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STRUCTURAL STABILITY IN A MINIMIZATION PROBLEM AND APPLICATIONS TO CONDUCTIVITY IMAGING

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Abstract. We consider problem of minimizing the functional $\int_\Omega a|\nabla u|dx$, with u in some appropriate Banach space and prescribed trace f on the boundary. For $a \in L^2(\Omega)$ and u in the sample space $H^1(\Omega)$, this problem appeared recently in imaging the electrical conductivity of a body when some interior data are available. When $a \in C(\Omega) \cap L^\infty(\Omega)$, the functional has a natural interpretation, which allows us to enlarge the sample space to $BV(\Omega)$. We show the stability of the minimum value with respect to a, in a neighborhood of a particular coefficient. In both cases the method of proof provides some convergent minimizing procedures. We also consider the minimization problem for the non-degenerate functional $\int_\Omega a \max\{|\nabla u|, \delta\}dx$, for some $\delta > 0$, and prove a stability result. Again, the method of proof constructs a minimizing sequence and we identify sufficient conditions for convergence. We apply the last result to the conductivity problem and show that, under an a posteriori smoothness condition, the method recovers the unknown conductivity.

1. Introduction.

In this paper we consider the question of stability in some non-smooth minimization problems occurring in the (electrical) conductivity imaging. For $\Omega \subset \mathbb{R}^d$ $(d \geq 2)$ a Lipschitz domain with connected boundary, $a \in L^2(\Omega)$ non-negative, and $f \in H^{1/2}(\Omega)$, we consider the minimization problem

$$\min\left\{\int_\Omega a\ |\nabla v|dx :\ v \in H^1(\Omega),\ v|_{\partial\Omega} = f\right\}, \tag{1}$$

and call the pair $(a, f) \in L^2(\Omega) \times H^{1/2}(\partial\Omega)$ *admissible* if (1) has a solution. Recall that $H^1(\Omega)$ is the space of functions with one derivative in $L^2(\Omega)$ and $H^{1/2}(\partial\Omega)$ is the space of traces on the boundary $\partial\Omega$ of functions in $H^1(\Omega)$.

We assume an admissible data (a, f) and study the continuous dependence on the coefficient a (for fixed f) of the minimum value and of the minimizer.

The minimization problem in (1) appeared recently in [16] in connection with the problem of reconstruction of the electrical conductivity of a body by using the interior knowledge of the magnitude of the current density field, generated

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while imposing a specific boundary voltage. More precisely, Ω is a conductive body with unknown conductivity $\sigma \in L^\infty(\Omega)$ bounded away from zero, $f \in H^{1/2}(\partial\Omega)$ is the voltage potential maintained at the boundary, and $u_\sigma \in H^1(\Omega)$ is the voltage potential inside generated by f (i.e., u_σ solves the Dirichlet problem $\nabla \cdot \sigma \nabla u_\sigma = 0$ with $u_\sigma|_{\partial\Omega} = f$). Let a denote the magnitude of the current density field $-\sigma\nabla u_\sigma$,

$$a = |\sigma \nabla u_\sigma|. \tag{2}$$

According to the result in [16, Proposition 1.2],

$$u_\sigma \in \operatorname{argmin}\{\int_\Omega a\ |\nabla v| dx :\ v \in H^1(\Omega),\ v|_{\partial\Omega} = f\}. \tag{3}$$

In particular, with a as in (2), the pair (a, f) is admissible. For brevity, we refer to the problem of determining σ from knowledge of the data (a, f) as the *conductivity imaging problem*. We remark that the use of interior measurements to image electrical conductivity is a fairly new trend in inverse problems ([9, 11, 10, 8, 13, 15, 5, 2, 16, 17]), which is driven, in part, by recent technological advances (e.g., [22, 20, 14]).

Since f is fixed throughout, it is convenient to encode the boundary condition in the functional. Accordingly, we seek solutions of (1) in the form $u + h$, where h ranges in $H^1_0(\Omega)$ and $u \in H^1(\Omega)$ is the harmonic map in Ω with trace f at the boundary. The functional to be minimized becomes

$$F[h; a] = \int_\Omega a|\nabla(u + h)|dx. \tag{4}$$

First, we show that the minimum value of F in (4) varies continuously with the coefficient a (for fixed f) nearby admissible data, and present a minimization algorithm, see Theorem 3.1. The proof is based on the regularization methods developed for the minimization of non-smooth functionals in nonreflexive Banach spaces in [18, 19]. One difficulty in this problem comes from the loss of admissibility for pairs $(\tilde{a}, f)$ nearby the admissible data (a, f).

Another difficulty comes from the fact that, from the point of view of calculus of variations, there is a mismatch between the structure of the functional in (4) and the sample space $H^1(\Omega)$, see Section 4. If the coefficient a is continuous, we show that the minimization problem (1) has a natural extension to the space of functions of bounded variations $BV(\Omega)$. Unlike the classical interpretation (of Radon measures) it is convenient to use the Riesz representation theorem and regard the total variation $\|Du\|$ as a linear functional on the space of bounded continuous maps, see Proposition 1.

In Section 5 we establish the stability of the minimum value of the extended functional

$$\tilde{F}[h; a] = \|D(u + h)\|(a), \tag{5}$$

over $h \in BV_0(\Omega)$ (functions of bounded variation with trace zero at the boundary) with respect to perturbations of the coefficient a, provided

$$\operatorname{essinf}_\Omega(a) = \alpha > 0, \tag{6}$$

for some $\alpha > 0$. We show that the algorithm in Theorem 3.1 also produces a minimizing sequence in $BV(\Omega)$, see Theorem 5.1.

In Sections 6 and 7 we remove the degeneracy in (1) under the assumption that the postulated solution u_0 of (1) satisfies

$$|\nabla u_0(x)| \geq \delta > 0, \ a.e. \ in \ \Omega, \tag{7}$$

for some $\delta > 0$. More precisely, we consider the functional

$$F^\delta[h;a] = \int_\Omega a \max\{|\nabla u|, \delta\} dx, \tag{8}$$

and show that it is lower semicontinuous in $H^1(\Omega)$ and it has a unique minimizer in the subspace $H^1(\Omega) \cap C(\overline{\Omega})$. In Theorem 6.1 we propose a regularized minimization algorithm for (8) and show stability of the minimum value with respect to $L^2(\Omega)$ perturbation of a. The algorithm produces a minimizer in $BV(\Omega) \cap L^q(\Omega)$, $q < d/(d-1)$.

In Section 8 we apply Theorem 6.1 to the conductivity imaging problem. The positivity assumption (7) is replaced by the positivity assumptions on the data (6).

Throughout the paper we use the following notations for the norms: $\|a\|$ denotes the $L^2(\Omega)$-norm, $\|h\|_{1,0} = \left(\int_\Omega |\nabla h|^2 dx\right)^{1/2}$ denotes the $H_0^1(\Omega)$-norm, and $\|f\|_{1/2}$ denotes the $H^{1/2}(\partial\Omega)$-norm. For $u \in BV(\Omega)$ we denote by $\|Du\|$ the positive Radon measure defined on any open set $U \subseteq \Omega$ by

$$\|Du\|(U) = \sup\{\int_\Omega u \nabla \cdot f : \ f = (f_1, ..., f_d) \in C_0^1(U; \mathbb{R}^d), \ |f| \leq 1\}, \tag{9}$$

where $|f| = \sqrt{f_1^2 + ... f_d^2}$. We denote strong convergence by "$\rightarrow$" and weak convergence by "$\rightharpoonup$".

2. **A regularized minimization problem.** Let $\tilde{a} \in L^2(\Omega)$ with $\tilde{a} \geq 0$ and $f \in H^{1/2}(\partial\Omega)$. The results in this section do not require $(\tilde{a}, f)$ to be admissible. With the harmonic choice of $u \in H^1(\Omega)$ in (4), from the classical theory of harmonic functions we have

$$\|\nabla u\| = \left(\int_\Omega |\nabla u|^2 dx\right)^{1/2} \leq \|u\|_{H^1(\Omega)} \leq C\|f\|_{1/2}, \tag{10}$$

for a constant C depending only on Ω.

Since $(\tilde{a}, f)$ is not necessarily admissible, the functional $h \mapsto F[h; \tilde{a}]$ may not have a minimizer in $H_0^1(\Omega)$. We regularize the functional such that the new functional has a unique minimizer in $H_0^1(\Omega)$. More precisely, for $\epsilon > 0$ arbitrarily fixed, define the regularization functional F_ϵ of F by

$$\begin{aligned} F_\epsilon[h; \tilde{a}] &:= \int_\Omega \tilde{a}|\nabla(u+h)|dx + \epsilon \int_\Omega |\nabla h|^2 dx \\ &= F[h; \tilde{a}] + \epsilon \|h\|_{1,0}^2. \end{aligned} \tag{11}$$

The following lemma shows that $h \mapsto F_\epsilon[h; \tilde{a}]$ is weakly lower semicontinuous on $H_0^1(\Omega)$.

