^{-1/2}(\Gamma)$$

is a bounded operator defined by

$$(3) \qquad \Lambda_{\gamma(\cdot, \omega)}(f) := \mathbf{n} \cdot \gamma(\cdot, \omega)\nabla u(\cdot, \omega)|_{\Gamma},$$

where $\mathbf{n}$ is the outer normal to the boundary. Moreover the map

$$\omega \mapsto \Lambda_{\gamma(\cdot, \omega)} \in C([0, \omega_0]; \mathcal{L}(H^{\frac{1}{2}}(\Gamma), H^{-\frac{1}{2}}(\Gamma))),$$

where $\mathcal{L}(H^{\frac{1}{2}}(\Gamma), H^{-\frac{1}{2}}(\Gamma))$ denotes the space of linear operators endowed with the strong topology.

The Calderón problem refers to the determination of $\gamma(\cdot, \omega)$ from $\Lambda_{\gamma(\cdot, \omega)}$, for each fixed frequency ω. At $\omega = 0$ this is the original question in

[6], which has been mostly settled in the affirmative: In dimensions $n \geq 3$, of direct relevance here, we mention the breakthrough result in [22], where $\Lambda_{\gamma(\cdot,0)}$ is shown to uniquely determine $\gamma(\cdot, 0)$, and the reconstruction method in [16]. Although not stated, these results extend to $\omega \neq 0$, case in which the coefficient $\gamma(\cdot, \omega)$ becomes complex valued. The analogous two dimensional problem (also at $\omega = 0$) was cracked in [17], refined in [3] (with reconstruction in [14]) and completely settled in [1]. The breakthrough result in [5] settles in two dimensions the uniqueness part: $\Lambda_{\gamma(\cdot,\omega)}$ uniquely determines $\gamma(\cdot, \omega)$ for each fixed ω (an earlier result [9] assumed $\gamma = \sigma_0(x) + i\omega\epsilon_0(x)$ with sufficiently small $\omega\epsilon_0$). For an understanding on the breadth of development of the Calderón problem we refer to the reviews [7], [4], [10], [2], and [23].

Recent biomedical advances [11, 12, 13, 18, 19] allow to measure discrepancies in boundary data at different frequencies, and produce images based on such differential data. Given that biological materials have frequency dependent electric-magnetic properties (they are dispersive) [8] and only one image is produced from multiple frequencies, the following questions arise:

What can be determined from the frequency derivative

$$\left.\frac{d^k}{d\omega^k}\Lambda_{\gamma(\cdot,\omega)}\right|_{\omega=\omega_0}$$

for a fixed order $k \neq 0$, or for several orders? Can one expect quantitative information from multi-frequency data? What do the images represent?

We address these questions when the coefficient $\gamma(\cdot, \omega)$ depends analytically in ω in dimensions three or higher. In [21] the authors considered the case $k = 1$ and the ansatz $\gamma = \sigma_0(x) + i\omega\epsilon_1(x)$ specific to media with frequency-independent electrical properties (i.e., non-dispersive).

For some $r, c > 0$, let us define the class $\mathcal{A}_{r,c}$ of functions $\gamma(x, \omega)$ such that

$$(4) \qquad \mathbb{Re}(\gamma(x,\omega)) = \sum_{n=0}^{\infty} \sigma_n(x)\omega^n, \quad \mathbb{Im}(\gamma(x,\omega)) = \sum_{n=1}^{\infty} \epsilon_n(x)\omega^n,$$

where $\sigma_0 \in C^{1,1}(\Omega)$ is real valued and constant near the boundary with

$$(5) \qquad\qquad 0 < c^{-1} \leq \sigma_0(x) \leq c,$$

and $\sigma_k, \epsilon_k \in C_0^{1,1}(\Omega)$ are real valued and compactly supported in Ω with

$$(6) \qquad\qquad \left\|\frac{\sigma_k}{\sigma_0}\right\|_\infty \leq r, \ \left\|\frac{\epsilon_k}{\sigma_0}\right\|_\infty \leq r, \ \forall k \geq 1.$$

Note that $\omega \mapsto \gamma(\cdot, \omega)$ is analytic in $|\omega| < 1/r$ with values in $L^\infty(\Omega)$,

We prove the following:

Theorem 1.1. *Let $n \geq 3$ and $\Omega \subset \mathbb{R}^n$ be open with $C^{1,1}$ boundary. Assume that $\gamma \in \mathcal{A}_{r,c}$, for some $r, c > 0$ and $N \geq 1$ be fixed. Then $\frac{d^k}{d\omega^k}\Lambda_{\gamma(\cdot,\omega)}\big|_{\omega=0}$ for $k = 0, ..., N$, uniquely determine $\sigma_0, \sigma_1, ..., \sigma_N$, and $\epsilon_1, ..., \epsilon_N$ inside Ω.*

Consequently, we also obtain:

