Differentiability with Respect to Lag Function for Nonlinear Pantograph Equations

Alexandru Tămăşan*

Differentiability with respect to $\lambda \in (0,1)$ is provided for the nonlinear pantograph equations $y'(t) = f(t, y(t), y(\lambda t))$ via the fibre contraction theorem and Picard operators' technique.

1 Introduction

The subject matter of this paper is the λ- dependence of the solution of the initial value problems:

$$\begin{cases} y'(t) = f(t, y(t), y(\lambda t)),\ t > 0 \\ x(0) = 0. \end{cases} \tag{1}$$

Occurring in number theory the linear version of (1), called the pantograph equation (PE), was first studied by Kato and Mc Leod in [9]. As well as the delay differential equations (DDE) $y'(t) = f(t, y(t), y(t-\tau))$, the PE belong to the class of functional differential equations (FDE). Differentiability with initial data for the FDE was first established by Hale in [1], but differentiability with respect to delays for DDE was proved relatively late by Hale and Ladeira in [2] using a generalized uniform contraction principle for quasi-Banach spaces. Also the soon to appear paper of Hokkanen and Moroşanu [4] gives a simple proof for DDE case. Neither of the methods cited above works on our problem. However the method presented by the author in [3] for DDE works out for PE too.
The Picard operators' technique proposed by Rus [5], [6] was first used on PE by V. Mureşan [10] in proving continuity with respect to λ.

*Faculty of Mathematics, Babeş-Bolyai University, 1 M. Kogalniceanu, 3400 Cluj-Napoca, Romania

Definition 1 *Let (X,τ) be a topological space and $T : X \longrightarrow X$ be a mapping. T is called* Picard *if the fixed point set of T namely $F_T = \{x^*\}$ and $T^n x_0 \longrightarrow x^*$ whichever the starting point $x_0 \in X$.*

Unfortunately by itself this method is not sufficient. The additional tool used is the fibre contraction theorem below [7], [8].

Theorem 1 (Fibre Contraction Theorem) *Let (X, d) be a metric space, (Y, ρ) be a complete metric space and $T : X \times Y \longrightarrow X \times Y$. Suppose that*

(i) $T(x, y) = (T_1(x), T_2(x, y))$;

(ii) $T_1 : X \longrightarrow X$ is Picard;

(iii) $\exists \lambda \in (0, 1)$ s. t. $\rho(T_2(x, y), T_2(x, z)) \leq \lambda \rho(y, z),\ \forall x \in X,\ y, z \in Y$.

Then T is Picard.

The Picard operators' technique consist in considering the unique solution $y^* = y^*(t, \lambda)$ of (1) as the unique fixed point of the operator

$$T_1 y(t, \lambda) = \int_0^t f(s, y(s, \lambda), y(\lambda s, \lambda))ds. \tag{2}$$

Let $y_n = T_1^n y_0$ be the n^{th}-iteration. In appropriate metric spaces we shall prove

$$y_n \longrightarrow y^*$$

and

$$\frac{\partial y_n}{\partial \lambda} \longrightarrow v^*.$$

Using an Weierstrass type argument we get v^* is differentiable with respect to λ and

$$v^* = \frac{\partial y^*}{\partial \lambda}.$$

2 Differentiability with respect to λ

Throughout this paper we assume that $f \in C[0, \alpha] \times \mathbf{R}^{2n}, \mathbf{R}^n)$ and

$$f(t, \cdot, \cdot) \in C^1(\mathbf{R}^{2n}, \mathbf{R}^n)\ \forall t \in [0, \alpha]$$

and

$$\frac{\partial f}{\partial y},\ \frac{\partial f}{\partial z} \in C([0,\alpha]\times \mathbf{R}^{2n}, \mathbf{R}^n).$$