Lemma 2.1. *Let $\tilde{a} \in L^2(\Omega)$ be non-negative, and $\{h_n\} \subset H_0^1(\Omega)$ be a sequence with $h_n \rightharpoonup h$ in $H_0^1(\Omega)$. Then*

$$F_\epsilon[h; \tilde{a}] \leq \liminf_{n\to\infty} F_\epsilon[h_n; \tilde{a}]. \tag{12}$$

Proof. Let $\{a_m\}$ be an increasing sequence of bounded continuous functions, which converges in $L^2(\Omega)$ sense to $\tilde{a}$.

For each fixed index m, let $f = (f_1, ..., f_d) \in C_0^1(\Omega; \mathbb{R}^d)$ be arbitrary with $|f| \leq a_m$. Since $h_n \rightharpoonup h$ in $L^2(\Omega)$ we have

$$\begin{aligned}\int_\Omega (h+u)\nabla\cdot f dx &= \lim_{n\to\infty}\int_\Omega (h_n+u)\nabla\cdot f dx = \liminf_{n\to\infty}\int_\Omega (h_n+u)\nabla\cdot f dx\\ &\leq \liminf_{n\to\infty}\ \sup\left\{\int_\Omega (h_n+u)\nabla\cdot g dx:\ g\in C_0^1(\Omega;\mathbb{R}^d),\ |g|\leq a_m\right\}\\ &= \liminf_{n\to\infty}\int_\Omega a_m|\nabla(h_n+u)|dx \leq \liminf_{n\to\infty}\int_\Omega \tilde{a}|\nabla(h_n+u)|dx. \end{aligned}\tag{13}$$

The last inequality above uses the fact that $a_m \leq \tilde{a}$.

By taking the supremum in (13) over all $f \in C_0^1(\Omega; \mathbb{R}^d)$ with $|f| \leq a_m$ we get

$$\begin{aligned}\int_\Omega a_m|\nabla(h+u)|dx &= \sup\left\{\int_\Omega (h+u)\nabla\cdot f dx:\ f\in C_0^1(\Omega;\mathbb{R}^d),\ |f|\leq a_m\right\}\\ &\leq \liminf_{n\to\infty}\int_\Omega \tilde{a}|\nabla(h_n+u)|dx.\end{aligned}\tag{14}$$

By letting $m \to \infty$ in (14) we obtain the lower semicontinuity for $F[h; \tilde{a}]$, the first term in (11).

The lower semicontinuity of the second term in (11) is a classical result in the theory of the calculus of variations which follows from Fatou's lemma applied to

$$\|h_n\|_{1,0}^2 - \|h\|_{1,0}^2 \geq \int_\Omega \nabla h\cdot\nabla(h_n - h)dx,\ \forall n\in\mathbb{N}.$$

□

Since $h \mapsto F_\epsilon[h; \tilde{a}]$ is coercive in $H_0^1(\Omega)$ and strictly convex (both due to the regularization), together with weakly lower semicontinuity, we have that F_ϵ has a unique minimizer in $H_0^1(\Omega)$, say

$$h_\epsilon := \text{argmin}\left\{F_\epsilon[h;\tilde{a}]\ : h\in H_0^1(\Omega)\right\},\tag{15}$$

see, e.g., [23]. We show next the continuous dependence of the minimizer on $\tilde{a}$.

Theorem 2.2. *Let $\epsilon > 0$ be fixed, $\{a_n\} \subset L^2(\Omega)$ be a convergent sequence $a_n \to \tilde{a}$ in $L^2(\Omega)$, and $\{h_{\epsilon,n}\}$ be the corresponding minimizing sequence of $F_\epsilon[\cdot\ ; a_n]$,*

$$h_{\epsilon,n} := \textit{argmin}\{F_\epsilon[h;a_n];\ h\in H_0^1(\Omega)\}.$$

Then $h_{\epsilon,n} \rightharpoonup h_\epsilon$ in $H_0^1(\Omega)$ as $n \to \infty$.

Proof. Let $\delta_n := \|\tilde{a} - a_n\|$ and consider n large enough so that

$$\delta_n \leq \|\tilde{a}\|.\tag{16}$$

We claim that the sequence $\{h_{\epsilon,n}\}$ is bounded in $H_0^1(\Omega)$. Indeed,

$$\begin{aligned}\epsilon\|h_{\epsilon,n}\|_{1,0}^2 \leq& F_\epsilon[h_{\epsilon,n};a_n] \leq F_\epsilon[0;a_n] = F[0;\tilde{a}] + F[0;a_n-\tilde{a}]\\ \leq& (\|\tilde{a}\|+\delta_n)\|\nabla u\| \leq 2C\|\tilde{a}\|\ \|f\|_{1/2}\end{aligned}$$

The second inequality uses the minimizing property defining $h_{\epsilon,n}$, and the last one uses (10) and (16). Then

$$\|h_{\epsilon,n}\|_{1,0} \leq \frac{\sqrt{2\|\tilde{a}\|C\|f\|_{1/2}}}{\sqrt{\epsilon}} =: C_\epsilon\tag{17}$$

and consequently the sequence $\{h_{\epsilon,n}\}$ has an H_0^1- weakly convergent subsequence; say $h_{\epsilon,n_k} \rightharpoonup \tilde{h}_\epsilon$, for some $\tilde{h}_\epsilon \in H_0^1(\Omega)$.

Let $h \in H_0^1(\Omega)$ be arbitrarily fixed. Since $h \mapsto F_\epsilon[h;\tilde{a}]$ is weakly lower semicontinuous on $H_0^1(\Omega)$, the following estimates hold:

$$\begin{aligned}
F_\epsilon[\tilde{h}_\epsilon;\tilde{a}] &\le \liminf_{k\to\infty} F_\epsilon[h_{\epsilon,n_k};\tilde{a}] = \liminf_{k\to\infty} \{F_\epsilon[h_{\epsilon,n_h};a_{n_k}] + F[h_{\epsilon,n_k};\tilde{a}-a_{n_k}]\} \\
&\le \liminf_{k\to\infty} \{F_\epsilon[h_{\epsilon,n_h};a_{n_k}] + \delta_{n_k}(\|h_{\epsilon,n_k}\|_{1,0} + \|\nabla u\|)\} \\
&\le \liminf_{k\to\infty} \left\{F_\epsilon[h_{\epsilon,n_k};a_{n_k}] + \delta_{n_k}(C_\epsilon + C\|f\|_{1/2})\right\} \\
&\le \liminf_{k\to\infty} \left\{F_\epsilon[h;a_{n_k}] + \delta_{n_k}(C_\epsilon + C\|f\|_{1/2})\right\} \\
&= \liminf_{k\to\infty} \left\{F_\epsilon[h;\tilde{a}] + F[h;a_{n_k}-\tilde{a}] + \delta_{n_k}(C_\epsilon + C\|f\|_{1/2})\right\} \\
&\le \liminf_{k\to\infty} \left\{F_\epsilon[h;\tilde{a}] + \delta_{n_k}(\|h\|_{1,0} + C_\epsilon + C\|f\|_{1/2})\right\} \\
&\le \limsup_{k\to\infty} \left\{F_\epsilon[h;\tilde{a}] + \delta_{n_k}(\|h\|_{1,0} + C_\epsilon + C\|f\|_{1/2})\right\} \\
&\le F_\epsilon[h;\tilde{a}] + \limsup_{k\to\infty} \delta_{n_k}(\|h\|_{1,0} + C_\epsilon + C\|f\|_{1/2}) \\
&= F_\epsilon[h;\tilde{a}].
\end{aligned}$$

In the estimate above the third inequality uses (10) and (17) and the fourth inequality uses the minimizing property defining $\{h_{\epsilon,n_k}\}$.

Therefore $\tilde{h}_\epsilon$ is a minimizer for F_ϵ in $H_0^1(\Omega)$. Since the minimizer is unique, $\tilde{h}_\epsilon = h_\epsilon$. Since any other weakly convergent subsequence of $\{h_{\epsilon,n}\}$ also converges to h_ϵ, the entire sequence is weakly convergent to h_ϵ. □

3. **Convergence of the regularized minimizers as $\epsilon \downarrow 0$.** In this section we make essential use of the fact that the pair $(a, f) \in L^2(\Omega) \times H^{1/2}(\partial\Omega)$ is admissible, i.e. there exists some $u_0 \in H^1(\Omega)$ with $u_0|_{\partial\Omega} = f$ solution of (1). Let u be the harmonic function of trace f at the boundary as before. Then

$$h_0 := u_0 - u \in H_0^1(\Omega) \tag{18}$$

is a minimizer of (4): $F[h_0; a] \le F[h; a]$ for all $h \in H_0^1(\Omega)$.

