On the Scattering Method for the $\overline{\partial}$-equation and Reconstruction of Convection Coefficients

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Abstract: In this paper we reconstruct convection coefficients from boundary measurements. We reduce the Beals and Coifman formalism from a linear first order system to a formalism for the $\overline{\partial}$-equation.

1 Introduction

The works of Nachman and Ablowitz [16], Nachman [17] and Henkin and Novikov [11] introduced the $\partial\overline{\partial}$- scattering methods to the parameter identification problems. In there, the linear Schrödinger equation in the physical space is paired with a pseudo-analytic equation in the complex space of the parameter. Another method, due to Beals and Coifman [2], pairs a first order $\overline{\partial}$-system in the physical space with a pseudo-analytic matrix equation in the parameter space. Sung analyzed lower regularity assumptions in [21, 22, 23]. This method was used by Brown and Uhlmann [4] in the Calderón problem [5] of determining the conductivity σ in $\nabla \cdot \sigma \nabla u = 0$, by Francini [9] in finding complex conductivities (with sufficiently small permittivity) and by Cheng and Yamamoto [6], [7] in proving unique determination of the convection coefficients b_1 and b_2 in

$$\Delta u + b_1 u_x + b_2 u_y = 0. \tag{1}$$

We consider here the scattering problem for the $\overline{\partial}$- equation (Theorems 1.1 and 1.2 below). Here $\overline{\partial} = (\partial_x + i\partial_y)/2$ is the Cauchy-Riemann operator. This can be seen as a diagonal version of the formalism in Beals and Coifman, see Lemma 2.1 below. Due to the symmetry between the scattered solutions in the physical space and the ones in the parameter space, we are able to present a non-linear analog of the Fourier inversion formula. Unlike Beals and Coifman's formalism, the $\partial\overline{\partial}$- scattering method for the $\overline{\partial}$-equation extends to a class of systems $\overline{\partial} u - Q\overline{u} = 0$, where Q is now a matrix-valued map whose eigenvectors have analytic entries. However, this extension is beyond the purpose of this paper.

As an application, we revisit the inverse problem proposed in [7] and present a reconstruction procedure. This problem has a potential industrial application in measuring magnetic fields inside inaccessible narrow slabs. One can introduce a sheet of (homogeneous) metal and apply currents on the sides measuring the resulting potential. The potential will then be distributed across the metal sheet according to the equation (1).

The reconstruction method is a combination of the boundary characterization of the exponentially growing solutions introduced by Knudsen and Tamasan in [13] with the $\partial\overline{\partial}$- scattering method for the $\overline{\partial}$-equation.

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For $k \in \mathbb{C}$ arbitrarily fixed, we say that u behaves like e^{izk} (written $u \sim e^{izk}$) in $L^r(\mathbb{R}_z^2)$ for large z, if $u(z,k)e^{-izk} - 1 \in L^r(\mathbb{R}_z^2)$. We use the notation $\langle k \rangle = (1+|k|^2)^{1/2}$. The scattering method is the content of the following two theorems.

Theorem 1.1 (Forward Scattering). *Assume that $q \in L_c^{\tilde{p}}(\mathbb{R}_z^2)$, $\tilde{p} > 2$ has compact support. For each $k \in \mathbb{C}$, the equation*

$$\frac{\partial \Psi}{\partial \overline{z}}(z) + q(z)\overline{\Psi}(z) = 0, \;\; z \in \mathbb{C}, \tag{2}$$

has unique solutions $\Psi_r(z,k) \sim e^{izk}$ and $\Psi_i(z,k) \sim ie^{izk}$ in $L^{\tilde{p}}(\mathbb{R}_z^2)$ for large z, and the scattering transform

$$t(k) = -\frac{i}{\pi}\int_{\mathbb{R}^2} e^{\overline{izk}}\overline{q}(z)\left(\Psi_r(z,k) - i\Psi_i(z,k)\right) d\mu(z) \tag{3}$$

is well defined. Moreover, if $q \in W_c^{\varepsilon,\tilde{p}}(\mathbb{R}_z^2)$ for some $\varepsilon > 0$ and $k \in \mathbb{C} - \{0\}$, we have

$$\|\Psi_r(z,k)e^{-izk} - 1\|_{L^{\tilde{p}}(\mathbb{R}_z^2)} + \|\Psi_i(z,k)e^{-izk} - i\|_{L^{\tilde{p}}(\mathbb{R}_z^2)} \leq C\langle k \rangle^{-\varepsilon} \tag{4}$$

and

$$\|[\Psi_r(z,k) - i\Psi_i(z,k)]e^{-izk} - 2\|_{W^{1,\tilde{p}}(\mathbb{R}_z^2)} \leq C\langle k \rangle^{-\varepsilon}, \tag{5}$$

and then $t \in L^r(\mathbb{R}_k^2)$ for each $r > 2/(\varepsilon+1)$. In particular $t \in L^r(\mathbb{R}_k^2) \cap L^{r'}(\mathbb{R}_k^2) \cap L^{\tilde{r}}(\mathbb{R}_k^2)$ for some $r < 2$, where $\tilde{r}^{-1} = r^{-1} - 1/2$ and $r'^{-1} + r^{-1} = 1$.