In particular for

$$f \in C^1([0,\alpha]\times \mathbf{R}^{2n}, \mathbf{R}^n) \tag{3}$$

all the above hold.
Consider

$$X = \{y \in BC([0,\alpha]\times(0,1), \mathbf{R}^n) :\ y(0,\lambda)=0,\ y(\cdot,\lambda)\in C^1[0,\alpha]\ \forall \lambda \in (0,1)\}$$

endowed with

$$\|y\|_{B_1} = \sup_{(t,\lambda)\in[0,\alpha]\times(0,1)} |y(t,\lambda)|e^{-\mu t}$$

and

$$Y = BC([0,\alpha]\times(0,1), \mathbf{R}^n)$$

with a similar norm

$$\|v\|_{B_2} = \sup_{(t,\lambda)\in[0,\alpha]\times(0,1)} |y(t,\lambda)|e^{-\nu t}$$

for some μ and ν to be specified later on.

Lemma 1 $T_1 : (X, \|\cdot\|_{B_1}) \longrightarrow (X, \|\cdot\|_{B_1})$ *given by (2) is a contraction for μ large enough whence T_1 is a Picard operator.*

Proof: Using the Lipschitz condition of $f(t,\cdot,\cdot)$ with a Lipschitz constant L independent of $t \in [0,\alpha]$ we have

$$|T_1y(t,\lambda) - T_1z(t,\lambda)| \le L\int_0^t |y(s,\lambda)-z(s,\lambda)|e^{-\mu s}e^{\mu s}ds +$$

$$\int_0^t |y(\lambda s,\lambda)-z(\lambda s,\lambda)|e^{-\mu\lambda s}e^{\mu\lambda s}ds \le \frac{2L}{\mu}\|y-z\|_{B_1}e^{\mu t}$$

whence

$$\|T_1y - T_1z\|_{B_1} \le \frac{2L}{\mu}\|y-z\|_{B_1}$$

and choose $\mu > 2L$. ▮
Consider now $B_R \subset X$ a closed ball of radius R. For α sufficiently small using the continuity of f we have B_R is invariant for T_1 i.e. $T_1(B_R) \subset B_R$.

In the sequel we consider such an α that B_R is invariant.
Consider now $T_2 : X \times Y \longrightarrow Y$ defined by

$$T_2(y,v)(t,\lambda) = \int_0^t \frac{\partial f}{\partial y}(s, y(s,\lambda), y(\lambda s,\lambda))v(s,\lambda)ds +$$

$$+ \int_0^t \frac{\partial f}{\partial z}(s, y(s,\lambda s), y(\lambda s,\lambda))\frac{\partial y}{\partial s}(\lambda s,\lambda)sds +$$

$$+ \int_0^t \frac{\partial f}{\partial z}(s, y(s,\lambda s), y(\lambda s,\lambda))v(\lambda s,\lambda)ds.$$

It is clear that T_2 is linear in v and is easy to see that $T_2 : B_R \times Y \longrightarrow Y$ is a contraction in the v-variable with respect the $\|\cdot\|_{B_2}$- norm with a Lipschitz constant independent on $x \in B_R$.
Using the above facts and applying the fibre contraction theorem we are lead to the following

Lemma 2 *$T : B_R \times Y \longrightarrow B_R \times Y$ defined by $T = (T_1, T_2)$ is a Picard operator.*

We are now ready to state and prove our main result.

Theorem 2 *Let $f \in C^1([0,h] \times \mathbf{R}^{2n}, \mathbf{R}^n)$ then the unique solution of (1) is differentiable with respect to λ.*

Proof: For the beginning we work with $T \in [0,\alpha]$.
Let (y^*, v^*) be the unique fixed point of T as given by Lemma 2. In particular y^* is the unique fixed point of T_1 hence the solution of (1). Consider now $y_0 = 0$ and $v_0 = 0$ then

$$y_1(t,\lambda) = T_1 0(t,\lambda) = \int_0^t f(s,0,0)ds$$

is independent on λ whence

$$\frac{\partial y_1}{\partial \lambda}(t,\lambda) = 0.$$

But also $v_1(t,\lambda) = T_2(0,0)(t,\lambda) = 0$.
Inductively, suppose that y_n is differentiable in λ and

$$\frac{\partial y_n}{\partial \lambda}(t,\lambda) = v_n(t,\lambda),\ \forall t \in [0,\alpha],\ \lambda \in (0,1).$$