Let $\{a_n\} \subset L^2(\Omega)$ be a sequence with $a_n \to a$ in $L^2(\Omega)$, and $\epsilon_n \downarrow 0$ be a decreasing sequence. Let $\{h_{\epsilon_n}\}$ be the sequence of the minimizers corresponding to the functionals $F_{\epsilon_n}[\cdot; a_n]$,

$$h_{\epsilon_n} := \operatorname{argmin}\{F_{\epsilon_n}[h; a_n] : \; h \in H_0^1(\Omega)\}. \tag{19}$$

In Section 2 we showed that each h_{ϵ_n} is well-defined. Note that both the regularized parameter and the coefficients in the functional are changing with n. In particular, the sequence h_{ϵ_n} may not be bounded in $H_0^1(\Omega)$, see (17). If the regularized parameter ϵ_n is chosen so that $\|a - a_n\| = o(\sqrt{\epsilon_n})$, we show below that $\{h_{\epsilon_n}\}$ is still a minimizing sequence for F in (4).

Theorem 3.1. *Assume that $(a, f) \in L^2(\Omega) \times H^{1/2}(\partial\Omega)$ is admissible with $a \ge 0$, and let $h_0 \in H_0^1(\Omega)$ be the minimizer defined in (18). Let $\{a_n\}$ be a sequence in $L^2(\Omega)$ with $a_n \to a$ in $L^2(\Omega)$, and choose $\epsilon_n \downarrow 0$ in such a way that*

$$\lim_{n\to\infty} \frac{\|a - a_n\|^2}{\epsilon_n} = 0. \tag{20}$$

Let $\{h_{\epsilon_n}\}$ *denote the sequence of the minimizers of* $F_{\epsilon_n}[\cdot\,; a_n]$ *as in* (19)*. Then*

$$\liminf_{n\to\infty} F_{\epsilon_n}[h_{\epsilon_n}; a_n] = \liminf_{n\to\infty} F[h_{\epsilon_n}; a_n] = \min\{F[h; a] : \ h \in H_0^1(\Omega)\}. \tag{21}$$

In addition, if a *is bounded from below as in* (6)*, then a subsequence of* $\{h_{\epsilon_n}\}$ *converges in* $L^q(\Omega)$*,* $1 \le q < d/(d-1)$*.*

Proof. Let $\delta_n := \|a - a_n\|$ be as before. Since $h_{\epsilon_n} \in H_0^1(\Omega)$ and h_0 is a minimizer of $F[\,\cdot\,; a]$ over $H_0^1(\Omega)$, we have

$$F[h_0; a] \le \liminf_{n\to\infty} F[h_{\epsilon_n}; a] \le \liminf_{n\to\infty} F_{\epsilon_n}[h_{\epsilon_n}; a]. \tag{22}$$

We claim that the reverse inequality also holds. Recall the definition of h_0 in (18). We have the estimate

$$\begin{aligned}
\liminf_{n\to\infty} F_{\epsilon_n}[h_{\epsilon_n}; a] &= \liminf_{n\to\infty} \{F_{\epsilon_n}[h_{\epsilon_n}; a_n] + F[h_{\epsilon_n}; a - a_n]\}\\
&\le \liminf_{n\to\infty} \{F_{\epsilon_n}[h_{\epsilon_n}; a_n] + \delta_n \|h_{\epsilon_n}\|_{1,0}\}\\
&\le \liminf_{n\to\infty} \left\{F_{\epsilon_n}[h_{\epsilon_n}; a_n] + \frac{\delta_n}{\sqrt{\epsilon_n}}\sqrt{2\|a\| C \|f\|_{1/2}}\right\}\\
&\le \liminf_{n\to\infty} \left\{F_{\epsilon_n}[h_0; a_n] + \frac{\delta_n}{\sqrt{\epsilon_n}}\sqrt{2\|a\| C \|f\|_{1/2}}\right\}\\
&= \liminf_{n\to\infty} \left\{F[h_0; a_n] + \epsilon_n \|h_0\|_{1,0} + \frac{\delta_n}{\sqrt{\epsilon_n}}\sqrt{2\|a\| C \|f\|_{1/2}}\right\}\\
&= \liminf_{n\to\infty} \left\{F[h_0; a] + F[h_0; a_n - a] + \epsilon_n \|h_0\|_{1,0} + \frac{\delta_n}{\sqrt{\epsilon_n}}\sqrt{2\|a\| C \|f\|_{1/2}}\right\}\\
&\le \liminf_{n\to\infty} \left\{F[h_0; a] + \delta_n \|\nabla(u + h_0)\| + \epsilon_n \|h_0\|_{1,0} + \frac{\delta_n}{\sqrt{\epsilon_n}}\sqrt{2\|a\| C \|f\|_{1/2}}\right\}\\
&\le \limsup_{n\to\infty} \left\{F[h_0; a] + \delta_n \|\nabla(u + h_0)\| + \epsilon_n \|h_0\|_{1,0} + \frac{\delta_n}{\sqrt{\epsilon_n}}\sqrt{2\|a\| C \|f\|_{1/2}}\right\}\\
&= F[h_0; a].
\end{aligned}$$

In the estimate above, the third inequality uses the minimizing property of $\{h_{\epsilon_n}\}$, and the last equality uses the hypothesis $\delta_n = o(\sqrt{\epsilon_n})$. This proves the identity (21).

Next we show that $\{h_{\epsilon_n}\}$ is bounded in $W_0^{1,1}(\Omega)$. From the positivity assumption (6) we have

$$\alpha \|h_{\epsilon_n}\|_{1,1} \le \int_\Omega a |\nabla h_{\epsilon_n}| dx \le F[h_{\epsilon_n}; a] + F[0; a] \le F[h_{\epsilon_n}; a] + C\|a\| \ \|f\|_{1/2}.$$

We next show that the right hand side is bounded uniformly in n:

$$\begin{aligned}
F[h_{\epsilon_n};a] \leq & F_{\epsilon_n}[h_{\epsilon_n};a] \\
= & F_{\epsilon_n}[h_{\epsilon_n};a_n] + F[h_{\epsilon_n};a-a_n] \\
\leq & F_{\epsilon_n}[h_0;a_n] + \delta_n(\|h_{\epsilon_n}\|_{1,0} + C\|f\|_{1/2}) \\
= & F[h_0;a_n] + \epsilon_n\|h_0\|_{1,0} + \delta_n(\|h_{\epsilon_n}\|_{1,0} + C\|f\|_{1/2}) \\
\leq & \|a_n\|C\|f\|_{1/2} + \epsilon_n\|h_0\|_{1,0} + \delta_n(\|h_{\epsilon_n}\|_{1,0} + C\|f\|_{1/2}) \\
\leq & (\delta_n + \|a\|)C\|f\|_{1/2} + \epsilon_n\|h_0\|_{1,0} + \delta_n(\|h_{\epsilon_n}\|_{1,0} + C\|f\|_{1/2}) \\
= & \|a\|C\|f\|_{1/2} + \epsilon_n\|h_0\|_{1,0} + \frac{\delta_n}{\sqrt{\epsilon_n}}\sqrt{2\|a\|C\|f\|_{1/2}} + 2\delta_n C\|f\|_{1/2} \\
\leq & 3\|a\|C\|f\|_{1/2} + \|h_0\|_{1,0} + \sqrt{2\|a\|C\|f\|_{1/2}}.
\end{aligned}$$

In the above estimate the second inequality uses the minimization property of h_{ϵ_n}, and the last inequality requires n to be sufficiently large and uses the hypothesis (20).

An application of Rellich-Kondrachov's compactness imbedding ([6]) shows the existence of a convergent subsequence $h_{\epsilon_n} \to h^*$ in $L^q(\Omega)$ for all $1 \leq q \leq d/(d-1)$. □

4. **Non-smooth minimization problems in non-reflexive Banach spaces.**

In this section we briefly discuss the appropriateness of the sample space in the minimization problems (1) from the point of view of the calculus of variations.

The traditional minimizing method in the calculus of variations is to establish precompactness of minimizing sequences and the lower semicontinuity of the functional. While it is true that a lower semi-continuous functional on a compact set S in a topological space attains its minimum in S, the norm topology is not appropriate since a closed ball is not compact in this norm. This motivates endowing a Banach space with a topology relative to which the set S becomes compact. In reflexive Banach spaces (such as $H^1(\Omega)$), the weak topology accomplishes this task. The role of the weak topology is derived from Alaoglu-Bourbaki-Kakutani's and Mazur's theorems. According to the first theorem, the closed ball in any dual space B^* is compact in the weak-star topology of B^*. For reflexive spaces, the weak topology of B coincide with the weak-star topology of B^{**}, and thus the ball is weakly compact. The second theorem ensures that every strongly closed convex set in a Banach space is weakly closed; hence in a reflexive Banach space every bounded closed convex set is weakly compact. On the basis of these results, if f is a weakly lower semicontinuous real valued function on a weakly closed subset S of a reflexive Banach space B, then f attains its infimum in S. In particular S can be a bounded, closed and convex subset of B. The utility of this result depends on establishing the weakly lower semicontinuity of the functional, and boundedness of the minimizing sequence.