Corollary 1.1. *Let $n \geq 3$ and $\Omega \subset \mathbb{R}^n$ be open with $C^{1,1}$ boundary. Assume $\gamma \in \mathcal{A}_{r,c}$, for some $r, c > 0$. Then the map $\omega \mapsto \Lambda_{\gamma(\cdot,\omega)}$ given in an arbitrarily small neighborhood of some fixed $\omega_0 \in [0, (1 + 2r)^{-1})$ recovers γ on $\overline{\Omega} \times [0, r^{-1})$.*

The proof of Theorem 1.1 is based on the complex geometrical optic solutions of Sylvester-Uhlmann in [22], whose existence is recalled below to establish notations. For $\mathbf{k}, \eta, \mathbf{l} \in \mathbb{R}^n$ with $\mathbf{k} \cdot \eta = \mathbf{k} \cdot \mathbf{l} = \mathbf{k} \cdot \eta = 0$, and $|\eta|^2 = \frac{|\mathbf{k}|^2}{4} + |\mathbf{l}|^2$, consider the vectors

$$(7) \quad \xi_1(\eta, \mathbf{k}, \mathbf{l}) := \eta - i\left(\frac{\mathbf{k}}{2} + \mathbf{l}\right), \quad \xi_2(\eta, \mathbf{k}, \mathbf{l}) := -\eta - i\left(\frac{\mathbf{k}}{2} - \mathbf{l}\right),$$

which satisfy $\xi_1 \cdot \xi_1 = \xi_2 \cdot \xi_2 = 0$, $|\xi_1|^2 = |\xi_2|^2 = 2\left(\left|\frac{\mathbf{k}}{2}\right|^2 + |\mathbf{l}|^2\right)$, and $\xi_1 + \xi_2 = -i\mathbf{k}$. The result is stated in the equivalent form:

Existence of CGOs (Theorem 2.3, [22]): *Let $n \geq 3$, and $\sigma_0 \in C^{1,1}(\mathbb{R}^n)$ be constant outside Ω. For $-1 < \delta < 0$ there are two constants $R, C > 0$ dependent only on δ, $\|\Delta\sqrt{\sigma_0}/\sqrt{\sigma_0}\|_{L^\infty(\Omega)}$, and Ω such that the following holds: For $\xi_j \in \mathbb{C}^n$ as in (7) with $|\mathbf{l}| > R$, there exist $w_j := w(\cdot, \xi_j) \in H^1_{loc}(\mathbb{R}^n)$ solutions in $\mathbb{R}^n$ of*

$$\nabla \cdot \sigma_0 \nabla w_j = 0$$

of the form

$$(8) \qquad w_j(x) = e^{x \cdot \xi_j}\sigma_0^{-1/2}(1 + \psi_j(x)),$$

with

$$(9) \qquad \|\psi_j\|_{L^2_\delta} \leq \frac{C}{|\xi_j|}, \quad j = 1, 2,$$

where $\|f\|^2_{L^2_\delta} := \int_{\mathbb{R}^n} |f(x)|^2(1 + |x|^2)^\delta dx$.

The assumptions on the coefficients may be relaxed as follows. The boundary values $\sigma_0|_\Gamma$ and $\frac{\partial\sigma_0}{\partial\mathbf{n}}\big|_\Gamma$ can be recovered from $\Lambda_{\gamma(\cdot,0)}$ as shown in [16] for this regularity (and earlier in [15] and [22] for C^∞-regular). Then σ_0 can be extended outside Ω with preserved regularity while making it constant near the boundary $\tilde{\Gamma}$ of a neighborhood of the original domain. Similarly, the compact support assumption on σ_n, ϵ_n, for $n = 1, 2, ...$ can be replaced by the knowledge of the boundary values of $\sigma_n|_\Gamma, \epsilon_n|_\Gamma, \frac{\partial\sigma_n}{\partial\mathbf{n}}\big|_\Gamma$, and $\frac{\partial\epsilon_n}{\partial\mathbf{n}}\big|_\Gamma$. With the same proof as in [17], the Dirichlet-to-Neumann map

$\Lambda_{\gamma(\cdot,\omega)}$ transfers from Γ to $\tilde{\Gamma}$ and we can consider the problem on the larger domain.

2. A FREQUENCY RANGE FOR ANALYTIC DEPENDENCE

In this section we show analytic dependence of the map $\omega \mapsto \Lambda_{\gamma(\cdot,\omega)}$ in the space of linear operators from $H^{1/2}(\Omega)$ to $H^{-1/2}(\Omega)$ endowed with the strong topology, and introduce two operators needed in the proof of Theorem 1.1.

Existence and uniqueness of solution for the conductivity problem (2) with a complex coefficient has been known for some time; e.g., in [20] the Lax-Milgram lemma is employed. Below we provide a solution which is frequency-analytic. Since the solution of (2) is unique, the result below also shows the analytic dependence of the solution of (2) for a frequency-analytic coefficient.