Theorem 1.2 (Inverse Scattering). *Let q, Ψ_r, Ψ_i and $t(k)$ and r, r', $\tilde{r}$ be as given in the forward scattering. Then the equation*

$$\frac{\partial \Phi}{\partial \overline{k}}(k) + t(k)\overline{\Phi}(k) = 0, \;\; k \in \mathbb{C}, \tag{6}$$

has unique solutions $\Phi_r \sim e^{izk}$ and $\Phi_i \sim ie^{izk}$ in $L^{\tilde{r}}(\mathbb{R}_k^2)$ for large $k \in \mathbb{C}$. Moreover, Ψ's and Φ's are related by

$$\begin{aligned} \operatorname{Re}\Phi_i &= -\operatorname{Im}\Psi_r, & \operatorname{Re}\Phi_r &= \operatorname{Re}\Psi_r, \\ \operatorname{Im}\Phi_i &= \operatorname{Im}\Psi_i, & \operatorname{Im}\Phi_r &= -\operatorname{Re}\Psi_i, \end{aligned} \tag{7}$$

in particular $\Phi_r - i\Phi_i = \Psi_r - i\Psi_i$ and

$$q(z) = -\frac{i}{\pi}\int_{\mathbb{R}^2} e^{\overline{izk}}\overline{t}(k)\left(\Phi_r(z,k) - i\Phi_i(z,k)\right) d\mu(k). \tag{8}$$

Let $\Omega \subset \mathbb{R}^2$ be a bounded, simple connected domain with Lipschitz boundary and $\tilde{p} > 2$. For $b_1, b_2 \in L^{\tilde{p}}(\Omega)$ and $g \in W^{2-1/\tilde{p},\tilde{p}}(\partial\Omega)$, let $u \in W^{2,\tilde{p}}$ be the unique solution of the boundary value problem

$$\begin{aligned} &\Delta u(x) + b_1(x)\frac{\partial u}{\partial x_1}(x) + b_2(x)\frac{\partial u}{\partial x_2}(x) = 0, \qquad x \in \Omega \\ &u(x) = g(x), \qquad x \in \partial\Omega. \end{aligned} \tag{9}$$

The Dirichlet to Neumann map $\Lambda_{b_1,b_2} : W^{2-1/\tilde{p},\tilde{p}}(\partial\Omega) \to W^{1-1/\tilde{p},\tilde{p}}(\partial\Omega)$ is given by

$$\Lambda_{b_1,b_2} g(x) = \nu_1(x)\frac{\partial u}{\partial x_1}(x) + \nu_2(x)\frac{\partial u}{\partial x_2}(x), \qquad x \in \partial\Omega,$$

where $(\nu_1(x), \nu_2(x))$ is the outer normal at x on the boundary.

We prove the following reconstruction result.

Theorem 1.3. *Let $\Omega \subset \mathbb{R}^2$ be bounded, simple connected domain with Lipschitz boundary. For some $\varepsilon > 0$ assume that $b_1, b_2 \in W_c^{\varepsilon,\tilde{p}}(\Omega)$ with support inside Ω. Then b_1, b_2 can be reconstructed from Λ_{b_1,b_2}.*

The coefficients do not need to vanish but rather their traces on the boundary need to be known (which is the case in practical applications). Then one can extend them across the boundary, preserving the regularity and have them vanish outside a larger disc. The Dirichlet-to-Neumann map can be pushed to an outside boundary via an argument similar to the one in [18], see also [13]. It has not been shown that the Dirichlet-to-Neumann for this problem uniquely determines the boundary values of the coefficients.

While $L^{\tilde{p}}(\Omega)$ is enough regularity to prove unique determination of b_1, b_2, see [6], for the reconstruction we need ϵ-extra regularity.

2 Proof of the theorems 1.1 and 1.2

We identify a point in $\mathbb{R}^2$ with a point in the complex plane by $x_1 + ix_2 = z$. By $\overline{\partial}^{-1}$ we denote the solid Cauchy transform $\overline{\partial}^{-1} f(z) = \frac{1}{\pi}\int_{\mathbb{R}^2} \frac{f(\zeta)}{z-\zeta} d\mu(\zeta)$, (here $d\mu(\zeta)$ is the Lebesgue area). We also denote by $e(z,k) = \exp(i(zk + \overline{zk}))$.

We look for solutions of (2) of the form $\Psi_r = \psi_r e^{izk}$ and $\Psi_i = i\psi_i e^{izk}$ with $\psi_r, \psi_i \in 1 + L^{\tilde{p}}(\mathbb{R}^2_z)$. The equations for ψ_r respectively ψ_i are

$$\begin{aligned} \frac{\partial}{\partial \overline{z}}\psi_r + qe(z,-k)\overline{\psi_r} = 0, \\ \frac{\partial}{\partial \overline{z}}\psi_i - qe(z,-k)\overline{\psi_i} = 0. \end{aligned} \tag{10}$$

The Hardy-Littlewood-Sobolev inequality gives $\overline{\partial}^{-1} : L^p(\mathbb{R}^2) \to L^{\tilde{p}}(\mathbb{R}^2)$ is bounded (see Stein [19]) for p and $\tilde{p}$ related by

$$\frac{1}{\tilde{p}} = \frac{1}{p} - \frac{1}{2}. \tag{11}$$

These indexes preserve this meaning throughout the paper.

Since $q \in L_c^{\tilde{p}}(\mathbb{R}^2) \subset L^2(\mathbb{R}^2)$ and $L^{\tilde{p}}(\mathbb{R}^2) \cdot L^2(\mathbb{R}^2) \subset L^p(\mathbb{R}^2)$ we have $\overline{\partial}^{-1}(q\cdot) : L^{\tilde{p}}(\mathbb{R}^2) \to L^{\tilde{p}}(\mathbb{R}^2)$ is also bounded. Moreover, as q has compact support we can use Rellich imbedding to conclude that $\overline{\partial}^{-1}(q\cdot) : L^{\tilde{p}}(\mathbb{R}^2) \to L^{\tilde{p}}(\mathbb{R}^2)$ is compact. Then we can apply Fredholm's alternative in $L^{\tilde{p}}(\mathbb{R}^2)$ to the equivalent integral equation

$$\{I + \overline{\partial}^{-1}[q(\cdot)e(\cdot,-k)\overline{(\cdot)}]\}(\psi_r(z) - 1) = \overline{\partial}^{-1}[qe(\cdot,-k)]. \tag{12}$$