While the functional in (1) is weakly lower semicontinuous in the Hilbert space $H^1(\Omega)$, the corresponding minimizing sequence may no longer stay bounded in there; see (17).

Under the positivity assumption (6) on the coefficient a, the minimizing sequences stay bounded in $W^{1,1}(\Omega)$. Unfortunately, due to its non-reflexivity, the weak topology of $W^{1,1}(\Omega)$ does not coincide with the weak-star topology of its dual, and the theorems of Alaoglu-Bourbaki-Kakutani and Mazur have no direct relevance.

Moreover, the map $u \to \int_\Omega a|\nabla u|dx$ is no longer weakly lower semicontinuous in $W^{1,1}(\Omega)$, and this is due to the fact that $L^1_{loc}(\Omega)$ limits of functions in $W^{1,1}(\Omega)$ may no longer be in $W^{1,1}(\Omega)$. However, we shall show below that, for a bounded continuous coefficient a, we can extend the functional in (1) to the space $BV(\Omega)$ of functions of bounded variation, while preserving the lower semicontinuity in this larger space (see Theorem 5.1). The regularized minimizing schemes in Theorems 3.1 and 7.2 produce minimizers which are in $BV(\Omega)$.

Moreover we show that, for the minimization problem (1) with admissible data (a, f), the (postulated) minimizer in $H^1(\Omega)$ minimizes the extended functional in $BV(\Omega)$.

5. **A global minimization property in $BV(\Omega)$.** For $u \in BV(\Omega)$ recall the definition of the total variation of in (9). By Riesz representation theorem see e.g. [3, 24], since $\|Du\|$ is a positive Radon measure, there is a linear bounded functional (denoted the same) on compactly supported continuous functions, such that for any continuous $g \in C_0(\Omega)$ of compact support in Ω:

$$\|Du\|(g) = \sup\left\{\int_\Omega u\nabla\cdot\varphi dx: \ \varphi \in C^1_0(\Omega; R^n),\ |\phi| \leq g\right\}. \tag{23}$$

We show first that $\|Du\|$ extends continuously to bounded continuous functions $C(\Omega)\cap L^\infty(\Omega)$. For the extension we consider the increasing sequence $\psi_n \in C_0(\Omega)$ with $0 \leq \psi_n \leq 1$ such that

$$\psi_n(x) = \begin{cases} 1, & dist(x, \partial\Omega) > 2^{-n}, \\ 0, & dist(x, \partial\Omega) < 2^{-n-1}. \end{cases} \tag{24}$$

Proposition 1. *Let $a \in C(\Omega)\cap L^\infty(\Omega)$ with $a \geq 0$. Then the sequence $\{\|Du\|(a\psi_n)\}$ is bounded and increasing. Define*

$$\|Du\|(a) := \lim_{n\to\infty}\|Du\|(\psi_n a). \tag{25}$$

Then

$$\|Du\|(a) = \sup\left\{\int_\Omega u\nabla\cdot\varphi dx: \ \varphi \in C^1_0(\Omega : R^n),\ |\varphi| \leq a\right\}. \tag{26}$$

In particular the definition (25) is independent of the choice of ψ_n.

Proof. Since $a\psi_n \leq a\psi_{n+1}$ and $\|Du\|$ is a positive Radon measure, we have $\|Du\|(a\psi_n) \leq \|Du\|(a\psi_{n+1})$. If $\|a\|_\infty = 0$ then $a \equiv 0$ and $\|Du\|(a\psi_n) = \|Du\|(a) = 0$.

Assume $\|a\|_\infty > 0$. Let $\varphi \in C^1_0(\Omega; R^n)$ be arbitrary with $|\varphi| \leq \psi_n a$. Then $\|a\|^{-1}_\infty\varphi \in C^1_0(\Omega; R^n)$ with $\|a\|^{-1}_\infty|\varphi| \leq 1$ and

$$\int_\Omega u\nabla\cdot\varphi dx \leq \|a\|_\infty\int_\Omega u\nabla\cdot\left(\frac{1}{\|a\|_\infty}\varphi\right)dx \leq \|a\|_\infty\ \|Du\|(\Omega),$$

where the last term is the total variation of u in Ω. By taking the supremum over $\varphi \in C^1_0(\Omega; R^n)$ we obtain the upper bound $\|Du\|(a\psi_n) \leq \|a\|_\infty\|Du\|(\Omega)$. Since the sequence increases the limit in (25) exists.

The fact that, for all $n \in N$,

$$\|Du\|(a\psi_n) \leq \sup\left\{\int_\Omega u\nabla\cdot\varphi dx: \ \varphi \in C^1_0(\Omega : R^n),\ |\varphi| \leq a\right\},$$

follows directly from the inclusion

$$\{\varphi \in C^1_0(\Omega : R^n): \ |\varphi| \leq a\psi_n\} \subset \{\varphi \in C^1_0(\Omega : R^n): \ |\varphi| \leq a\}.$$

To show the converse, let $\varphi \in C_0^1(\Omega; R^n)$ with $|\varphi| \leq a$ be arbitrarily fixed. If n is such that $2^{-n} < dist(\text{supp}(\varphi), \partial\Omega)$, then

$$\text{supp}(\varphi) \subset \{x : \ dist(x, \partial\Omega) > 2^{-n}\}$$

and $|\varphi| \leq a\psi_n$. Consequently,

$$\int_\Omega u\nabla \cdot \varphi dx \leq \|Du\|(a\psi_n) \leq \|Du\|(a).$$

The conclusion in (26) follows by taking the supremum in $\varphi \in C_0^1(\Omega; R^n)$. □

Proposition 2 (lower semicontinuity)**.** *Let $\{u_n\} \subset BV(\Omega)$, be a sequence convergent in $L^1_{loc}(\Omega)$ to some $u \in L^1_{loc}$ and let $a \in C(\Omega) \cap L^\infty(\Omega)$ be nonnegative. Then*

$$\|Du\|(a) \leq \liminf_{n\to\infty} \|Du_n\|(a). \tag{27}$$

Proof. Let $\varphi \in C_0^1(\Omega; \mathbb{R}^n)$, with $|\varphi| \leq a$, then

$$\int_\Omega u\nabla \cdot \varphi dx = \lim_{n\to\infty} \int_\Omega u_n \nabla \cdot \varphi dx \leq \liminf_{n\to\infty} \|Du_n\|(a). \tag{28}$$

Now take the supremum over all such φ. □

For the minimization problem (1) with admissible data (a, f), the minimizer in $u_0 \in H^1(\Omega)$ minimizes the extended functional in $BV(\Omega)$ as follows.

Proposition 3. *Let $a \in C(\Omega) \cap L^\infty(\Omega)$ with $a \geq 0$. Assume that $u_0 \in H^1(\Omega)$ is such that*

$$\int_\Omega a|\nabla u|dx \leq \int_\Omega a|\nabla v|dx, \tag{29}$$

for all $v \in H^1(\Omega)$ with $\text{supp}(v - u_0) \subset \Omega$. Then

$$\|Du_0\|(a) \leq \|Dv\|(a), \tag{30}$$

for all $v \in BV(\Omega)$ with $\text{supp}(v - u) \subset \Omega$.

Proof. If $u \in W^{1,1}(\Omega)$ then $\int_\Omega a|\nabla u|dx = \|Du\|(a)$.

From the density result in [7, Remark 2.12] it follows that, for any $v \in BV(\Omega)$, there is a sequence of functions $v_n \in C^\infty(\Omega)$ such that

(i) $v_n \to v$ in $L^1(\Omega)$,
(ii) $\|Dv_n\| \rightharpoonup \|Dv\|$ (in the sense of measures),
(iii) $supp(v_n - v) \subset \Omega$.

From (29) applied to v_n, we get

$$\int_\Omega a|\nabla u|dx \leq \int_\Omega a|\nabla v_n|dx = \|Dv_n\|(a).$$

Since $\|Dv_n\|(a) \to \|Dv\|(a)$ as $n \to \infty$, the inequality (30) follows. □

Let $BV_0(\Omega)$ be the subspace of functions of bounded variations with zero trace at the boundary. We are ready now to formulate the extension of Theorem 3.1

Theorem 5.1. *Assume that* $(a, f) \in (C(\Omega) \cap L^\infty(\Omega)) \times C(\partial\Omega)$ *is admissible and* a *is bounded from below as in* (6). *Let* $\{a_n\}$ *be a sequence in* $L^2(\Omega)$ *with* $a_n \to a$ *in* $L^2(\Omega)$.

Choose $\epsilon_n \downarrow 0$ *in such a way that* $\|a - a_n\| = o(\sqrt{\epsilon_n})$ *and let* h_{ϵ_n} *be the minimizer of the regularized functional* $F_{\epsilon_n}[\cdot\,; a_n]$ *as in* (19).

