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RECONSTRUCTION OF LESS REGULAR CONDUCTIVITIES IN THE PLANE

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Abstract

We consider the inverse conductivity problem of how to reconstruct an isotropic electric conductivity distribution in a conductive body from static electric measurements on the boundary of the body. An exact algorithm for the reconstruction of a conductivity in a planer domain from the associated Dirichlet-to-Neumann map is given. We assume that the conductivity has essentially one derivative, and hence we improve earlier reconstruction results. The method relies on a reduction of the conductivity equation to a first order system, to which the $\overline{\partial}$-method of inverse scattering theory can be applied.

Key Words: inverse conductivity problem, electrical impedance tomography, reconstruction method, $\overline{\partial}$-method

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I. Introduction

The inverse conductivity problem is the mathematical problem behind a new method for medical imaging called Electrical Impedance Tomography (EIT). In EIT one has a conductive body with unknown conductivity, and from static electric measurements on the boundary, i.e. by applying a voltage potential on the boundary and measuring the current flux through the boundary, one would like to image the interior conductivity. Since muscle tissue, fat tissue, bones, inner organs, lungs etc. have different conductive properties (see [1]), an image of the conductivity distribution inside a body may be used for medical diagnostics. We refer to [2] and [3] for a review of the mathematical methods and applications of EIT.

To pose the mathematical problem we let $\Omega \subset \mathbb{R}^n$ be a bounded and smooth domain and assume that the conductivity $\gamma \in L^\infty(\Omega)$ satisfies $0 < c \leq \gamma$ for some constant c. The application of a voltage potential f on the boundary $\partial\Omega$ induces a voltage potential u inside Ω given as the unique solution to

$$\begin{aligned} \nabla \cdot \gamma \nabla u &= 0, \text{ in } \Omega, \\ u &= f, \text{ on } \partial\Omega. \end{aligned} \tag{1}$$

When $f \in H^{1/2}(\partial\Omega)$ the solution $u \in H^1(\Omega)$. Moreover, u has a well defined normal derivative at the boundary defined coherently as an element in $H^{-1/2}(\partial\Omega)$ by

$$\langle \gamma(\partial_\nu u)|_{\partial\Omega}, g \rangle = \int_\Omega \gamma \nabla u \cdot \nabla v dx,$$

where ν is the outer unit normal defined in a neighborhood of $\partial\Omega$, $v \in H^1(\Omega)$ is any function with $v|_{\partial\Omega} = g$, and $\langle \cdot, \cdot \rangle$ is the dual pairing of $H^{1/2}(\partial\Omega)$ and $H^{-1/2}(\partial\Omega)$. The distribution $\gamma(\partial_\nu u)|_{\partial\Omega}$ is the current flux through the boundary, and it is the natural Neumann data for the equation (1). Thus we can define the Dirichlet-to-Neumann (or voltage-to-current) map $\Lambda_\gamma \colon H^{1/2}(\partial\Omega) \to H^{-1/2}(\partial\Omega)$ by

$$\Lambda_\gamma f = \gamma(\partial_\nu u)|_{\partial\Omega}. \tag{2}$$

This map encodes all possible boundary measurements which can in principle be considered for EIT. The inverse conductivity problem, as it was posed by Calderón [4], concerns the inversion of the map $\gamma \mapsto \Lambda_\gamma$. There are different aspects of the problem; here we are mainly interested in the question of reconstruction, i.e. finding an algorithm for the computation of γ from Λ_γ.

There is a huge literature on the inverse conductivity problem. For a recent review on the history and developments we refer to [5]. We will outline only the main results:

Calderón [4] solved the linearized problem and gave an approximate method for the reconstruction of conductivities close to constant. The first global result was by Kohn and Vogelius [6], who showed that when $\gamma \in C^\infty(\overline{\Omega})$ then Λ_γ determines $\gamma|_{\partial\Omega}$ and all derivatives on $\partial\Omega$. This solved the uniqueness question for real-analytic conductivities. Later the result was extended to include piecewise real-analytic conductivities [7].