Theorem 2.1. *Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$, be a domain with Lipschitz boundary. Consider the Dirichlet problem (2) for $f \in H^{1/2}(\Gamma)$ real valued and*

$$(10) \qquad \gamma(x,\omega) := \sigma_0(x) + \sum_{k=1}^{\infty}(\sigma_k(x) + i\epsilon_k(x))\omega^n,$$

where σ_0 is real valued satisfying (5) for some $c > 0$, and $\sigma_k, \epsilon_k \in L^{\infty}(\Omega)$, $k \geq 1$, are real valued satisfying (6) for some $r > 0$. Then, for

$$(11) \qquad |\omega| < \frac{1}{1+2r},$$

the Dirichlet problem (2) has a unique solution $u(\cdot,\omega) \in H^1(\Omega)$ with the series representation

$$(12) \qquad \mathbb{Re}\{u(x,\omega)\} = \sum_{k=0}^{\infty} v_k(x)\omega^k, \quad \mathbb{Im}\{u(x,\omega)\} = \sum_{k=0}^{\infty} h_k(x)\omega^k$$

absolutely convergent in $H^1(\Omega)$, where $h_0 \equiv 0$, v_0 solves

$$(13) \qquad \nabla \cdot \sigma_0 \nabla v_0 = 0 \text{ in } \Omega, \quad v_0|_{\Gamma} = f,$$

and, for $k = 1, 2, ...,$ the following recurrence holds:

$$(14) \quad \nabla \cdot \sigma_0 \nabla v_k = -\sum_{j=1}^{k} \nabla \cdot (\sigma_j \nabla v_{k-j} - \epsilon_j \nabla h_{k-j}) \text{ in } \Omega, \quad v_k|_{\Gamma} = 0,$$

$$(15) \quad \nabla \cdot \sigma_0 \nabla h_k = -\sum_{j=1}^{k} \nabla \cdot (\sigma_j \nabla h_{k-j} + \epsilon_j \nabla v_{k-j}) \text{ in } \Omega, \quad h_k|_{\Gamma} = 0.$$

We stress here that the recurrence (14) and (15) uses the reality of the Dirichlet boundary data f. The proof of the theorem is a consequence of the estimate in Lemma 2.1 below. We start by a simple remark, which can be justified by induction.

Remark 2.1. *Let* $\{a_k\}_{k=0}^\infty$ *be a sequence of non-negatives and* $m > 0$ *such that* $a_1 \le a_0$ *and*

$$a_k \le a_0 + m(a_1 + ... + a_{k-1}), \quad \forall k \ge 2.$$

Then

$$a_k \le (1+m)^k a_0, \quad \forall k \ge 0.$$

We denote by $\| \cdot \|$ the $L^2(\Omega)$-norm, and by $\| \cdot \|_1$ the $H^1(\Omega)$-norm.

Lemma 2.1. *Let* v_0 *satisfy* (13) *and assume the sequences* $\{v_n\}_{n=1}^\infty$, $\{h_n\}_{n=1}^\infty$ *in* $H^1(\Omega)$ *satisfy the corresponding problems* (14) *and* (15). *Then, for* $k = 1, 2, ...$ *we have*

(16) $$\|v_k\|_1 + \|h_k\|_1 \le c(1 + 2r)^k \|\nabla v_0\|.$$

Proof. Let k be a fixed positive index. Since $v_k|_\Gamma = 0$, by applying the divergence theorem to (14) we estimate

$$\|\sqrt{\sigma_0}|\nabla v_k|\|^2 = -\sum_{j=1}^k \int_\Omega (\sigma_j \nabla v_{k-j} - \epsilon_j \nabla h_{k-j}) \cdot \nabla v_k dx$$

$$\le r \sum_{j=1}^k \int_\Omega \sigma_0 |\nabla v_{k-j} \cdot \nabla v_k| + \sigma_0 |\nabla h_{k-j} \cdot \nabla v_k| dx$$

$$\le r \|\sqrt{\sigma_0}|\nabla v_k|\| \sum_{j=1}^k \left(\|\sqrt{\sigma_0}|\nabla v_{k-j}|\| + \|\sqrt{\sigma_0}|\nabla h_{k-j}|\|\right),$$

where the first inequality uses (6). Thus

$$\|\sqrt{\sigma_0}|\nabla v_k|\| \le r \sum_{j=1}^k \left(\|\sqrt{\sigma_0}|\nabla v_{k-j}|\| + \|\sqrt{\sigma_0}|\nabla h_{k-j}|\|\right).$$

Similarly, using $h_k|_\Gamma = 0$ and the divergence theorem in (15) we obtain

$$\|\sqrt{\sigma_0}|\nabla h_k|\| \le r \sum_{j=1}^k \left(\|\sqrt{\sigma_0}|\nabla v_{k-j}|\| + \|\sqrt{\sigma_0}|\nabla h_{k-j}|\|\right).$$

Now apply Remark 2.1 for $a_k = \|\sqrt{\sigma_0}|\nabla v_k|\| + \|\sqrt{\sigma_0}|\nabla h_k|\|$ and $m = 2r$ to obtain

$$(17) \qquad \| \sqrt{\sigma_0} |\nabla v_k| \| + \| \sqrt{\sigma_0} |\nabla h_k| \| \leq (1 + 2r)^k \| \sqrt{\sigma_0} |\nabla v_0| \|.$$

Since $v_k, h_k \in H_0^1(\Omega)$ the estimate (16) follows from the lower and upper bounds on σ_0 in (5).