Then, $\{h_{\epsilon_n}\}$ *has a subsequence which is convergent in* $L^q(\Omega)$ *to some* h^*, $1 \leq q < \frac{d}{d-1}$. *Moreover* $h^* \in BV_0(\Omega)$ *and*

$$\|D(u + h^*)\|(a) = min\{\|D(u + h)\|(a) : \; h \in BV_0(\Omega)\}, \tag{31}$$

where u *is the harmonic function with trace* f *on the boundary.*

Proof. Following the proof of Theorem 3.1 we obtain, as before, the sequence $\{h_{\epsilon_n}\} \subset H_0^1(\Omega)$, which, on a subsequence, converges in $L^q(\Omega)$ to some h^*, $1 \leq q \leq d/(d-1)$. From the lower semicontinuity (27), we obtain

$$\begin{aligned}
\|D(u + h^*)\|(a) &\leq \liminf_{n\to\infty} \|D(u + h_{\epsilon_n})\|(a) \\
&= \liminf_{n\to\infty} \int_\Omega a|\nabla(u + h_{\epsilon_n})|dx \\
&= \liminf_{n\to\infty} F[h_{\epsilon_n}; a] \\
&= \min\{\int_\Omega a|\nabla(u + h)|dx : \; h \in H_0^1(\Omega)\} \\
&= \min\{\|D(u + h)\|(a) : \; h \in BV_0(\Omega)\}.
\end{aligned}$$

The third equality above uses (21), and the last equality uses Proposition 3. □

6. **A non-degenerate minimization problem.** Let (a, f) be an admissible pair. In this section we assume that the minimization problem (1) has a solution satisfying

$$|\nabla u_0(x)| \geq \delta, \quad a.e. \quad \Omega, \tag{32}$$

for some $\delta > 0$.

Theorem 6.1. *Let* $u_0 \in H^1(\Omega)$ *be a solution of* (1) *satisfying* (32). *Then* u_0 *also solves*

$$\min\left\{\int_\Omega a\max\{|\nabla v|, \delta\}dx : \; v \in H^1(\Omega), \; v|_{\partial\Omega} = f\right\}. \tag{33}$$

In addition, if $u_0 \in C(\overline{\Omega})$, *then it is a unique minimizer for* (33) *within the set* $H^1(\Omega) \cap C(\overline{\Omega})$.

Proof. For any $v \in H^1(\Omega)$, we have

$$\int_\Omega a\max\{|\nabla u_0|, \delta\}dx = \int_\Omega a|\nabla u_0|dx \leq \int_\Omega a|\nabla v|dx \leq \int_\Omega a\max\{|\nabla v|, \delta\}dx.$$

Now let w be another solution of the minimization problem (33), and define

$$\Omega_\delta = \{x \in \Omega : \; |\nabla w(x)| > \delta\}.$$

Since

$$\begin{aligned}\int_\Omega a\max\{|\nabla w|,\delta\}dx &= \int_\Omega a\max\{|\nabla u_0|,\delta\}dx = \int_\Omega a|\nabla u_0|dx \\ &\leq \int_\Omega a|\nabla w|dx = \int_{\Omega_\delta} a|\nabla w|dx + \int_{\Omega\setminus\Omega_\delta} a|\nabla w|\ dx \\ &\leq \int_{\Omega_\delta} a|\nabla w|dx + \delta\int_{\Omega\setminus\Omega_\delta} a\ dx \\ &= \int_\Omega a\max\{|\nabla w|,\delta\}dx,\end{aligned} \tag{34}$$

all the inequalities are, in fact, equalities. In particular,

$$\int_{\Omega\setminus\Omega_\delta} a|\nabla w|dx = \delta\int_{\Omega\setminus\Omega_\delta} a\ dx. \tag{35}$$

Since $a>0$ a.e. in Ω, and $|\nabla w(x)|\leq\delta$ a.e. in $\Omega\setminus\Omega_\delta$, the identity (35) implies that either $\Omega\setminus\Omega_\delta$ is (Lebesgue)-negligible, or $|\nabla w(x)|=\delta$ a.e. in $\Omega\setminus\Omega_\delta$. In both cases we obtain $|\nabla w(x)|\geq\delta$ a.e. in Ω. Also, since the first inequality in (34) is also an equality, $w\in H^1(\Omega)$ is another minimizer of the problem in (1).

If both u_0 and w belong to $H^1(\Omega)\cap C(\overline{\Omega})$ then the uniqueness result in [16, Theorem 1.3] implies $u_0=w$. □

Next we derive a stable, regularized minimization method for the functional in (33) similar to the minimization method for (1). The key ingredient is the lower semi-continuity property for the functional in (33). We prove first a weak formulation of the functional in (33). By $\mathcal{O}(\Omega)$ we denote the set of all open subsets of Ω.

Proposition 4. *Let $a\in L^2(\Omega)$ be positive a.e. in Ω, $\delta>0$, and $v\in H^1(\Omega)$. Then*

$$\int_\Omega a\max\{|\nabla v|,\delta\}dx = \sup_{U,f}\left(\int_U v\nabla\cdot f dx + \delta\int_{\Omega\setminus U} a\ dx\right), \tag{36}$$

where $U\in\mathcal{O}(\Omega)$ ranges over all open subsets of Ω, and $f=(f_1,...,f_d)\in C_0^1(U;\mathbb{R}^d)$ with $|f|=\sqrt{f_1^2+...+f_d^2}\leq a$.

The proof follows from the two lemmas below.

Lemma 6.2. *For any $v\in H^1(\Omega)$,*

$$\int_\Omega a\max\{|\nabla v|,\delta\}dx = \sup_{U}\left(\int_U a|\nabla v|dx + \delta\int_{\Omega\setminus U} a\ dx\right), \tag{37}$$

where $U\in\mathcal{O}(\Omega)$ ranges over all open subsets of Ω.

Proof. Let $\epsilon>0$ be arbitrarily fixed. Since $|\nabla v|\in L^2(\Omega)$, by Lusin's theorem, there is a continuous map $g\in C(\Omega)$, and a closed set K_ϵ of Lebesgue measure $|K_\epsilon|<\epsilon$, such that $|\nabla v|=g$ on $\Omega\setminus K_\epsilon$.

Let $\Omega_\delta=\{x\in\Omega:\ |\nabla u(x)|>\delta\}$ and $G_\delta=\{x\in\Omega:\ g(x)>\delta\}$. Note that G_δ is open and

$$\omega_\epsilon := \Omega_\delta\setminus K_\epsilon = G_\delta\setminus K_\epsilon = \{x\in\Omega\setminus K_\epsilon:\ g(x)>\delta\} \tag{38}$$

is an open set (although Ω_δ may not be open). Note also the disjoint decomposition

$$\Omega\setminus\omega_\epsilon = \Omega\setminus(\Omega_\delta\setminus K_\epsilon) = (\Omega\setminus\Omega_\delta)\cup(K_\epsilon\cap\Omega_\delta)$$

and its corollary

$$\int_{\Omega\setminus\Omega_\delta} a\ dx = \int_{\Omega\setminus\omega_\epsilon} a\ dx - \int_{K_\epsilon\cap\Omega_\delta} a\ dx. \tag{39}$$

We show below that

$$\int_{\omega_\epsilon} a|\nabla v|dx + \delta\int_{\Omega\setminus\omega_\epsilon} a\ dx = \int_\Omega a\max\{|\nabla v|,\delta\}dx - \mathcal{G}(\epsilon),$$

for some $\mathcal{G}(\epsilon)\to 0$ with $\epsilon\to 0^+$.

Using the definition (38) and the relation (39), we obtain

$$\begin{aligned}
&\int_\Omega a\max\{|\nabla v|,\delta\}dx = \int_{\Omega_\delta} a|\nabla v|dx + \delta\int_{\Omega\setminus\Omega_\delta} a\ dx\\
&= \int_{\Omega_\delta\setminus K_\epsilon} a|\nabla v|dx + \int_{\Omega_\delta\cap K_\epsilon} a|\nabla v|dx + \delta\int_{\Omega\setminus\omega_\epsilon} a\ dx - \delta\int_{\Omega_\delta\cap K_\epsilon} a\ dx\\
&= \int_{\omega_\epsilon} a|\nabla v|dx + \delta\int_{\Omega\setminus\omega_\epsilon} a\ dx + \mathcal{G}(\epsilon),
\end{aligned}$$

where

$$\mathcal{G}(\epsilon) := \int_{\Omega_\delta\cap K_\epsilon} a(|\nabla v|-\delta)dx. \tag{40}$$

Note that $\mathcal{G}(\epsilon)\geq 0$. Since $|K_\epsilon|\to 0$ as $\epsilon\to 0$, and both a and $a|\nabla v|$ are integrable, we have $\mathcal{G}(\epsilon)\to 0$ as $\epsilon\to 0$. □