A similar equation holds for ψ_r. The homogeneous equation has only the null solution due to Liouville's theorem for pseudo-analytic functions with coefficients in $L^{\tilde{p}}(\mathbb{R}^2_z) \cap L^p(\mathbb{R}^2_z)$ shown by Vekua [V62]. These prove the (uniform in k) bounded-ness of the map $[I - \overline{\partial}^{-1} qe(\cdot,-k)\cdot]^{-1}$ from $L^{\tilde{p}}(\mathbb{R}^2_z)$ to itself. The estimate

$$||\overline{\partial}^{-1}(e(\cdot,-k)q)||_{L^{\tilde{p}}(\mathbb{R}^2_z)} \leq C\langle k\rangle^{-\epsilon}||q||_{W^{\varepsilon,p}(\mathbb{R}^2_z)} \tag{13}$$

from (Proposition 2.3) [13] concludes the decay estimates in (4).

Next, we clarify the relation with the formalism of Beals and Coifman. This will help us show the regularity property of $\Psi_r + i\Psi_i$ in (5) and prove the theorem 1.2. Let us define $m_1(z,k)$ and $m_2(z,k)$ in terms of the ψ's by

$$m_1(z,k) = \frac{1}{2}\left(\psi_r(z,k) + \psi_i(z,k)\right), \tag{14}$$
$$m_2(z,k) = \frac{1}{2}e(z,-k)\left(\overline{\psi_i}(z,k) - \overline{\psi_r}(z,k)\right).$$

The result below shows that $(m_1, m_2)^t$ is the first column of the Jost matrix in the exponentially growing solutions for the first order $\overline{\partial}$ system considered by Beals and Coifman.

Lemma 2.1. *Let m_1 and m_2 defined in* (14)*. Then $m_1(\cdot,k)-1, m_2(\cdot,k) \in L^{\tilde{p}}(\mathbb{R}_z)$, and they satisfy*

$$\overline{\partial} m_1 = q m_2, \tag{15}$$
$$(\partial + ik) m_2 = \overline{q} m_1.$$

Moreover, the following estimates hold:

$$||m_1(\cdot,k) - 1||_{W^{1,\tilde{p}}(\mathbb{R}^2_z)} \leq C\langle k\rangle^{-\epsilon}, \tag{16}$$

$$||m_2(\cdot,k)||_{L^{\tilde{p}}(\mathbb{R}^2_z)} \leq C\langle k\rangle^{-\epsilon}. \tag{17}$$

Proof. From their definition $m_1(\cdot,k) - 1, m_2(\cdot,k) \in L^{\tilde{p}}(\mathbb{R}^2_z)$ since $\psi_r, \psi_i \in L^{\tilde{p}}(\mathbb{R}^2_z)$ and $|e(z,k)| = 1$. The fact that they solve the system (15) comes from a straightforward calculation and the equations (10). The $L^{\tilde{p}}(\mathbb{R}^2_z)$-estimates of decay in k for both m_1 and m_2 come from the estimates (4) for ψ_r and ψ_i proven above. We are left to justify the extra smoothness gained by m_1. From the first equation we have that $m_1 - 1 = \overline{\partial}^{-1}(q m_2)$. Since $q \in L^{\tilde{p}}_c(\mathbb{R}^2) \subset L^2(\mathbb{R}^2)$ we have $\overline{\partial}^{-1}(q m_2) \in W^{1,\tilde{p}}(\mathbb{R}^2_z)$ with an imbedding constant independent of k. From where $||m_1(\cdot,k) - 1||_{W^{1,\tilde{p}}(\mathbb{R}^2_z)} = \|\overline{\partial}^{-1}(q m_2)\|_{W^{1,\tilde{p}}(\mathbb{R}^2_z)} \leq C\|q\|_{L^2(\mathbb{R}^2)}\|m_2(\cdot,k)\|_{L^{\tilde{p}}(\mathbb{R}^2_z)} \leq C < k >^{-\epsilon}$. ☐

Since $2m_1 = [\Psi_r(z,k) - i\Psi_i(z,k)]e^{-izk} - 2$ the lemma above shows the extra regularity estimate in (5), thus finishing the proof of the Theorem 1.1.

The inverse scattering method of Beals and Coifman regards the behavior in k of the unique solutions $m_1(z,k)$, $m_2(z,k)$ of (15) in the lemma above. More precisely, the map $k \to m(\cdot,k)$ is differentiable (in the strong operator norm topology from L^p_α to $L^{\tilde{p}}_\beta$, with $\alpha > 2/p'$ and $\beta > 2/\tilde{p}$) as shown by Nachman in [18] and for any in $z \in \mathbb{C}$ we have

$$\frac{\partial}{\partial \overline{k}} m_1(z,k) = t(k) e(z,-k)\overline{m_2(z,k)}, \tag{18}$$
$$\frac{\partial}{\partial \overline{k}} m_2(z,k) = t(k) e(z,-k)\overline{m_1(z,k)},$$

where

$$t(k) = -\frac{i}{\pi}\int_{\mathbb{R}^2} e(z,k)\overline{q}(z) m_1(z,k) d\mu(z). \tag{19}$$

For the proof of this see Beals and Coifman [2], Brown and Uhlmann [4] or Sung [21].