We have by the induction hypothesis

$$\begin{aligned} y_{n+1} = & \; T_1 y_n \\ v_{n+1} = & \; T_2(y_n, v_n) = T_2(y_n, \tfrac{\partial y_n}{\partial \lambda}) \end{aligned} \tag{4}$$

But

$$T_2(y_n, \frac{\partial y_n}{\partial \lambda})(t,\lambda) = \int_0^t \frac{\partial f}{\partial y}(s, y(s,\lambda), y(\lambda s, \lambda)) \frac{\partial y_n}{\partial \lambda}(s,\lambda) ds +$$
$$+ \int_0^t \frac{\partial f}{\partial z}(s, y(s,\lambda s), y(\lambda s, \lambda)) \frac{\partial y_n}{\partial s}(\lambda s,\lambda) s ds +$$
$$+ \int_0^t \frac{\partial f}{\partial z}(s, y(s,\lambda s), y(\lambda s, \lambda)) \frac{\partial y_n}{\partial \lambda}(\lambda s,\lambda) ds.$$

It's a simple computation that

$$T_2(y_n, \frac{\partial y_n}{\partial \lambda})(t,\lambda) = \frac{\partial T_1 y_n}{\partial \lambda}(t,\lambda) = \frac{\partial y_{n+1}}{\partial \lambda}(t,\lambda)$$

Therefore

$$\frac{\partial y_n}{\partial \lambda} = v_n, \; \forall n \in \mathbf{N}.$$

So far we got a sequence (y_n) in X such that

$$y_n \xrightarrow{\|\cdot\|_{B_1}} y^*$$

and

$$\frac{\partial y_n}{\partial \lambda} \xrightarrow{\|\cdot\|_{B_2}} y^*.$$

Using a Weierstrass argument we conclude v^* is differentiable and

$$v^* = \frac{\partial y_n}{\partial \lambda}.$$

In order to consider the solution extended on larger t note for $t \in [\alpha, \frac{\alpha}{\lambda}]$ we have $\lambda t \in [\lambda\alpha, \alpha]$ where we already know the solution is differentiable in λ. Also we know that differentiability with respect to parameters for the solutions of ODE holds.

Assuming differentiability for solutions defined on $I_n = [0, \frac{\alpha}{\lambda^n}]$ we prove it for solutions defined on $I_{n+1} = [0, \frac{\alpha}{\lambda^n + 1}]$.

Let ψ be the solution of (1) restricted to I_n i.e.

$$\psi(t) = y^*(t), \; \forall t \in I_n$$

For $t \in I_{n+1}$ we have y^* satisfies the following ODE

$$y'(t) = f(t, y(t), \psi(\lambda t))$$

Remains to remark that the application

$$(t, \lambda) \in I_n \times (0, 1) \longrightarrow \psi(\lambda t)$$

is differentiable. Applying now the classical result for ODE we got differentiability for I_{n+1}. ∎

3 Remarks

(i) The existence result and continuity with respect to λ result are hidden in our proof in the convergence

$$y_n \xrightarrow{\|\cdot\|_{B_1}} y^*$$

and do not require any small α so that the result apply on the whole interval $[0, h]$.
(ii) For the linear case studied by Kato and McLeod [9]

$$y'(t) = ay(t) + by(\lambda t)$$

we have

$$\frac{\partial f}{\partial y}(t, y, z) \text{ and } \frac{\partial f}{\partial y}(t, y, z)$$

are bounded on the whole $[0, h] \times \mathbf{R}^{2n}$ whence we can work with the operator T_1 defined on the entire space X rather than on a closed ball and so we can also drop the assumption of α small enough from the very beginning.

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