<div align="right">□</div>

Proof. [of Theorem 2.1] We seek the solution in the ansatz

$$(18) \qquad u(x, \omega) := v_0(x) + \sum_{n=1}^{\infty} (v_n(x) + ih_n(x))\omega^n$$

and then the recurrence in (14) and (15) is obtained by identifying the same order coefficients in a formal series. It is in here that the reality of the Dirichlet boundary data f (hence of v_0) is used.

We construct the sequences $\{v_n\}, \{h_n\}$ inductively by solving the Dirichlet problems (14) and (15). Part of the inductive step, the right hand hand sides (in divergence form) of the equations in (14) and (15) belong to $H^{-1}(\Omega)$. Thus the recurrence is well defined to construct $v_0 \in H^1(\Omega)$ and the two sequences $\{v_n\}_{n=1}^{\infty}$ and $\{h_n\}_{n=1}^{\infty}$ in $H_0^1(\Omega)$.

The estimate (16) and $|\omega| < 1/(1 + 2r)$ shows

$$(19) \qquad \|u\|_1 \leq c\|v_0\|_1 \sum_{n=0}^{\infty} (1 + 2r)^k \omega^k < \infty.$$

Therefore $u(\cdot, \omega)$ defined in (18) is in $H^1(\Omega)$. A direct calculation involving (14) and (15) shows that it solves the Dirichlet problem (2) for

$$\gamma(x, \omega) := \sigma_0(x) + \sum_{k=1}^{\infty} (\sigma_k(x) + i\epsilon_k(x))\omega^n.$$

<div align="right">□</div>

As a consequence of the theorem above we can explicit the real and imaginary part of the operators $\frac{d^k}{d\omega^k} \Lambda_{\gamma(\cdot, \omega)} \Big|_{\omega=0} (f)$ for a real valued f as follows.

Corollary 2.1. *Let $\gamma(x, \omega)$ be as in the Theorem 2.1 and $f \in H^{1/2}(\Gamma)$ be real valued. Then, with $h_0 \equiv 0$, v_0 solution of (2), and v_k and h_k, $k = 1, 2, ...$ given by the recurrences (14) and (15), we obtain*

$$(20) \qquad \frac{1}{k!} \mathbb{Re} \left\{ \frac{d^k}{d\omega^k} \Lambda_{\gamma(\cdot, \omega)} \Big|_{\omega=0} (f) \right\} = \sum_{j=0}^{k} \left(\sigma_j \frac{\partial v_{k-j}}{\partial \mathbf{n}} - \epsilon_j \frac{\partial h_{k-j}}{\partial \mathbf{n}} \right) \Big|_{\Gamma},$$

$$(21) \qquad \frac{1}{k!} \mathbb{Im} \left\{ \frac{d^k}{d\omega^k} \Lambda_{\gamma(\cdot, \omega)} \Big|_{\omega=0} (f) \right\} = \sum_{j=0}^{k} \left(\sigma_j \frac{\partial h_{k-j}}{\partial \mathbf{n}} + \epsilon_j \frac{\partial v_{k-j}}{\partial \mathbf{n}} \right) \Big|_{\Gamma}.$$

Proof. For $\omega < (1+2r)^{-1} < r^{-1}$ the series in (10) defining γ is absolutely convergent in $L^\infty(\Omega)$, and the series (18) defining u is absolutely convergent in $H^1(\Omega)$. Therefore the two series below

$$\gamma(\cdot,\omega)\frac{\partial u(\cdot,\omega)}{\partial n} = \sum_{k=0}^\infty \omega^k \sum_{j=0}^k \left(\sigma_j \frac{\partial v_{k-j}}{\partial \mathbf{n}} - \epsilon_j \frac{\partial h_{k-j}}{\partial \mathbf{n}} \right)$$
$$+ i \sum_{k=0}^\infty \omega^k \sum_{j=0}^k \left(\sigma_j \frac{\partial h_{k-j}}{\partial \mathbf{n}} + \epsilon_j \frac{\partial v_{k-j}}{\partial \mathbf{n}} \right)$$

are absolutely convergent in $H^{-1/2}(\Gamma)$. The formulas (20) and (21) now follow by differentiation in ω.