Look now for solutions of (6) in the form $\Phi(z,k) = ie^{izk}\phi_r(z,k)$ respectively $\Phi_i = e^{izk}\phi_r(z,k)$. As in the forward problem, they must satisfy an integral formulation analogous to (12) where the role of k and z is reversed. Since $t(k) \in L^r(\mathbb{R}^2_k) \cap L^2(\mathbb{R}^2_k)$ (for an $r < 2$ as in the Theorem 1.1) we have existence and uniqueness for their solution in $L^{\tilde{r}}(\mathbb{R}^2_k)$, where $\tilde{r}^{-1} = r^{-1} - 1/2$. Using the equations (18) it is easy to check that

$$\frac{\partial}{\partial \bar{k}}(m_1 - m_2)(z,k) = -t(k)e(z,-k)\overline{m_1 - m_2}(z,k),$$
$$\frac{\partial}{\partial \bar{k}}(m_1 + m_2)(z,k) = t(k)e(z,-k)\overline{m_1 + m_2}(z,k).$$

By the uniqueness result for solutions of such systems we must have

$$\begin{aligned}\phi_i(z,k) &= m_1(z,k) + m_2(z,k),\\ \phi_r(z,k) &= m_1(z,k) - m_2(z,k).\end{aligned} \tag{20}$$

The following equalities show the relation between solutions of the forward and inverse equation.

$$\begin{aligned}\Phi_i = ie^{izk}\phi_i = ie^{izk}(m_1+m_2) &= \frac{ie^{izk}}{2}(\psi_r+\psi_i) + \frac{ie^{izk}}{2}e(z,-k)(\overline{\psi_i} - \overline{\psi_r}),\\ &= \frac{i}{2}\Psi_r + \frac{1}{2}\Psi_i - \frac{1}{2}\overline{\Psi_i} - \frac{i}{2}\overline{\Psi_r} = -\operatorname{Im}\Psi_r + i\operatorname{Im}\Psi_i.\end{aligned}$$

Similarly, $\Phi_r = \operatorname{Re}\Psi_r - i\operatorname{Re}\Psi_i$. These prove the identities (7).

The reconstruction formula (8) is due to a symmetry argument as follows. Starting with q produce Ψ_r and Ψ_i by solving (2) subject to the exponential behavior. Define $t_q(k)$ by (3) and solve now (6) subject to the exponential behavior and produce Φ_r and Φ_i. Define

$$q_1(z) = -\frac{i}{\pi}\int_{\mathbb{R}^2} e^{i\overline{zk}}\overline{t_q}(k)\left(\Phi_r(z,k) - i\Phi_i(z,k)\right)d\mu(k).$$

Due to the identities (7) we have that $\overline{\partial}\Psi_r(z,k) + q_1(z)\overline{\Psi_r}(z,k) = 0$ in the z-plane for any parameter $k \in \mathbb{C}$. On the other hand we knew that Ψ_r solves $\overline{\partial}\Psi_r(z,k) + q(z)\overline{\Psi_r(z,k)} = 0$, as we started that way. In particular we have $(q(z) - q_1(z))\overline{\Psi_r}(z,k) = 0$ for all $k \in \mathbb{C}$ and all $z \in \mathbb{C}$. Hence $q = q_1$.

3 Reconstructing convection coefficients

In this section we apply the above scattering method to reconstruction of the convection coefficients b_1, b_2 in

$$\Delta u(x) + b_1\frac{\partial u}{\partial x}(x) + b_2(x)\frac{\partial u}{\partial x}(x) = 0, x \in \Omega \tag{21}$$

from the Dirichlet-to-Neumann map Λ_{b_1,b_2}. Here $\Omega \subset \mathbb{R}^2$ is a bounded, simply connected domain with Lipschitz boundary. We assume that $b_1, b_2 \in W_c^{\epsilon,\tilde{p}}(\Omega)$, $\tilde{p} > 2$ are real valued maps with compact support in Ω and set $b = (b_1 + ib_2)/4$.

The following result from Vekua [V62] makes the reduction of (21) to a $\overline{\partial}$-equation. If u is a solution of (21) then $w = \partial u$ solves

$$\overline{\partial}w(z) + \overline{b}(z)\overline{w}(z) + b(z)w(z) = 0. \tag{22}$$

Lemma 3.1. *Let Ω be simply connected with Lipschitz boundary. If $u \in W^{2,\tilde{p}}(\Omega)$ is a solution of (21), then $w = \partial u \in W^{1,\tilde{p}}(\Omega)$ is a solution of (22). Conversely, if $w \in W^{1,\tilde{p}}(\Omega)$ is a solution of (22) then there exists an $u \in W^{2,\tilde{p}}(\Omega)$ solution of (21) and such that $\partial u = w$ in Ω.*

Proof. By Sobolev imbedding $u \in W^{2,\tilde{p}} \subset C^{2-2/\tilde{p}}(\Omega)$, hence $w \in C^{1-2/\tilde{p}}(\Omega)$. As a direct consequence of the Poincaré lemma, notice that if $\bar{\partial} w$ is real valued, then $w = \partial u$ for some real valued u. Indeed $2\bar{\partial} w = (\partial_x + i\partial_y)(f + ig) = (\partial_x f - \partial_y g) + i(\partial_x g + \partial_y f)$. By assumption $\partial_x g = -\partial_y f$, from where the one-form $gdy - fdx$ is exact. Therefore, there exists a real valued F such that $dF = (-f)dx + gdy$. We have $w = f + ig = \partial_x(-u) - i\partial_y(-u) = \partial(-2F)$. The equivalence is now apparent. $\square$

Now we extend $b \in W_c^{\varepsilon,\tilde{p}}(\Omega)$ by zero outside Ω. Its extension denoted also by b preserves regularity $b \in W_c^{\varepsilon,\tilde{p}}(\mathbb{R}^2)$. From now on we shall work with solutions of (22) in the whole plane.