Lemma 6.3. *Let $a\in L^2(\Omega)$ be positive a.e. in Ω, $v\in H^1(\Omega)$, and let U be an open subset of Ω. Then*

$$\int_U a|\nabla v|dx = \sup\{\int_U v\nabla\cdot f\ dx:\ f\in C_0^1(U;\mathbb{R}^d),\ |f|\leq a\}. \tag{41}$$

Proof. For any competitor f,

$$\int_U v\nabla\cdot f = -\int_\Omega f\cdot\nabla v\ dx \leq \int_\Omega |f|\ |\nabla v|\ dx \leq \int_\Omega a|\nabla v|\ dx.$$

This shows that the right hand side in (41) is smaller than the left hand side. To show the reverse, define $g\in L^2(\Omega;\mathbb{R}^d)$ by

$$g = \begin{cases} a\frac{\nabla v}{|\nabla v|}, & \text{where} \quad |\nabla v|>0,\\ 0, & \text{where} \quad |\nabla v|=0.\end{cases}$$

Note that $|g|\leq a$. Let $\{f_n\}\subset C_0^1(\Omega;\mathbb{R}^d)$ be a sequence with $|f_n|\leq a$, and $f_n\to g$ in $L^2(\Omega;\mathbb{R}^d)$. Then

$$\begin{aligned}
\lim_{n\to\infty}\int_U \nabla v\cdot f_n\ dx &= \int_U \nabla v\cdot g\ dx = \int_{\{|\nabla v|>0\}} \nabla v\cdot a\frac{\nabla u}{|\nabla v|}dx\\
&= \int_{\{|\nabla v|>0\}} a|\nabla v|dx = \int_U a|\nabla v|dx.
\end{aligned}$$

□

We are now ready to prove the lower semicontinuity property.

Theorem 6.4 (lower semicontinuity)**.** *Let $a \in L^2(\Omega)$ be positive, $\{v_n\}$ be a sequence in $H^1(\Omega)$, and $v \in H^1(\Omega)$ with $v_n \rightharpoonup v$ in $L^2(\Omega)$. Then*

$$\int_\Omega a \max\{|\nabla v|, \delta\} dx \le \liminf_{n\to\infty} \int_\Omega a \max\{|\nabla v_n|, \delta\} dx. \tag{42}$$

Proof. Let $U \in \mathcal{O}(\Omega)$ be arbitrarily fixed open subset of Ω, and $f \in C_0^1(U; \mathbb{R}^d)$ with $|f| \le a$. From the L^2-weak convergence of v_n we obtain

$$\begin{aligned}
&\int_U v\nabla\cdot f\ dx + \delta \int_{\Omega\setminus U} a\ dx = \lim_{n\to\infty}\left(\int_U v_n\nabla\cdot f\ dx + \delta\int_{\Omega\setminus U} a\ dx\right)\\
&= \liminf_{n\to\infty}\left(\int_U v_n\nabla\cdot f\ dx + \delta\int_{\Omega\setminus U} a\ dx\right)\\
&\le \liminf_{n\to\infty}\sup_{V,g}\left\{\int_V v_n\nabla\cdot g\ dx + \delta\int_{\Omega\setminus V} a\ dx : V\in\mathcal{O}(\Omega),\ g\in C_0^1(V;\mathbb{R}^d),\ |g|\le a\right\}\\
&= \liminf_{n\to\infty}\int_\Omega a\max\{|\nabla v_n|,\delta\}dx.
\end{aligned}$$

The last equality above uses Proposition 4.

By taking the supremum over all $U \in \mathcal{O}(\Omega)$ and $f \in C_0^1(\Omega; \mathbb{R}^d)$ with $|f| \le a$, and by Proposition 4, we obtain (42). □

7. **A regularized, stable scheme for the non-degenerate minimization problem.** In this section we show stability with respect to the coefficient a of the minimum value in the problem (33), and show that an $H^1(\Omega) \cap C(\overline{\Omega})$-smooth minimizer depends continuously in a. With minor modifications, the arguments are the same as in the Sections 2 and 3.

Recall that we assume existence of a minimizer $u_0 \in H^1(\Omega)$ of (1), whose gradient stays away from zero as in (32). As before, we seek a minimizer in $\{u\} + H_0^1(\Omega)$, where $u \in H^1(\Omega)$ is the harmonic map with trace f at the boundary, and study the minimization of the functional $F^\delta[\cdot\,; a]$ in (8). Theorem 6.1 shows that

$$u_0 - u \in \operatorname{argmin}\{F^\delta[h; a] :\ h \in H_0^1(\Omega)\}.$$

For $\tilde{a} \in L^2(\Omega)$ nearby a, the pair $(\tilde{a}, f)$ may not be admissible, hence $F^\delta[\,\cdot\,; \tilde{a}]$ may not have a minimizer. This is the reason why we regularize F^δ as follows.

Proposition 5. *Let $\tilde{a} \in L^2(\Omega)$ be nonnegative, and $\epsilon > 0$ be arbitrarily fixed. Then*

$$F_\epsilon^\delta[h; \tilde{a}] := F^\delta[h; \tilde{a}] + \epsilon \int_\Omega |\nabla h|^2 dx \tag{43}$$

is weakly lower semicontinuous in $H_0^1(\Omega)$ and has a unique minimizer

$$h_\epsilon = \mathit{argmin}\{F_\epsilon^\delta[h; \tilde{a}] :\ h \in H_0^1(\Omega)\}.$$

Proof. Let $\{h_n\} \subset H_0^1(\Omega)$ be such that $h_n \rightharpoonup h$ in $H_0^1(\Omega)$. Then $(u + h_n) \rightharpoonup (u + h)$ in $L^2(\Omega)$. By Theorem 6.4,

$$\int_\Omega \tilde{a} \max\{|\nabla(u + h)|, \delta\} dx \le \liminf_{n\to\infty} \int_\Omega \tilde{a} \max\{|\nabla(u + h_n)|, \delta\} dx.$$

Since $h \to \|h\|_{1,0}$ is also lower semicontinuity in $H_0^1(\Omega)$, we obtain

$$F_\epsilon^\delta[h; \tilde{a}] \le \liminf_{n\to\infty} F^\delta[h_n; \tilde{a}] + \liminf_{n\to\infty} \epsilon \int_\Omega |\nabla h_n|^2 dx \le \liminf_{n\to\infty} F_\epsilon^\delta[h_n; \tilde{a}].$$

As before, due to the regularization, the functional (43) is both coercive in $H^1_0(\Omega)$ and strictly convex, and thus, it has a unique minimizer in $H^1_0(\Omega)$. □

The next result shows that, for fixed $\epsilon > 0$, the unique minimizer depends continuously on the coefficient $\tilde{a}$.

Theorem 7.1. *Let $\epsilon > 0$ be fixed. Let $\{a_n\} \subset L^2(\Omega)$ be a sequence with $a_n \to \tilde{a}$ in $L^2(\Omega)$. Consider the minimizers*

$$h_{n,\epsilon} = argmin\{F^\delta_\epsilon[h; a_n] : \ h \in H^1_0(\Omega)\}, \text{ and}$$
$$h_\epsilon = argmin\{F^\delta_\epsilon[h; \tilde{a}] : \ h \in H^1_0(\Omega)\}$$

as given by Proposition 5. Then $h_{\epsilon,n} \rightharpoonup h_\epsilon$ in $H^1_0(\Omega)$.

Proof. We show first that the sequence $\{h_{\epsilon,n}\}$ is bounded in $H^1_0(\Omega)$. We estimate

$$F^\delta[0; \tilde{a}] = \int_\Omega \tilde{a} \max\{|\nabla u|, \delta\} dx \leq \int_\Omega \tilde{a}(|\nabla u| + \delta) dx \leq \|\tilde{a}\|(\|\nabla u\| + \delta\sqrt{|\Omega|}),$$

where $|\Omega|$ is the Lebesgue measure of Ω. Then, using the minimizing property defining $h_{\epsilon,n}$,

$$\begin{aligned}\epsilon\|h_{\epsilon,n}\|^2_{1,0} \leq& F^\delta_\epsilon[h_{\epsilon,n}; a_n] \leq F^\delta_\epsilon[0; a_n] = F^\delta[0; \tilde{a}] + F^\delta[0; a_n - \tilde{a}]\\ \leq& (\|\tilde{a}\| + \|a_n - \tilde{a}\|)(\|\nabla u\| + \delta\sqrt{|\Omega|}) \leq 2\|\tilde{a}\|(C\|f\|_{1/2} + \delta\sqrt{|\Omega|}).\end{aligned}$$

The last inequality uses (10) and assumes n large enough that $\|\tilde{a} - a_n\| \leq \|\tilde{a}\|$. Therefore the sequence $\{h_{\epsilon,n}\}$ is bounded,

$$\|h_{\epsilon,n}\|_{1,0} \leq \frac{\sqrt{2\|\tilde{a}\|(C\|f\|_{1/2} + \delta\sqrt{|\Omega|})}}{\sqrt{\epsilon}} =: C_\epsilon, \tag{44}$$

and there exists $\tilde{h}_\epsilon \in H^1_0(\Omega)$ a weak limit of a subsequence of $\{h_{\epsilon,n_k}\}$.

