Extremal Solutions for the Discontinuous Delay- Equations

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Rezumat: Solutii extremale pentru ecuatii diferentiale cu argument intârziat. Folosind tehnici iterative monotone se demonstreaza existenta solutiilor extremale ale problemelor cu valori initiale pentru ecuatii cu argument intârziat.

Abstract: We prove the existence of the extremal solutions of the initial value problem, for short IVP: $y^{'}(t) = f(t, y(t), y(\theta(t))),\ \ y(0) = y_0$ with f discontinuous, using some monoton iterative technique.

1 Introduction

The subject matter of the present article is the delay-differential equation :

$$\begin{cases} y^{'}(t) = f(t, y(t), y(\theta(t))), a.e. t \in [0, T] \\ y(0) = y_0 \end{cases} \tag{1}$$

with f satisfying some Caratheodory's type conditions and monotony. The lag θ is an absolutely continuous function with $\theta(0) = 0$, $0 \leq \theta(t) \leq t$ a. e. $t \in [0, T]$.

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This problem is dealt with in several papers (see [3], [4], [5]). The main merit of this paper consists of allowing discontinuous right hand side. The idea is to treat the delay term as a new variable. We are looking for solutions in the space of absolutely continuous functions denoted by $AC[0,T]$.

The monotone iterative method used in section 2 is the one presented for ODE in [1]. In particular we use the following two results:

Proposition 1 *(Theorem 1.5.1, [1]) Consider the IVP* $x^{'}(t) = f(t,x)$ $x(0) = x_0$*, where* $f : [0,T] \times \mathbf{R} \longrightarrow \mathbf{R}$*. Let* $\alpha, \beta \in AC[0,T]$ *be a lower and respectively an upper solution, such that* $\alpha \leq \beta$*. Consider* f *is a Caratheodory function in* $\Omega = \{(t,x) : \alpha(t) \leq x \leq \beta(t),\ t \in [0,T]\}$*. If there exists an* $m \in L^1([0,T], \mathbf{R}_+)$ *such that:* $|f(t,x)| \leq m(t)$ *for all* $x \in [\alpha(t), \beta(t)]$ *and a. e.* $t \in [0,T]$ *then the IVP has the extremal solutions in the order interval* $[\alpha, \beta]$.

Proposition 2 *(Proposition 1.4.4, [1]) Given a nonempty order interval* $[\alpha, \beta] \subset AC[0,T]$*, a nondecreasing mapping* $G : [\alpha, \beta] \longrightarrow [\alpha, \beta]$ *and assume there exists* $v \in L^1([0,T], \mathbf{R}_+)$ *such that* $|(Gx)^{'}(s)| \leq v(t),\ x \in [\alpha, \beta]$*, a. e.* $t \in [0,T]$ *then the chain* $\{G^n\alpha : n \in \mathcal{N}\}$ *has a maximum* x_* *and the chain* $\{G^n\beta : n \in \mathcal{N}\}$ *has a minimum* x^* *and* $x_* = \min\{x : Gx \leq x\}$ *and* $x^* = \max\{x : x \leq Gx\}$*. In particular* x_**,* x^* *are the extremal fixed points of* G.

We call a lower solution of (1) a function $\alpha \in AC[0,T]$ which satisfies:

$$\begin{cases} \alpha^{'}(t) \leq F(t, \alpha(t), \alpha(\theta(t))) a.e. t \in [0,T] \\ \alpha(0) \leq y_0. \end{cases}$$

By duality we get an upper solution.

In the beginning we assume the existence of a lower and a upper solution but later on we shall drop it under some additional conditions on F. The monotony dependence on data of the extremal solutions is also pointed out.

In the last paragraph we apply the results to a discontinuous pantograph-like equation.

2 Existence of the extremal solutions

Consider the IVP (1) with $f : [0,T] \times \mathbf{R}^2 \longrightarrow \mathbf{R}$ obeying the followings:

$(H1)$ there exist $\alpha, \beta \in AC[0,T]$ a lower and a upper solution for (1) with $\alpha(t) \leq \beta(t)$, $t \in [0,T]$;

$(H2)$ there exist a $N \in L^1([0,T], \mathbf{R}_+)$ such that $|f(t,x,y)| \leq N(t)$ a. e. $t \in [0,T]$, $x \in [\alpha(t), \beta(t)]$ and $y \in [\alpha(\theta(t)), \beta(\theta(t))]$;

$(H3)$ $f(\cdot, x, y(\cdot))$ is measurable for all $x \in \mathbf{R}$ and $y \in AC[0,T]$;

$(H4)$ $f(t, \cdot, y)$ is continuous a. e. $t \in [0,T]$ and $y \in \mathbf{R}$;

$(H5)$ $f(t, x, \cdot)$ is nondecreasing a. e. $t \in [0,T]$ and $x \in \mathbf{R}$.