Lemma 3.2. *The equation* (22) *has unique solutions in the whole plane* $W_r(z,k) \sim e^{izk}$ *respectively* $W_i(z,k) \sim ie^{izk}$ *in* $L^{\tilde{p}}(\mathbb{R}^2_z)$ *for large* z. *Moreover,* $e^{-izk}W_r - 1, e^{-izk}W_i - i \in W^{1,\tilde{p}}(\mathbb{R}^2_z)$ *and* $W_r(\cdot,k), W_i(\cdot,k) \in W^{1,\tilde{p}}_{loc}(\mathbb{R}^2)$.

Proof. As in the proof of theorem 1.1, we look for solutions $W(z,k) = e^{izk}w(z,k)$ with $w - 1 \in L^{\tilde{p}}(\mathbb{R}^2_z)$. The equation for w is

$$\overline{\partial}(w(z) - 1) + b(z)(w(z) - 1) + e(z,-k)\overline{b}(z)(\overline{w}(z) - 1) = -b(z) - e(z,-k)\overline{b}. \tag{23}$$

Using the fact that $\overline{\partial}^{-1} : f \in L^{\tilde{p}}_c(\mathbb{R}^2) \mapsto W^{1,\tilde{p}}(\mathbb{R}^2)$ together with b of compact support we get $\overline{\partial}^{-1}(b\cdot) : L^{\tilde{p}}(\mathbb{R}^2_z) \to L^{\tilde{p}}(\mathbb{R}^2_z)$ is a compact operator. We apply Fredholm's alternative in $L^{\tilde{p}}(\mathbb{R}^2_z)$ to the equivalent integral equation $\{[I + \overline{\partial}^{-1}[b(\cdot) + e(\cdot,-k)\overline{b}\overline{(\cdot)}]\}(w(z) - 1) = -\overline{\partial}^{-1}[b + e(\cdot,-k)\overline{b}]$. As mentioned earlier, uniqueness comes from Liouville's theorem for the $\overline{\partial}$-equation. By construction we already have that $g = w_r - 1 \in W^{1,\tilde{p}}(\mathbb{R}^2_z)$. Then $W_r(z,k) = e^{izk}(g+1) \in L^{\tilde{p}}_{loc}(\mathbb{R}^2_z)$, $\partial W_r = ike^{izk}g(z,k) + ike^{izk} + e^{izk}\partial g \in L^{\tilde{p}}_{loc}(\mathbb{R}^2_z)$ and $\overline{\partial} W_r = e^{izk}\overline{\partial} g \in L^{\tilde{p}}_{loc}(\mathbb{R}^2_z)$. Similar relations hold for W_i. $\square$

To simplify notations, let

$$q(z) = \overline{b}(z)e^{\overline{\partial}^{-1}b(z) - \partial^{-1}\overline{b}(z)} \tag{24}$$

denote a new potential and notice that if w is a solution of (22) then $v = e^{\overline{\partial}^{-1}b}w$ is a solution of

$$\overline{\partial} v + q\overline{v} = 0. \tag{25}$$

Since $b \in L^{\tilde{p}}(\mathbb{R}^2) \cap L^p(\mathbb{R}^2)$ we have that $\overline{\partial}^{-1}b \in L^\infty(\mathbb{R}^2) \cap C^{1-2/\tilde{p}}(\mathbb{R}^2)$, see Vekua [V62]. Then $e^{-\overline{\partial}^{-1}b} \in L^\infty(\mathbb{R}^2)$ and so $q \in L^{\tilde{p}}(\mathbb{R}^2) \cap L^p(\mathbb{R}^2)$.

The next theorem relates scattering solutions of (22) to scattering solutions of (25) and gives the behavior in k of $W_r(z,k)$ and $W_i(z,k)$.

Proposition 3.3. *Let* $b \in W_c^{\varepsilon,\tilde{p}}(\mathbb{R}^2)$, *for some* $\varepsilon > 0$. *Let* W_r *and* W_i *be the scattering solutions for* (22) *as given by the lemma above, and let* Ψ_r *and* Ψ_i *be the scattering solutions of* (25) *as given by the theorem 1.1. Then* $W_r = e^{-\overline{\partial}^{-1}b}\Psi_r$, $W_i = e^{-\overline{\partial}^{-1}b}\Psi_i$ *and*

$$\begin{aligned} \|W_r(z,k)e^{-izk} - e^{-\overline{\partial}^{-1}b}\|_{L^{\tilde{p}}(\mathbb{R}^2_z)} + \|W_i(z,k)e^{-izk} - ie^{-\overline{\partial}^{-1}b}\|_{L^{\tilde{p}}(\mathbb{R}^2_z)} \leq C\langle k\rangle^{-\varepsilon}, \\ \left\|[W_r(z,k) - iW_i(z,k)]e^{-izk} - 2e^{-\overline{\partial}^{-1}b}\right\|_{W^{1,\tilde{p}}(\mathbb{R}^2_z)} \leq C\langle k\rangle^{-\varepsilon}. \end{aligned} \tag{26}$$

Proof. The fact that W_r and W_i solve (22) is trivial. Uniqueness result of lemma 3.2 ensures that they are the scattering solutions of (22). The estimates follow directly from the estimates for ψ_r and ψ_i in (4) and (5) and from the fact that $e^{-\overline{\partial}^{-1}b} \in L^\infty(\mathbb{R}^2_z)$ as noticed before. Again, the imbedding $W^{1,\tilde{p}}(\mathbb{R}^2) \subset C^{1-2/\tilde{p}}(\mathbb{R}^2)$ shows that the estimates (26) hold pointwise in $z \in \mathbb{C}$. $\square$