$\square$

The proof of Theorem 1.1 requires applications of $\frac{d^k}{d\omega^k}\Lambda_{\gamma(\cdot,\omega)}\Big|_{\omega=0}$ to traces of the complex (valued) geometrical optics solutions. However, if the Dirichlet data in (13) is complex valued, the formulas (20) and (21) no longer hold for v_k and h_k obtained by the recurrence (14) and (15) . In order to preserve this compatibility we extend the real and imaginary part of $\frac{d^k}{d\omega^k}\Lambda_{\gamma(\cdot,\omega)}\Big|_{\omega=0}$ by complex linearity: For $k \geq 0$ and real valued $f, g \in H^{1/2}(\Gamma)$ we define $\mathcal{R}^{(k)}, \mathcal{I}^{(k)} : H^{1/2}(\Gamma) \to H^{-1/2}(\Gamma)$ by

(22)
$$\mathcal{R}^{(k)}(f+ig) := \frac{1}{k!}\,\mathbb{R}\mathrm{e}\left\{ \frac{d^k\Lambda_{\gamma(\cdot,\omega)}}{d\omega^k}\Big|_{\omega=0}(f) \right\} + \frac{i}{k!}\,\mathbb{R}\mathrm{e}\left\{ \frac{d^k\Lambda_{\gamma(\cdot,\omega)}}{d\omega^k}\Big|_{\omega=0}(g) \right\},$$

(23)
$$\mathcal{I}^{(k)}(f+ig) := \frac{1}{k!}\,\mathbb{I}\mathrm{m}\left\{ \frac{d^k\Lambda_{\gamma(\cdot,\omega)}}{d\omega^k}\Big|_{\omega=0}(f) \right\} + \frac{i}{k!}\,\mathbb{I}\mathrm{m}\left\{ \frac{d^k\Lambda_{\gamma(\cdot,\omega)}}{d\omega^k}\Big|_{\omega=0}(g) \right\}.$$

Proposition 2.1. *Let $\mathcal{R}^{(k)}, \mathcal{I}^{(k)} : H^{1/2}(\Gamma) \to H^{-1/2}(\Gamma)$ be as defined in* (22) *and* (23)*. Then*

(24)
$$\mathcal{R}^{(k)}(f+ig) = \sum_{j=0}^k \left(\sigma_j \frac{\partial v_{k-j}}{\partial \mathbf{n}} - \epsilon_j \frac{\partial h_{k-j}}{\partial \mathbf{n}} \right)\Big|_\Gamma$$

(25)
$$\mathcal{I}^{(k)}(f+ig) = \sum_{j=0}^k \left(\sigma_j \frac{\partial h_{k-j}}{\partial \mathbf{n}} + \epsilon_j \frac{\partial v_{k-j}}{\partial \mathbf{n}} \right)\Big|_\Gamma,$$

where $h_0 \equiv 0$, v_0 solves

$$\nabla \cdot \sigma_0 \nabla v_0 = 0, \quad v_0|_\Gamma = f + ig$$

and h_k and v_k are defined inductively by the recurrence (14) *and* (15) *for $k \geq 1$.*

Proof. The identities hold when $g \equiv 0$ by Corollary 2.1. The left hand sides of (24) and (25) are complex linear by definition. But the right hand side is also complex linear since the sequences $\{h_k\}$ and $\{v_k\}$ produced by the recurrence (14) and (15) are complex linear with the Dirichlet data.

$\square$

3. PROOF OF THEOREM 1.1 AND ITS COROLLARY

In this section we assume the hypotheses in Theorem 1.1. The proof is inductive and relies on the basic identities in the lemma below which relate the quantities

$$(26) \qquad Q_N^r := \frac{\nabla \cdot (\nabla \sigma_N - \sigma_N \nabla \ln \sigma_0)}{\sigma_0},$$

$$(27) \qquad Q_N^i := \frac{\nabla \cdot (\nabla \epsilon_N - \epsilon_N \nabla \ln \sigma_0)}{\sigma_0}, \text{ for } N \geq 1,$$

with the boundary operator $\mathcal{R}^{(N)}$ in (22), respectively $\mathcal{I}^{(N)}$ in (23), and some interior data previously determined in the inductive step.

In the following lemma σ_j, ϵ_j for $j = 1, ..., N$, need not be compactly supported in Ω, and σ_0 need not be constant near the boundary, but rather its normal derivative be vanishing at the boundary.