We are able now to state the following:

Theorem 1 *If the hypothesis $(H1)$ to $(H5)$ hold then the problem (1) has the extremal solutions in the order interval $[\alpha, \beta]$ for each $y_0 \in [\alpha(0), \beta(0)]$. The minimal solution $y_* = \max\{G^n\alpha : n \in \mathcal{N}\} = \min\{y \in [\alpha, \beta] : Gy \leq y\}$ and the maximal solution $y^* = \min\{G^n\beta : n \in \mathcal{N}\} = \max\{y \in [\alpha, \beta] : Gy \geq y\}$.*

Proof Let $y_0 \in [\alpha(0), \beta(0)]$ and $y \in [\alpha, \beta]$ be given. Consider the following IVP:

$$\begin{cases} x'(t) = F_y(t, x(t)) \\ x(0) = y_0 \end{cases} \tag{2}$$

where $F_y(t,x) = f(t, x, y(\theta(t)))$.

It is easy that α and β are a lower solution and respectively an upper solution for (2). Also we have

$$|F_y(t,x)| = |f(t,x,y(\theta(t))| \leq N(t)$$

for $x \in [\alpha(t), \beta(t)]$ and $F_y(t,x)$ is a Caratheodory function on $\Omega = \{(t,x) : \alpha(t) \leq x \leq \beta(t),\ t \in [0,T]\}$. Using Proposition 1 the IVP (2) has for each $y \in [\alpha, \beta]$ the extremal solutions in $[\alpha, \beta]$. We set $G : [\alpha, \beta] \longrightarrow [\alpha, \beta]$ by $Gy = x$, where x is the maximal solution of (2) for each $y \in [\alpha, \beta]$. Since $(Gy)'(t) = F_y(t, Gy(t))$ for all $y \in [\alpha, \beta]$ and a. e. $t \in [0,T]$ we get the bounding condition $|(Gy)'(t)| \leq N(t)$. Moreover G is a nondecreasing operator. Indeed let $y_1, y_2 \in [\alpha, \beta]$ be such that $y_1 \leq y_2$ and $x_i = Gy_i,\ i = 1, 2$. Since

$$x_1' = F_{y_1}(t, x_1) = f(t, x_1, y_1(\theta(t))) \leq f(t, x_1, y_2(\theta(t))) = F_{y_2}(t, x_1)$$

we have x_1 is a lower solution for $x' = F_{y_2}(t,x)$. But x_2 is a maximal solution of it whence (cf. [1]) $x_1 \leq x_2$ or $Gy_1 \leq Gy_2$. Thus G satisfies the hypothesis of Proposition 2. Therefore $x_* = \max_{n \in \mathbf{N}} G^n\alpha$ is the minimal solution and $x^* = \min_{n \in bfN} G^n\beta$ is the maximal solution of (1).

Remark (i) If we use a criteria to solve the problem (2) and so to build up the operator G one can approach the extremal solutions by successive iteration starting from any lower and respectively upper solution;

(ii) Assuming

$$|f(t,x,y)| \leq H(t,|x|,|y|) \text{ for } x,y \in \mathbf{R} \text{ and a. a. } t \in [0,T] \tag{3}$$

where $H : [0,T] \times \mathbf{R}_+^2 \longrightarrow \mathbf{R}_+$ is a nondecreasing on the second and the third variable and the IVP: $w^{'} = H(t,w,w),\ w(0) = |x_0|$ has an upper solution w^* then $-w^*$ and $+w^*$ are lower and upper solution for (1);

(iii) The existence of the extremal solutions are within the order interval $[\alpha,\beta]$ provided $y_0 \in [\alpha(0),\beta(0)]$. We do not know anything about solutions for $y_0 \notin [\alpha(0),\beta(0)]$.

The main assumption we made is the existence of a lower and an upper solution. Under an additional assumption we can avoid this inconvenient. The same arguments as given in [1] work out for our delay differential equation and give us a sufficient condition to ensure lower and upper solutions for all $y_0 \in \mathbf{R}$ as follows:

$(H6)$ $|f(t,x,y)| \leq p(t)h(|x|,|y|)$ a. e. $t \in [0,T]$ and $x,y \in \mathbf{R}$ where $p \in L^1([0,T],\mathbf{R}_+)$, $h : [0,\infty) \times [0,\infty) \longrightarrow (0,\infty)$ is a nondecreasing function in both of its arguments and

$$\int_0^\infty \frac{du}{h(u,u)} = \infty$$

Suppose f confined to $(H6)$, then a lower and an upper solutions for (1) are given by:

$$\begin{aligned} \alpha(t) &= y_0 + |y_o| - w(t) \\ \beta(t) &= y_0 - |y_o| + w(t) \end{aligned} \tag{4}$$

where $w \in AC[0,T]$ is the only solution of

$$\begin{cases} w' = p(t)h(w,w) \\ w(0) = |y_0| \end{cases}$$

and moreover all the solutions of (1) will lie within these lower and upper solutions. We conclude:

Theorem 2 *Consider the IVP* (1) *with* f *satisfying* $(H2)$ *to* $(H6)$. *Then for each* $y_0 \in \mathbf{R}$ *the IVP (1) has the extremal solutions which lie together with all the other solutions in the order interval* $[\alpha,\beta]$ *where* α,β *are given by (4).*

3 Monotony dependence on data

The result stated in **Theorem 1** can be used in studying the dependence of the extremal solutions on the initial value y_0 and on f. We remained that we refer to extremal solution within the order interval $[\alpha, \beta]$.