We have now all the ingredients necessary for reconstruction. Since q in (24) has compact support in Ω, the scattering transform depends only on the traces on $\partial\Omega$ of the scattering solutions Ψ_r and Ψ_i. Let $\nu = \nu_1 + i\nu_2$ be the complex-normal to the boundary. Then

$$\begin{aligned} t(k) &= -\frac{i}{\pi}\int_\Omega e^{i\overline{zk}}\overline{q}(z)\left(\Psi_r(z,k) - i\Psi_i(z,k)\right)d\mu(z) = \frac{i}{\pi}\int_\Omega e^{i\overline{zk}}\left(\partial\overline{\Psi_r}(z,k) - i\partial\overline{\Psi_i}(z,k)\right)d\mu(z) \\ &= \frac{i}{2\pi}\int_{\partial\Omega} e^{i\overline{zk}}\overline{\nu}(z)\left(\overline{\Psi_r}(z,k) - i\overline{\Psi_i}(z,k)\right)d\sigma(z), \end{aligned} \tag{27}$$

The last equality uses the fact that $\partial(e^{i\overline{zk}}) = 0$.

Next we show how to reconstruct traces of Ψ_r and Ψ_i to $\partial\Omega$ from the Dirichlet to Neumann map Λ_{b_1,b_2}. First we reconstruct traces of W_r and W_i to $\partial\Omega$.

As in Knudsen and Tamasan [13], we consider the single layer potential operator $\mathcal{S}_k : C^\alpha(\partial\Omega) \to C^\alpha(\partial\Omega)$, $\alpha = 1 - 2/\tilde{p}$, defined by

$$\overline{\mathcal{S}_k}f(z) = \frac{1}{2\pi i}p.v.\int_{\partial\Omega} f(\zeta)\overline{g_k}(\zeta - z)d\overline{\zeta}, \qquad z \in \partial\Omega,$$

where $g_k(z) = e^{-izk}/(\pi z)$ is a Cauchy kernel for $\overline{\partial}$ which also takes into account the exponential behavior at infinity. For Lipschitz boundary $\mathcal{S}_k$ is a bounded operator (e.g. see Muskhelishvili [15]). Since q has compact support we have that W_r and W_i are analytic outside Ω and behaves like e^{izk} at infinity. Traces of such functions will satisfy a singular boundary equations involving $\mathcal{S}_k$. Inside Ω they satisfy a pseudo-analytic equation. This will impose constrains (in terms of Λ_{b_1,b_2}) on their trace. We will prove that these two conditions are sufficient to determine the traces.

We noticed already that $W_r(\cdot,k), W_i(\cdot,k) \in C^\alpha(\mathbb{R}^2)$ with $\alpha = 1 - 2/\tilde{p}$, hence their traces on $\partial\Omega$ are in $C^\alpha(\partial\Omega)$. Let

$$C_0^\alpha(\partial\Omega) := \{h \in C^\alpha(\partial\Omega) : \int_{\partial\Omega} h(s)ds = 0\}.$$

Define now a right inverse of the tangential vector field ∂_s (here s is the arc length) on $\partial\Omega$ by

$$\partial_s^{-1}f(t) = \int_0^t f(s)ds, \tag{28}$$

for $f \in C_0^\alpha(\partial\Omega)$. In the above integral we fixed an arbitrary point on $\partial\Omega$ from where we measure the arc length counter-clockwise. Notice that $\partial_s^{-1} : C_0^\alpha(\partial\Omega) \to C^{1+\alpha}(\partial\Omega)$ is a well defined (independent of the reference point) bounded operator. The following result defines a Hilbert transform for the pseudo-analytic maps.

Lemma 3.4. *$\mathcal{H}_b \equiv -\Lambda_{b_1,b_2}\partial_s^{-1} : C_0^\alpha(\partial\Omega) \to C^\alpha(\partial\Omega)$ is a bounded operator.*

Proof. Let $g = \partial_s^{-1}f \in C^{\alpha+1}(\partial\Omega) \subset W^{2-1/\tilde{p},\tilde{p}}$. Classical theory of PDE (e.g. Gilbarg and Trudinger [10]) gives that the boundary value problem

$$\begin{aligned} &\Delta u(x) + b_1\frac{\partial u}{\partial x}(x) + b_2(x)\frac{\partial u}{\partial x}(x) = 0, \;\; x \in \Omega \\ &u|_{\partial\Omega}(x) = g(x), \;\; x \in \partial\Omega \end{aligned} \tag{29}$$

has a unique solution up to a constant in $W^{2,\tilde{p}}(\Omega)$ and $||u||_{W^{2,\tilde{p}}(\Omega)} \leq C||g||_{W^{2-1/\tilde{p},\tilde{p}}(\partial\Omega)}$. Using the mapping properties of the Dirichlet to Neumann map we have

$$\begin{aligned}\|\mathcal{H}_b f\|_{C^\alpha(\partial\Omega)} &\leq \|\Lambda_{b_1,b_2} g\|_{W^{1,\tilde{p}}(\partial\Omega)} \leq \|\nabla u\|_{W^{1,\tilde{p}}(\Omega)} \leq ||u||_{W^{2,\tilde{p}}(\Omega)} \\ &\leq C||g||_{W^{2-1/\tilde{p},\tilde{p}}(\partial\Omega)} \leq C||g||_{C^{1+\alpha}(\partial\Omega)} \leq C\|f\|_{C^\alpha(\partial\Omega)}.\end{aligned}$$

□

Next we show that $\mathcal{H}_b$ reconstructs traces of the exponentially growing solutions on $\partial\Omega$.