Lemma 3.1. *Let* $w_1, w_2 \in H^1(\Omega)$ *be* σ_0-*harmonic. Then*

$$\int_\Omega \nabla \cdot (\nabla \sigma_1 - \sigma_1 \nabla \ln \sigma_0) w_1 w_2 dx = 2 \int_\Gamma \mathcal{R}^{(1)}[w_1] w_2 ds$$

$$(28) \qquad\qquad\qquad - \int_\Gamma (\sigma_1 \frac{\partial(w_1 w_2)}{\partial \mathbf{n}} - \frac{\partial \sigma_1}{\partial \mathbf{n}} w_1 w_2) ds,$$

and, for $N \geq 2$,

$$\int_\Omega \nabla \cdot (\nabla \sigma_N - \sigma_N \nabla \ln \sigma_0) w_1 w_2 dx$$

$$= 2 \int_\Gamma \mathcal{R}^{(N)}[w_1] w_2 ds - \int_\Gamma (\sigma_N \frac{\partial(w_1 w_2)}{\partial \mathbf{n}} - \frac{\partial \sigma_N}{\partial \mathbf{n}} w_1 w_2) ds$$

$$(29) \qquad - 2 \sum_{j=1}^{N-1} \int_\Omega (\sigma_j \nabla v_{N-j} - \epsilon_j \nabla h_{N-j}) \nabla w_2 dx.$$

Also

$$\int_\Omega \nabla \cdot (\nabla \epsilon_1 - \epsilon_1 \nabla \ln \sigma_0) w_1 w_2 dx = 2 \int_\Gamma \mathcal{I}^{(1)}[w_1] w_2 ds$$

$$(30) \qquad\qquad\qquad - \int_\Gamma (\epsilon_1 \frac{\partial(w_1 w_2)}{\partial \mathbf{n}} - \frac{\partial \epsilon_1}{\partial \mathbf{n}} w_1 w_2) ds,$$

and, for $N \geq 2$,

$$\int_\Omega \nabla \cdot (\nabla \epsilon_N - \epsilon_N \nabla \ln \sigma_0) w_1 w_2 dx = 2 \int_\Gamma \mathcal{I}^{(N)}[w_1] w_2 ds$$

$$- \int_\Gamma \left(\epsilon_N \frac{\partial(w_1 w_2)}{\partial \mathbf{n}} - \frac{\partial \epsilon_N}{\partial \mathbf{n}} w_1 w_2 \right) ds$$

(31)
$$- 2 \sum_{j=1}^{N-1} \int_\Omega (\sigma_j \nabla h_{N-j} + \epsilon_j \nabla v_{N-j}) \cdot w_2 dx.$$

Proof. From Proposition 2.1,

$$(32) \qquad \mathcal{R}^{(N)}[w_1] = \sum_{j=0}^{N} \left(\sigma_j \frac{\partial v_{N-j}}{\partial \mathbf{n}} - \epsilon_j \frac{\partial h_{N-j}}{\partial \mathbf{n}} \right)$$

where $v_0 = w_1$, $h_0 = \epsilon_0 = 0$, and $v_j, h_j, j = 1, ..., N$ are defined recursively in (14) and (15).

Upon multiplication of (32) by w_2 and integration by parts we obtain

$$\int_\Gamma \mathcal{R}^{(N)}[w_1] w_2 ds = \sum_{j=0}^{N} \int_\Gamma \left(\sigma_j \frac{\partial v_{N-j}}{\partial \mathbf{n}} - \epsilon_j \frac{\partial h_{N-j}}{\partial \mathbf{n}} \right) w_2 ds$$

$$= \int_\Omega w_2 \nabla \cdot \left(\sum_{j=0}^{N} (\sigma_j \nabla v_{N-j} - \epsilon_j \nabla h_{N-j}) \right) dx$$

$$+ \sum_{j=0}^{N} \int_\Omega (\sigma_j \nabla v_{N-j} - \epsilon_j \nabla h_{N-j}) \cdot \nabla w_2 dx$$

$$= \int_\Omega \sigma_0 \nabla v_N \cdot \nabla w_2 dx + \sum_{j=1}^{N-1} \int_\Omega (\sigma_j \nabla v_{N-j} - \epsilon_j \nabla h_{N-j}) \cdot \nabla w_2 dx$$

$$+ \int_\Omega \sigma_N \nabla w_1 \cdot \nabla w_2 dx$$

(33)

$$= \sum_{j=1}^{N-1} \int_\Omega (\sigma_j \nabla v_{N-j} dx - \epsilon_j \nabla h_{N-j}) \cdot \nabla w_2 dx + \int_\Omega \sigma_N \nabla w_1 \cdot \nabla w_2 dx,$$

where in the third equality we use the recurrence (14), and for the last equality we use the facts that w_2 is σ_0-harmonic and $v_N \in H_0^1(\Omega)$. Note that for $N = 1$ the identity (33) reduces to

$$(34) \qquad \int_\Gamma \mathcal{R}^{(1)}[w_1] w_2 ds = \int_\Omega \sigma_1 \nabla w_1 \cdot \nabla w_2 dx.$$

Since $\sigma_0 \in C^{1,1}(\Omega)$, w_j also solves

$$(35) \qquad \Delta w_j = -\nabla \ln \sigma_0 \cdot \nabla w_j, \quad j = 1, 2.$$