Let $\delta_n = \|a_n - \tilde{a}\|$, and $h \in H^1_0(\Omega)$ be arbitrary. The estimate below is needed later:

$$\begin{aligned}F^\delta[h; a_n - \tilde{a}] &= \int_\Omega (a_n - \tilde{a}) \max\{|\nabla(u + h)|, \delta\} dx\\ &\leq \int_\Omega |a_n - \tilde{a}|(|\nabla u| + |\nabla h| + \delta) dx\\ &\leq \|a_n - \tilde{a}\|(\|\nabla u\| + \|h\|_{1,0} + \delta\sqrt{|\Omega|})\\ &\leq \delta_n(C\|f\|_{1/2} + \|h\|_{1,0} + \delta\sqrt{|\Omega|}).\end{aligned} \tag{45}$$

Since the map $h \mapsto F_\epsilon^\delta[h;\tilde{a}]$ is weakly lower semi-continuous on $H_0^1(\Omega)$, for each $h \in H_0^1(\Omega)$ arbitrarily fixed the following estimates hold:

$$\begin{aligned}
F_\epsilon^\delta[\tilde{h}_\epsilon;\tilde{a}] \leq & \liminf_{k\to\infty} F_\epsilon^\delta[h_{\epsilon,n_k};\tilde{a}] = \liminf_{k\to\infty} \left\{F_\epsilon^\delta[h_{\epsilon,n_h};a_{n_k}] + F^\delta[h_{\epsilon,n_k};\tilde{a}-a_{n_k}]\right\} \\
\leq & \liminf_{k\to\infty} \left\{F_\epsilon^\delta[h_{\epsilon,n_h};a_{n_k}] + \delta_{n_k}(C\|f\|_{1/2} + \|h_{\epsilon,n_k}\|_{1,0} + \delta\sqrt{|\Omega|})\right\} \\
\leq & \liminf_{k\to\infty} \left\{F_\epsilon^\delta[h_{\epsilon,n_k};a_{n_k}] + \delta_{n_k}(C\|f\|_{1/2} + C_\epsilon + \delta\sqrt{|\Omega|})\right\} \\
\leq & \liminf_{k\to\infty} \left\{F_\epsilon^\delta[h;a_{n_k}] + \delta_{n_k}(C\|f\|_{1/2} + C_\epsilon + \delta\sqrt{|\Omega|})\right\} \\
= & \liminf_{k\to\infty} \left\{F_\epsilon^\delta[h;\tilde{a}] + F^\delta[h;a_{n_k}-\tilde{a}] + \delta_{n_k}(C\|f\|_{1/2} + C_\epsilon + \delta\sqrt{|\Omega|})\right\} \\
\leq & \liminf_{k\to\infty} \left\{F_\epsilon^\delta[h;\tilde{a}] + \delta_{n_k}(2C\|f\|_{1/2} + \|h\|_{1,0} + C_\epsilon + 2\delta\sqrt{|\Omega|})\right\} \\
\leq & \limsup_{k\to\infty} \left\{F_\epsilon^\delta[h;\tilde{a}] + \delta_{n_k}(2C\|f\|_{1/2} + \|h\|_{1,0} + C_\epsilon + 2\delta\sqrt{|\Omega|})\right\} \\
\leq & F_\epsilon^\delta[h;a] + \limsup_{k\to\infty} \delta_{n_k}(2C\|f\|_{1/2} + \|h\|_{1,0} + C_\epsilon + 2\delta\sqrt{|\Omega|}) \\
= & F_\epsilon^\delta[h;\tilde{a}].
\end{aligned}$$

The second and fifth inequalities use (45), the third inequality uses (10) and (17), and the fourth inequality uses the minimizing property defining $\{h_{\epsilon,n_k}\}$.

Therefore $\tilde{h}_\epsilon$ is a minimizer for F_ϵ^δ in $H_0^1(\Omega)$. Since the minimizer is unique, $\tilde{h}_\epsilon = h_\epsilon$. Since any other weakly convergent subsequence of $\{h_{\epsilon,n}\}$ also converges to h_ϵ, the entire sequence is weakly convergent to h_ϵ. □

Next, we prove an analogue of Theorem 3.1 for the non-degenerate functional (8).

Theorem 7.2. *Let $(a,f) \in L^2(\Omega) \times H^{1/2}(\partial\Omega)$ be admissible, $u_0 \in H^1(\Omega)$ be a solution of (1) satisfying (32) for some $\delta > 0$, and u be the harmonic function with trace f at the boundary. Let $\{a_n\} \subset L^2(\Omega)$ be a sequence with $a_n \to a$ in $L^2(\Omega)$. Choose $\epsilon_n \downarrow 0$ in such a way that*

$$\lim_{n\to\infty} \frac{\|a-a_n\|^2}{\epsilon_n} = 0,$$

and, for each n, let h_{ϵ_n} be the solution of the minimization problem

$$h_{\epsilon_n} = \mathit{argmin}\{F_{\epsilon_n}^\delta[h;a_n] : \ h \in H_0^1(\Omega)\},$$

with $F_{\epsilon_n}^\delta[h;a_n]$ defined in (43). Then

$$\liminf_{n\to\infty} F_{\epsilon_n}^\delta[h_{\epsilon_n};a_n] = \liminf_{n\to\infty} F^\delta[h_{\epsilon_n};a_n] = \min\{F^\delta[h;a] : \ h \in H_0^1(\Omega)\}.$$

In addition, if a is bounded from below as in (6), then a subsequence of $\{h_{\epsilon_n}\}$ converges in $L^q(\Omega)$ to some h^*, for $1 \leq q < d/(d-1)$. Furthermore, provided that $u_0 \in C(\overline{\Omega})$ and $h^* \in H_0^1(\Omega) \cap C(\overline{\Omega})$, we have

$$h^* = u_0 - u. \tag{46}$$

Proof. Let $\delta_n = \|a-a_n\|$ be as before. Since $h_0 := u_0 - u$ is a minimizer of $F^\delta[\,\cdot\,;a]$ over $H_0^1(\Omega)$, we have

$$F^\delta[h_0;a] \leq \liminf_{n\to\infty} F^\delta[h_{\epsilon_n};a] \leq \liminf_{n\to\infty} F_{\epsilon_n}^\delta[h_{\epsilon_n};a].$$

Next we show that the reverse inequality also holds:

$$\begin{aligned}
\liminf_{n\to\infty} F^{\delta}_{\epsilon_n}[h_{\epsilon_n}; a] =& \liminf_{n\to\infty}\{F^{\delta}_{\epsilon_n}[h_{\epsilon_n}; a_n] + F^{\delta}[h_{\epsilon_n}; a - a_n]\} \\
\leq& \liminf_{n\to\infty}\{F^{\delta}_{\epsilon_n}[h_{\epsilon_n}; a_n] + \delta_n(C\|f\|_{1/2} + \|h_{\epsilon_n}\|_{1,0} + \delta\sqrt{|\Omega|})\} \\
\leq& \liminf_{n\to\infty}\left\{F^{\delta}_{\epsilon_n}[h_{\epsilon_n}; a_n] + \delta_n(C\|f\|_{1/2} + C_{\epsilon_n} + \delta\sqrt{|\Omega|})\right\} \\
\leq& \liminf_{n\to\infty}\left\{F^{\delta}_{\epsilon_n}[h_0; a_n] + \delta_n(C\|f\|_{1/2} + C_{\epsilon_n} + \delta\sqrt{|\Omega|})\right\} \\
=& \liminf_{n\to\infty}\left\{F^{\delta}[h_0; a_n] + \epsilon_n\|h_0\|_{1,0} + \delta_n(C\|f\|_{1/2} + C_{\epsilon_n} + \delta\sqrt{|\Omega|})\right\} \\
=& \liminf_{n\to\infty}\left\{F^{\delta}[h_0; a] + F^{\delta}[h_0; a_n - a] + \epsilon_n\|h_0\|_{1,0} + \delta_n(C\|f\|_{1/2} + C_{\epsilon_n} + \delta\sqrt{|\Omega|})\right\} \\
\leq& \liminf_{n\to\infty}\left\{F^{\delta}[h_0; a] + \epsilon_n\|h_0\|_{1,0} + \delta_n(2C\|f\|_{1/2} + \|h_0\|_{1,0} + C_{\epsilon_n} + 2\delta\sqrt{|\Omega|})\right\} \\
\leq& \limsup_{n\to\infty}\left\{F^{\delta}[h_0; a] + \epsilon_n\|h_0\|_{1,0} + \delta_n(2C\|f\|_{1/2} + \|h_0\|_{1,0} + C_{\epsilon_n} + 2\delta\sqrt{|\Omega|})\right\} \\
=& F^{\delta}[h_0; a].
\end{aligned}$$

In the estimate above, the first and fourth inequalities use (45), the third inequality uses the minimizing property of $\{h_{\epsilon_n}\}$, and the last equality uses the hypothesis $\delta_n = o(\sqrt{\epsilon_n})$ to conclude $\delta_n C_{\epsilon_n} \to 0$. This proves the identity (21).