Theorem 3 *Let* $f, \widetilde{f} : [0,T] \times \mathbf{R}^2 \longrightarrow \mathbf{R}$ *be satisfying* $(H1)$ *to* $(H5)$ *with* $f(t,x,y) \leq \widetilde{f}(t,x,y)$ *for all* $t \in [0,T]$ *and* $x, y \in \mathbf{R}$ *and* $y_0, \widetilde{y}_0 \in [\alpha(0), \beta(0)]$ *be such that* $y_0 \leq \widetilde{y}_0$. *Consider the two corresponding IVP:*

$$\begin{cases} y^{'}(t) = f(t, y(t), y(\theta(t))), a.e.t \in [0,T] \\ y(0) = y_0 \end{cases} \tag{5}$$

and

$$\begin{cases} \widetilde{y}^{'}(t) = \widetilde{f}(t, \widetilde{y}(t), \widetilde{y}(\theta(t))), a.e.t \in [0,T] \\ \widetilde{y}(0) = \widetilde{y}_0 \end{cases} \tag{6}$$

If y_*, y^* *respectively* $\widetilde{y}_*, \widetilde{y}^*$ *are the extremal solutions of the above problems within the order interval* $[\alpha, \beta]$ *then* $y_* \leq \widetilde{y}_*$ *and* $y^* \leq \widetilde{y}^*$.

Proof It is obvious that y_* is a lower solution for (6) but a lower solution is less then any solution hence is less then the minimal solution $\widetilde{y}_*$. The dual fact holds similarly.

4 Application

Let us consider the IVP of the discontinuous Pantograph-like equation:

$$\begin{cases} y^{'}(t) = ay(t) + [y(qt)]_*, \ \ t \in [0,T] \\ y(0) = y_0 \end{cases} \tag{7}$$

where $[x]_*$ is the integer part of x, a is a real constant and $0 < q < 1$.

For $y_0 = 0$ the only continuous solution of (5) is the null one. Whereas for $y_0 \neq 0$ we have

$$\alpha(t) = y_0 + |y_0| - |y_0|e^{2ct}$$
$$\beta(t) = y_0 - |y_0| + |y_0|e^{2ct}$$

with $c = \max\{|a|, 1\}$ are respectively some lower and upper solutions. Since the above IVP satisfies $(H1)$ to $(H5)$ we guarantee the existence of the extremal solutions and moreover the minimal solution $y_*(t) = \lim_{n \to \infty} G^n \alpha(t)$ and the maximal solution $y^*(t) = \lim_{n \to \infty} G^n \beta(t)$.

For this simple example we can compute analyticaly some iteration of α through G. We have $G\alpha$ is the only solution of:

$$\begin{cases} y'(t) = ay(t) + [y_0 + |y_0| - |y_0|e^{2cqt}]_*, \;\; t \in [0,T] \\ y(0) = y_0 \end{cases} \tag{8}$$

For the sake of simplicity let us set $\lambda = -\frac{[y_0]_*}{|y_0|} + \frac{y_0}{|y_0|} + 1$. For $0 \le \frac{1}{2cq}\ln(\lambda + \frac{1}{|y_0|})$ the only solution of (8) is the solution of

$$\begin{cases} y'(t) = ay(t) + [y_0]_*, \;\; t \in [0,T] \\ y(0) = y_0 \end{cases} \tag{9}$$

$$\text{i. e. } y_1(t) = e^{at}\left(y_0 + \frac{[y_0]_*}{a}\right) - \frac{[y_0]_*}{a}\ .$$

Iteratively we get for

$$\frac{1}{2cq}\ln(\lambda + \frac{i}{|y_0|}) \le t < \frac{1}{2cq}\ln(\lambda + \frac{i+1}{|y_0|})$$

the solution of (8) is the solution of

$$\begin{cases} y'(t) = ay(t) + [y_0]_* - i, \;\; t \in [0,T] \\ y(a_i) = y_i(a_i - 0) \end{cases} \tag{10}$$

where

$$a_i = \frac{1}{2cq}\ln(\lambda + \frac{i}{|y_0|})$$

and y_i is the solution of the previous problem (10) with i is replaced by $i-1$.

It is worth noting that the null set of non-differentiable points changes on each iteration.

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