By using the general identity $2\nabla w_1 \cdot \nabla w_2 = \Delta(w_1 w_2) - w_1 \Delta w_2 - w_2 \Delta w_1$, and (35) we get

$$2\int_\Omega \sigma_N \nabla w_1 \cdot \nabla w_2 dx = \int_\Omega \sigma_N (\Delta(w_1 w_2) - w_1 \Delta w_2 - w_2 \Delta w_1) dx$$

$$= \int_\Omega \sigma_N (\Delta(w_1 w_2) + \nabla \ln \sigma_0 \cdot \nabla(w_1 w_2)) dx$$

$$= \int_\Gamma \left(\sigma_N \frac{\partial(w_1 w_2)}{\partial \mathbf{n}} - \frac{\partial \sigma_N}{\partial \mathbf{n}} w_1 w_2 \right) ds + \int_\Omega (\Delta \sigma_N) w_1 w_2 dx$$

$$+ \int_\Omega \sigma_N \nabla \ln \sigma_0 \cdot \nabla(w_1 w_2) dx$$

$$= \int_\Gamma \left(\sigma_N \frac{\partial(w_1 w_2)}{\partial \mathbf{n}} - \frac{\partial \sigma_N}{\partial \mathbf{n}} w_1 w_2 \right) ds + \int_\Gamma \sigma_N \frac{\partial \ln \sigma_0}{\partial \mathbf{n}} w_1 w_2 ds$$

$$+ \int_\Omega \nabla \cdot (\nabla \sigma_N - \sigma_N \nabla \ln \sigma_0) w_1 w_2 dx$$

$$= \int_\Gamma \left(\sigma_N \frac{\partial(w_1 w_2)}{\partial \mathbf{n}} - \frac{\partial \sigma_N}{\partial \mathbf{n}} w_1 w_2 \right) ds$$

$$(36) \qquad + \int_\Omega \nabla \cdot (\nabla \sigma_N - \sigma_N \nabla \ln \sigma_0) w_1 w_2 dx,$$

where, for the last equality, the vanishing of the second boundary integral accounts for σ_0 being constant near the boundary Γ.

The use of (34), respectively (36), in (33) proves the identity (28), respectively (29).

The identity (31) follows similarly: An integration by parts and use of (15) yield

$$(37)$$

$$\int_\Gamma \mathcal{I}^{(N)}[w_1] w_2 ds = \sum_{j=1}^{N-1} \int_\Omega (\sigma_j \nabla h_{N-j} + \epsilon_j \nabla v_{N-j}) \cdot w_2 dx + \int_\Omega \epsilon_N \nabla w_1 \nabla w_2 dx,$$

and the calculation in (36) with ϵ_N replacing σ_N shows

$$2\int_\Omega \epsilon_N \nabla w_1 \cdot \nabla w_2 dx = \int_\Gamma \left(\epsilon_N \frac{\partial(w_1 w_2)}{\partial \mathbf{n}} - \frac{\partial \epsilon_N}{\partial \mathbf{n}} w_1 w_2 \right) ds$$

$$(38) \qquad + \int_\Omega \nabla \cdot (\nabla \epsilon_N - \epsilon_N \nabla \ln \sigma_0) w_1 w_2 dx, .$$

By using (38) in (37) we obtain (31).

$$\square$$

We prove Theorem 1.1 by induction on the order of derivative in frequency. We will employ the basic identities in the lemma above with w_1, w_2 being the CGO solutions in (8). Recall the dependence of w_1 and w_2 on the vectors $\eta, \mathbf{k}, \mathbf{l}$, which we do not made explicit to keep the formulas readable.

Case $N = 0$: Assume that $\Lambda_{\gamma(\cdot,0)}$ is known, then we appeal to [16, Lemmae 2.7 and 2.12 b)] to conclude that $\Lambda_{\gamma(\cdot,0)}$ recovers the traces $w_j|_\Gamma$ for $j = 1, 2$, and to [16, Theorem 5.1] to recover coefficient σ_0 in Ω.

Case $N = 1$: Assume that $\mathcal{R}^{(1)}$ and $\mathcal{I}^{(1)}$ are known. By using the CGO solutions w_1, w_2 in (30), and accounting for the compact support of σ_1, ϵ_1, we get

$$(39) \qquad \int_{\mathbb{R}^n} e^{-ix \cdot \mathbf{k}} Q_1^r(x)(1 + \psi_1(x))(1 + \psi_2(x))dx = \int_\Gamma \mathcal{R}^{(1)}[w_1]w_2 ds.$$

Due to the decay estimate (9), the left hand side in (39) has a limit with $|\mathbf{l}| \to \infty$, thus so does the right hand side, and the Fourier transform $\mathcal{F}[Q_1^r]$ is determined. The Fourier inversion yields

$$Q_1^r = \mathcal{F}^{-1}\left[\lim_{|\mathbf{l}| \to \infty} \int_\Gamma \mathcal{R}^{(1)}[w_1]w_2 ds\right],$$

Starting with (30) and proceeding similarly with the boundary data $\mathcal{I}^{(1)}$, we find by Fourier inversion the $L^\infty(\Omega)$-function

$$Q_1^i = \mathcal{F}^{-1}\left[\lim_{|\mathbf{l}| \to \infty} \int_\Gamma \mathcal{I}^{(1)}[w_1]w_2 ds\right].$$

In the case $N = 1$ we note that $\frac{d}{d\omega}\big|_{\omega=0} \Lambda_{\gamma(\cdot,\omega)}$ uniquely determines the quantities Q_1^r and Q_1^i, independent of knowledge of the DtN map $\Lambda_{\gamma(\cdot,0)}$.