Theorem 3.5 (Trace theorem). *Let $b \in W_c^{\varepsilon,\tilde{p}}(\Omega)$. Consider the class of functions*

$$\mathcal{B} = \{h \in C^\alpha(\partial\Omega) : \mathrm{Im}(\nu h) \in C_0^\alpha(\partial\Omega)\}.$$

Then, for each $k \in \mathbb{C}$ arbitrarily fixed, the traces $h_r = W_r(\cdot, k)|_{\partial\Omega}$., respectively $h_i = W_i(\cdot, k)|_{\partial\Omega}$ are the unique solution in $\mathcal{B}$ of the systems

$$(I - i\mathcal{S}_k)h_r(z) = 2e^{izk}, \qquad z \in \partial\Omega, \tag{30}$$

$$\mathcal{H}_b(\mathrm{Im}(\nu h_r))(z) = \mathrm{Re}(\nu h_r)(z), \qquad z \in \partial\Omega, \tag{31}$$

respectively,

$$(I - i\mathcal{S}_k)h_i(z) = 2ie^{izk}, \qquad z \in \partial\Omega,$$

$$\mathcal{H}_b(\mathrm{Im}(\nu h_i))(z) = \mathrm{Re}(\nu h_i)(z), \qquad z \in \partial\Omega.$$

Proof. We argue only for W_r, the arguments for W_i are similar.

We prove first the necessity. The arguments for (30) are identical to the ones in [13] reason for which we do not reproduce them here. They are based on the Green -Gauss and Plemelj's formulae.

We show first the necessity of (31). Recall from lemma 3.2 that $W_r(z) = \partial u(z)$ for some $u \in W^{2,\tilde{p}}(\Omega)$ which solve the equation (21). Therefore

$$h_r = W_r|_{\partial\Omega} = \frac{1}{2}(\partial_x - i\partial_y)u|_{\partial\Omega}. \tag{32}$$

For $z \in \partial\Omega$ let $(\nu_1(z), \nu_2(z))$ be the unit outer normal, we also let $\nu(z) = \nu_1(z) + i\nu_2(z)$. Next we express the partial derivatives for points on the boundary in terms of the tangent ∂_s and the normal ∂_ν derivatives

$$\nabla u(x) = \begin{pmatrix} -\nu_2 & \nu_1 \\ \nu_1 & \nu_2 \end{pmatrix} \begin{pmatrix} \partial_s u \\ \Lambda_{b_1,b_2} u \end{pmatrix}, \tag{33}$$

where we recall $\partial_\nu u = \Lambda_{b_1,b_2} u$. Therefore $2h_r = (\partial_x - i\partial_y)u = -i\overline{\nu}\partial_s u + \overline{\nu}\Lambda_{b_1,b_2} u$, or, using $\nu\overline{\nu} = 1$,

$$2\nu h_r = \Lambda_{b_1,b_2} u - i\partial_s u. \tag{34}$$

Note that $\mathrm{Im}(\nu h_r) = -\partial_s u/2$ and thus $h_r \in \mathcal{B}$ and $\partial_s^{-1}(\mathrm{Im}(\nu h_r))$ makes perfect sense. Identifying the real part in (34) gives (31). Notice not only that we proved necessity but also we provided existence of solutions for (31) and (30).

Conversely, let $h \in \mathcal{B}$ be a solution of the system (31) and (30). We extend h inside Ω by the following procedure. Inspired by (34) define $g = -\partial_s^{-1}\,\mathrm{Im}(2\nu h) \in C^\alpha(\partial\Omega)$ then uniquely solve the boundary value problem (9) for $u \in W^{2,\tilde{p}}(\Omega)$. Notice g is real valued hence u has also real values. Define $W_r(z) = \partial u(z)$ inside Ω and notice that $\partial u|_{\partial\Omega} \in C^\alpha(\partial\Omega)$. Now check that $\partial u|_{\partial\Omega} = h$.

Indeed, as before,$2\partial u = -i\overline{\nu}\partial_s u + \overline{\nu}\Lambda_{b_1,b_2} u = i\overline{\nu}\,\mathrm{Im}(2\nu h) + \overline{\nu}\,\mathrm{Re}(2\nu h)$. The last equality used the fact that h is a solution of (31). Multiplication by ν gives $\partial u = h$.

Extend h_r analytically outside Ω by

$$W_r(z) = e^{izk} - \frac{1}{2\pi i}\int_{\partial\Omega} \frac{e^{-i(\zeta - z)k} h(\zeta) d\zeta}{\zeta - z}.$$

The fact that h solves (30) implies that $\lim_{z\to z_0\in\partial\Omega} W_r(z) = h(z_0)$. Thus W_r is an outside continuous extension of h. Moreover, $e^{-izk}W_r - 1 = O(1/z)$ for z large, hence $W_r \in L^{\tilde{p}}(\mathbb{C} - \Omega)$.

We produced a continuous map in $\mathbb{R}^2$ which solves (22) both inside and outside Ω and behaves like e^{izk} for z large. We need to check that it solves the equation (22) across the boundary. Since b has compact support inside Ω we have that W_r is in fact analytic in both sides of the boundary and continuous across. Morera's theorem asserts that W_r must be in fact analytic across. Therefore W_r solves (22) in the whole plane and has the right behavior at infinity. Uniqueness in lemma 3.2 concludes the proof. □

Immediate consequence to the proposition 3.3 and to the pointwise estimates (26) we can determine the traces on $\partial\Omega$ of Ψ_r and Ψ_i. Moreover by formula (27) we determine the scattering transform.