Next we show that $\{h_{\epsilon_n}\}$ is bounded in $W^{1,1}_0(\Omega)$. From the positivity assumption (6) we have

$$\begin{aligned}
\alpha\|h_{\epsilon_n}\|_{1,1} \leq& \int_\Omega a|\nabla h_{\epsilon_n}|dx \leq \int_\Omega a|\nabla(u + h_{\epsilon_n})|dx + \int_\Omega a|\nabla u|dx \\
\leq& F^{\delta}[h_{\epsilon_n}; a] + \|a\|\,\|\nabla u\| \leq F^{\delta}[h_{\epsilon_n}; a] + C\|a\|\,\|f\|_{1/2},
\end{aligned}$$

and show that the right hand side is bounded uniformly in n:

$$\begin{aligned}
F^{\delta}[h_{\epsilon_n}; a] \leq& F^{\delta}_{\epsilon_n}[h_{\epsilon_n}; a] \\
=& F^{\delta}_{\epsilon_n}[h_{\epsilon_n}; a_n] + F^{\delta}[h_{\epsilon_n}; a - a_n] \\
\leq& F^{\delta}_{\epsilon_n}[h_0; a_n] + \delta_n(C\|f\|_{1/2} + \|h_{\epsilon_n}\|_{1,0} + \delta\sqrt{|\Omega|}) \\
=& F^{\delta}[h_0; a_n] + \epsilon_n\|h_0\|_{1,0} + \delta_n(C\|f\|_{1/2} + C_{\epsilon_n} + \delta\sqrt{|\Omega|}) \\
\leq& (\|a_n\| + \delta_n)(C\|f\|_{1/2} + \delta\sqrt{|\Omega|}) + (\|a_n\| + \epsilon_n)\|h_0\|_{1,0} + \delta_n C_{\epsilon_n} \\
\leq& 2\|a\|(C\|f\|_{1/2} + \delta\sqrt{|\Omega|}) + 2\|a\|\|h_0\|_{1,0} + 1,
\end{aligned}$$

In the above estimate the second inequality uses the minimization property of h_{ϵ_n} and (45), and the last inequality requires n to be sufficiently large and uses the hypothesis (20).

An application of Rellich-Kondrachov's compactness imbedding shows the existence of a convergent subsequence $h_{\epsilon_n} \to h^*$ in $L^q(\Omega)$ for all $1 \leq q \leq d/(d-1)$.

If $u_0 \in C(\overline{\Omega})$, then $f \in C(\partial\Omega)$ and the harmonic map $u \in C(\overline{\Omega})$ ([6]). If $h^* \in H^1_0(\Omega) \cap C(\overline{\Omega})$, then both $u + h^*$ and u_0 are minimizers of the problem (33) and we can apply the uniqueness part of Theorem 6.1 to conclude (46). □

8. **Application to the conductivity imaging problem.** In this section we show how Theorem 7.2 can be applied to the conductivity imaging problem. The domain $\Omega \subset \mathbb{R}^d$ is bounded with Lipschitz boundary that satisfies the exterior cone condition at each point $x \in \partial\Omega$ ([6]).

A priori assumptions:

- (H1) the (unknown) conductivity σ has a (known) upper bound: $\sigma(x) \leq M$, a.e. $x \in \Omega$;
- (H2) the magnitude a of the current density field is bounded away from zero as in (6), for some $\alpha > 0$.

Proposition 6. *Assume that the data $(a, f) \in L^2(\Omega) \times C(\partial\Omega)$ satisfy (H1) and (H2). Let $\{a_n\} \subset L^2(\Omega)$ be a sequence with $a_n \to a$ in $L^2(\Omega)$. Define $\epsilon_n = \|a - a_n\|$ and $\delta = \alpha/M$.*

For $n \in \mathbb{N}$, let h_n be the unique solution of the minimization problem:

$$h_n = \mathit{argmin}\{F^{\delta}_{\epsilon_n}[h; a_n] : \ h \in H^1_0(\Omega)\},$$

with $F^{\delta}_{\epsilon_n}[h; a_n]$ given in (43) with u being the harmonic function with trace f at the boundary.

Then, on a subsequence, $\{h_n\}$ converges in $L^q(\Omega)$ to some $h^* \in L^q(\Omega) \cap BV_0(\Omega)$, $1 \leq q < d/(d-1)$.

Provided that $h^* \in H^1_0(\Omega) \cap C(\overline{\Omega})$, we recover the voltage potential by

$$u_\sigma = u + h^*, \tag{47}$$

and the conductivity by

$$\sigma = a/|\nabla u_\sigma|. \tag{48}$$

Proof. The voltage potential u_σ is uniquely defined by the Dirichlet problem

$$\nabla \cdot \sigma \nabla u_\sigma = 0, \quad u_\sigma|_{\partial\Omega} = f.$$

Since f is continuous, the elliptic regularity (up to the boundary) implies $u_\sigma \in H^1(\Omega) \cap C(\overline{\Omega})$ ([6, Theorem 8.30]). Moreover,

$$|\nabla u_\sigma| = \frac{a}{\sigma} \geq \frac{\alpha}{M},$$

and (32) is satisfied for $\delta = \alpha/M$.

The specific choice of ϵ_n made above implies that $\|a - a_n\| = o(\sqrt{\epsilon_n})$. Now we apply Theorem 7.2 and use (46) to recover u_σ. The reconstruction of σ follows from the structure of a in (2). □

For simply connected planar domains, an almost two-to-one boundary voltage f guarantees an interior voltage potential u_σ which satisfies (7), for some $\delta > 0$ ([1, 15]), and, consequently, the positivity relation (6) holds.

Corollary 1. *Let $\Omega \subset \mathbb{R}^2$ be simply connected, $a \in L^2(\Omega)$ be the magnitude of the current density field (as in (2)) generated while maintaining an almost two-to-one boundary voltage $f \in C(\partial\Omega)$. Then there exists $\delta > 0$ sufficiently small such that*

$$u_\sigma = \mathit{argmin}\{F^{\delta}[h; a] : \ h \in H^1_0(\Omega)\}, \tag{49}$$

where F^δ is given by (8), with u being the harmonic map of trace f at the boundary. Moreover, for any $\{a_n\} \subset L^2(\Omega)$ sequence, with $a_n \to a$ in $L^2(\Omega)$, define $\epsilon_n = \|a - a_n\|$, and let h_n be the unique solution of the minimization problem:

$$h_n = \mathit{argmin}\{F^{\delta}_{\epsilon_n}[h; a_n] : \ h \in H^1_0(\Omega)\},$$

with $F^{\delta}_{\epsilon_n}$ as in (43).

Then, on a subsequence, $\{h_n\}$ converges in $L^q(\Omega)$ to some $h^* \in L^q(\Omega) \cap BV_0(\Omega)$, $1 \leq q < d/(d-1)$.

Moreover, if $h^* \in H_0^1(\Omega)\cap C(\overline{\Omega})$*, then* u_σ *is recovered via the formula* $u_\sigma = u+h^*$.

9. **Concluding remarks.** We considered the problem of minimizing the functional $F[u;a] = \int_\Omega a|\nabla u|dx$ over functions in $H^1(\Omega)$ with fixed trace f at the boundary. We showed stability of the minimum value with respect to a, in a neighborhood of a specific coefficient: a is such that the pair (a, f) is admissible. If a is bounded continuous and positive, the functional above has a natural extension to the space of functions of bounded variation $BV(\Omega)$. The stability of the extended functional is preserved in this larger space. In both cases, the method of proof constructs a minimizing sequence out of $H^1(\Omega)$ minimizers of some regularized functionals.

Under the assumption that F has a solution $u \in H^1(\Omega)$ with $|\nabla u| \geq \delta$, for some $\delta > 0$, we prove a similar stability result for the non-degenerate version of the functional $F^\delta[u;a] = \int_\Omega a\max\{|\nabla u|, \delta\}dx$ over maps in $H^1(\Omega)$. Such problems occur in electrical conductivity imaging when the magnitude of the current density field is known in the interior of Ω. The positivity assumption above is implied by the positivity in the coefficient a. In two dimensional domains, the positivity assumption can be insured a priori by imposing an almost two-to-one boundary voltage f. The minimization problem for F^δ has a unique solution in $H^1(\Omega)\cap C(\overline{\Omega})$. Consequently, if the minimizer (obtained as the limit in $L^q(\Omega)$ of minimizers of regularized functionals) lies in $H^1(\Omega) \cap C(\overline{\Omega})$ we can recover the voltage potential inside, and therefore, the conductivity inside.

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