However, since σ_0 is known, we can also recover σ_1 and ϵ_1 as the unique solutions of the respective Dirichlet problems:

$$(40) \qquad \Delta\sigma_1 - \nabla\sigma_1 \cdot \nabla \ln \sigma_0 - \sigma_1 \Delta \ln \sigma_0 = \sigma_0 Q_1^r \text{ in } \Omega, \quad \sigma_1|_\Gamma = 0,$$

$$(41) \qquad \Delta\epsilon_1 - \nabla\epsilon_1 \cdot \nabla \ln \sigma_0 - \epsilon_1 \Delta \ln \sigma_0 = \sigma_0 Q_1^i \text{ in } \Omega, \quad \epsilon_1|_\Gamma = 0.$$

The inductive step: Assume that σ_0, and σ_j, ϵ_j, $j = 1, ..., N-1$ have been determined, and $\mathcal{R}^{(N)}, \mathcal{I}^{(N)}$ are known. Below we reconstruct σ_N and ϵ_N.

By solving the recurrence (14) and (15) starting with $v_0|_\Gamma = w_1|_\Gamma$, we can determine $v_j, h_j \in H_0^1(\Omega)$ for $j = 1, ..., N-1$ (recall $v_0 = w_1$ and $h_0 = 0$).

Since σ_N is assumed compactly supported in Ω, the identity (29) yields

$$\int_{\mathbb{R}^n} e^{-ix\cdot\mathbf{k}} Q_N^r(x)(1+\psi_1(x))(1+\psi_2(x))dx$$

$$(42) \qquad = 2\int_\Gamma \mathcal{R}^{(N)}[w_1]w_2 ds - 2\sum_{j=1}^{N-1}\int_\Omega \left(\sigma_j \nabla v_{N-j} - \epsilon_j \nabla h_{N-j}\right)\nabla w_2 dx.$$

Note that the right hand side of (42) is known. Due to the decay estimate (9) the left hand side of (42) has a limit with $|\mathbf{l}| \to \infty$, thus so does the right hand side, and the Fourier transform $\mathcal{F}[Q_N^r]$ is recovered from

$$\mathcal{F}[Q_N^r](\mathbf{k}) = \lim_{|\mathbf{l}|\to\infty}\left\{\int_\Gamma \mathcal{R}^{(N)}[w_1]w_2 ds \right.$$

$$(43) \qquad\qquad \left. -2\sum_{j=1}^{N-1}\int_\Omega \left(\sigma_j \nabla v_{N-j} - \epsilon_j \nabla h_{N-j}\right)\nabla w_2 dx\right\}.$$

By Fourier inversion in (42), we recovered $Q_N^r \in L_0^\infty(\Omega)$.

By a similar reasoning starting from (31) with w_1, w_2 being the CGO solutions in (8), we determine $Q_N^i \in L_0^\infty(\Omega)$.

Finally, the coefficients σ_N and ϵ_N, are recovered as the unique solution to the respective Dirichlet problems:

$$(44) \quad \Delta\sigma_N - \nabla\sigma_N \cdot \nabla\ln\sigma_0 - \sigma_N\Delta\ln\sigma_0 = \sigma_0 Q_N^r \text{ in } \Omega, \quad \sigma_N|_\Gamma = 0,$$

$$(45) \quad \Delta\epsilon_N - \nabla\epsilon_N \cdot \nabla\ln\sigma_0 - \epsilon_N\Delta\ln\sigma_0 = \sigma_0 Q_N^i \text{ in } \Omega, \quad \epsilon_N|_\Gamma = 0.$$

This finishes the proof of Theorem 1.1.

For the proof of the corollary, we note that the analyticity of the map $\omega \to \Lambda_{\gamma(\cdot,\omega)}$ allows for each derivative $\frac{d^k}{d\omega^k}\Lambda_{\gamma(\cdot,\omega)}$ at $\omega = 0$ to be determined from the derivatives $\left.\frac{d^j}{d\omega^j}\Lambda_{\gamma(\cdot,\omega)}\right|_{\omega_0}$, $j \geq 0$, taken at any other frequency ω_0 within the domain of analyticity $[0, (1+2r)^{-1})$.

4. ACKNOWLEDGMENT

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