Corollary 3.6 (Reconstruction of the scattering transform). *Under the assumptions of the proposition 3.3 we have*

$$e^{-\overline{\partial}^{-1}b}(z) = \lim_{k\to\infty} W_r(z,k), \qquad z \in \partial\Omega. \tag{35}$$

and for any $k \in \mathbb{C}$ we recover

$$\Psi_r(z,k) = e^{\overline{\partial}^{-1}b}(z) W_r(z,k), \qquad z \in \partial\Omega, \tag{36}$$

$$\Psi_i(z,k) = e^{\overline{\partial}^{-1}b}(z) W_i(z,k), \qquad z \in \partial\Omega. \tag{37}$$

Moreover,

$$t(k) = \frac{i}{2\pi}\int_{\partial\Omega} e^{i\overline{zk}}\overline{\nu}(z)\left(\overline{\Psi_r}(z,k) - i\overline{\Psi_i}(z,k)\right) d\sigma(z), \tag{38}$$

is a function in $L^r(\mathbb{R}^2) \cap L^{\tilde{r}}(\mathbb{R}^2) \cap L^{r'}(\mathbb{R}^2)$ for some $r < 2$, $\tilde{r}^{-1} = r^{-1} - 1/2$ and $r^{-1} + r'^{-1} = 1$.

Now we use the inverse scattering method of theorem 1.2 to reconstruct q.

Corollary 3.7. *Let $\Phi_r \sim e^{izk}$ and $\Phi_i \sim ie^{izk}$ in $L^{\tilde{p}}$ for large $k \in \mathbb{C}$ be the unique solutions*

$$\frac{\partial\Phi}{\partial\overline{k}}(k) + t(k)\overline{\Phi}(k) = 0,\ k \in \mathbb{C}. \tag{39}$$

Then

$$q(z) = -\frac{i}{\pi}\int_{\mathbb{R}^2} e^{i\overline{zk}}\overline{t}(k)\left(\Phi_r(z,k) - i\Phi_i(z,k)\right) d\mu(k). \tag{40}$$

Knowing q we also know $|b|$ since from (24) we have $|q| = |b|$. Moreover, Cheng and Yamamoto showed that (24) has a unique solution [6]. We will show in the lemma below how to find this solution.

Lemma 3.8 (Phase unwrapping). *Let $v \in 1 + L^{\tilde{p}}(\mathbb{R}^2)$ be the unique solution of*

$$\overline{\partial} v = \overline{qv} \tag{41}$$

then v vanishes on a set of measure zero. Define $b = \overline{qv}/v$ on the set where v does not vanish, else we can set $b = q$. Then $b \in L^{\tilde{p}}(\Omega)$ is the unique solution of

$$q(z) = \overline{b}(z) e^{\overline{\partial}^{-1} b(z) - \partial^{-1} \overline{b}(z)}.$$

Proof. Existence and uniqueness of v follows from the Fredholm alternative as before. It is known from Vekua [V62] that the set of zeroes of pseudo-analytic functions has measure zero . Since $\overline{bv} = qv$ we have that v also solves $\overline{\partial} v = bv$ in the whole plane. Equivalently $\overline{\partial}(e^{-\overline{\partial}^{-1} b} v) = 0$. Thus $e^{-\overline{\partial}^{-1} b} v$ is analytic and also goes to 1 as $|z| \to \infty$. By Liouville's theorem we have $v = e^{\overline{\partial}^{-1} b}$. From its definition we have

$$b = \overline{q} e^{-\partial^{-1} \overline{b} + \overline{\partial}^{-1} b}.$$

□

4 Remarks

In order to solve the inverse problem, one needs to find first the traces of the exponentially growing solutions. It is a question of unique continuation from the boundary and such it is severely ill-posed. In theory one could solve the integral equation (30) subject to constrains given by (31) by minimization techniques which regularize. However, accuracy is also important since there is only a logarithmic type stability, see Barcelo et. al. in [1]. A similar problem was carried out numerically by Siltanen et. al. [20] in the radially symmetric case, see also Knudsen [14].

For radially symmetric problems, (i.e. Ω is a disc, $b_1 = c(r)\cos\theta$ and $b_2 = c(r)\sin\theta$) one can show that $\mathcal{H}_b(z^n) = \lambda_n z^n$ for all $n \in \mathbb{Z}$ and compute λ_n accurately. Moreover, any solution of (30) can be represented as a series $W(z,k) = e^{izk} \sum_{n=0}^{\infty} a_n z^{-n}$, with unknown coefficients a_n. Using (31) one ends up with a linear system for a_n. This system is again severely ill-conditioned and it is not yet clear how to effectively overcome this problem.

The second step consists in constructing the scattering transform $t(k)$ via the formulae of corollary 3.6. Next we solve the weakly singular integral equations (6) in the k-space. This part is stable. It is here that we need the ϵ-extra regularity. One needs $t \in L^r(\mathbb{R}^2_k)$ for some $r < 2$ in order to solve (6). If q is only in $L^{\tilde{p}}_c$ then $t \in L^2(\mathbb{R}^2)$ (according to Sung [22] as corrected by Brown and Uhlmann [4]) and this suffices for uniqueness. This covers the uniqueness result of Cheng and Yamamoto. It is not clear how to find solutions of (6) when $t \in L^2(\mathbb{R}^2_k)$.

Reconstruct q from the formula (8). Notice that we have estimates of decay in k for $t \in L^r(\mathbb{R}^2_k)$ as well as for $e^{-izk}(\Phi_r - i\Phi_i) - 2$ as given in (5). These can lead to estimates of the truncation error in the integral in (8).

In section 2 we exhibit the one-to-one connection between pseudo-analytic functions with exponentially growing behavior at infinity and solutions of the first order $\overline{\partial}$-system of Beals and Coifman. Since we characterized traces of the former in terms of a generalized Hilbert transform (31), we also characterized the Cauchy data of the solutions of the first order system, thus answering a question in [25]. We point out that a partial answer was given before in [13] for potentials of a special type ($q = \partial f$ for some real valued f).

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