In dimension $n \geq 3$ global uniqueness was proved by Sylvester and Uhlmann [8] for conductivities $\gamma \in C^\infty(\overline{\Omega})$. The assumption on γ for uniqueness to hold has since been relaxed by a number of people ([9, 10, 11, 12, 13]); the most general global uniqueness result so far is due to Brown and Torres [14] for $\gamma \in W^{3/2,p}(\Omega), p > 2n$. Also for the higher dimensional problem Nachman [10] and Novikov [15] gave a reconstruction algorithm and Alessandrini [9] showed conditional stability.

In dimension $n = 2$, local uniqueness for $\gamma \in W^{3,\infty}(\Omega)$ close to constant was proved by Sylvester and Uhlmann [16], but the general question remained open until 1996, when Nachman [17] gave a uniqueness proof and a reconstruction algorithm for $\gamma \in W^{2,p}(\Omega), p > 1$. Nachman's method relies on the reduction of the conductivity equation to a Schrödinger equation, a reduction that requires essentially two derivatives on the conductivity. For this equation the $\overline{\partial}$-method of inverse scattering can be adapted and give a reconstruction algorithm. The algorithm has been tried out numerically by Siltanen, Mueller and Isaacson [18, 19, 20, 21]. In 1997 Brown and Uhlmann [22] improved the uniqueness result to cover $\gamma \in W^{1,p}(\Omega), p > 2$. The method of proof relies on a reduction of the conductivity equation to a first order system for which only one derivative on the conductivity is required. For this system the $\overline{\partial}$-method is again applicable. Also in two dimensions there are conditional stability results due to Liu [23] and Barceló, Barceló and Ruiz [24].

In this paper we will show how the uniqueness proof in [22] for the two-dimensional inverse conductivity problem can be turned into a reconstruction method. The main result is the following:

Theorem I.1. *Let $\Omega \subset \mathbb{R}^2$ be a bounded and smooth domain. Let $2 < p, 0 < \epsilon < 1$ and assume $\gamma \in W^{1+\epsilon,p}(\Omega)$ satisfies $0 < c \leq \gamma$ for some constant c. Then γ can be reconstructed from Λ_γ.*

We note that the assumption $\gamma \in W^{1+\epsilon,p}(\Omega)$ is slightly more restrictive than what is sufficient for uniqueness to hold. However, Theorem I.1 is a sharp improvement of the reconstruction result due to Nachman [17].

Generally we assume that γ satisfies the hypothesis in Theorem I.1 and that $\gamma = 1$ near $\partial\Omega$. It was proved by Nachman [17] that for $\gamma \in W^{2,p}(\Omega), p > 1$, this assumption is not restrictive; we will show in section V that this is also true in our case.

The reconstruction algorithm is based on the reduction of the conductivity equation to a first order system. Let $p > 2$ and define $\beta = 1 - 1/p$. Then the Sobolev embedding theorem implies that $W^{1,p}(\Omega) \subset C^\beta(\overline{\Omega})$, and hence it is well known [25, Theorem 8.34] that for $f \in C^{1+\beta}(\partial\Omega)$, the solution u to (1) is in $C^{1+\beta}(\overline{\Omega})$. Define the vector-valued function (v, w) by

$$\begin{pmatrix} v \\ w \end{pmatrix} = \gamma^{1/2} \begin{pmatrix} \partial u \\ \overline{\partial} u \end{pmatrix} \in C^\beta(\overline{\Omega}) \times C^\beta(\overline{\Omega}) \tag{3}$$

with $\partial = (\partial_x - i\partial_y)/2$ and $\overline{\partial} = (\partial_x + i\partial_y)/2$. Using (1) it is straightforward to verify that

$$\begin{aligned} \bar{\partial} v &= q w \\ \partial w &= \overline{q} v, \end{aligned} \tag{4}$$

where

$$q = -\gamma^{-1/2} \partial \gamma^{1/2}. \tag{5}$$

The system (4) is interesting in its own right. An inverse problem for the system is whether the Cauchy data defined by

$$\mathcal{C}_q = \{(v|_{\partial\Omega}, w|_{\partial\Omega})\colon (v,w) \in C^\beta(\overline{\Omega}) \times C^\beta(\overline{\Omega}) \text{ solves (4)}\}$$

determines the potential q. We will answer this question affirmatively:

Theorem I.2. *Let $q_1, q_2 \in W^{\epsilon,p}(\overline{\Omega}), p > 2$. Then $\mathcal{C}_{q_1} = \mathcal{C}_{q_2}$ implies $q_1 = q_2$.*

The proofs of the Theorem I.1 and Theorem I.2 go through a certain intermediate function S, the so-called non-physical scattering transform of the potential. To define this function we introduce the vector-valued function $m = (m_1, m_2)$, which is defined uniquely by the properties

$$\begin{aligned} &\Psi(z,k) = m(z,k)e^{izk} \text{ solves (4) in } \mathbb{R}^2 \text{ with } q = 0 \text{ in } \mathbb{C} \setminus \overline{\Omega}, \\ &m \sim (1,0), \text{ i.e. } \lim_{|z|\to\infty} m(z,k) = (1,0), \end{aligned} \tag{6}$$

where we have identified $z = x + iy \in \mathbb{C}$ and $(x,y) \in \mathbb{R}^2$. We call the solution Ψ an exponentially growing solution or a complex geometrical optics solution (see section II for the exact construction and properties of this special solution). The non-physical scattering transform of the potential q is then defined by

$$S(k) = -\frac{i}{\pi} \int_{\mathbb{R}^2} e(z,k)\overline{q}(z) m_1(z,k)\, d\mu(z), \quad k \in \mathbb{C}, \tag{7}$$

where $e(z,k) = \exp(izk + i\overline{zk})$ and μ denotes the Lebesgue measure in the plane.

The method of proof can now be decomposed into the two steps

$$\Lambda_\gamma / \ \mathcal{C}_q \xrightarrow{1} S \xrightarrow{2} \gamma / \ q.$$

Concerning the first step in the reconstruction procedure, it was shown in [22] that Λ_γ determines S uniquely. The innovation here is that we give an explicit method for the computation of S in terms of Λ_γ. Concerning the second step it was shown in [22] that S determines q uniquely and then using (5) that q determines γ. We will combine these results with a result from [24] and give a direct method for the computation of γ from S. Note that this second step is where the extra smoothness for our algorithm is required.

We emphasize that the proposed algorithm can be implemented numerically (see [26, 27]), and hence it might be of use in practical EIT. Note that the stability analysis in [24] shows that the first step of computing the scattering transform from the Dirichlet-to-Neumann map has logarithmic conditional stability while the second step of computing the conductivity from the scattering transform has linear conditional stability. Hence the first step carries the ill-posedness of the problem, and in practice this step has to be regularized in an appropriate way.

The paper is organized as follows. In section II we review the scattering theory for (4) based on the construction and properties of the function m, and we give a few new results, which fit our regularity assumptions. Then in section III we show how to compute the scattering transform from the boundary data, and finally in section IV we show how to compute the conductivity directly from the scattering transform. The paper concludes with section V, where the reduction to the case $\gamma = 1$ near $\partial\Omega$ is considered.

II. Direct and Inverse Scattering for the First Order System

The direct and inverse scattering theory for (4) due to Beals and Coifman [28, 29] is an application of the so-called $\overline{\partial}$-method. This method was initially introduced in the study of some one-dimensional non-linear evolution equations [30, 31], which could be linearized by the scattering transform, and the method was later developed and extended to higher dimensional problems by a number of people. For a general introduction to the $\overline{\partial}$-method we refer to the review papers [28, 32]. The scattering theory for (4) has been considered in various contexts by a number of authors [29, 33, 34, 35, 22, 24], however, since we have reduced the notation and also need to improve a few estimates slightly, we will be explicit about the constructions.

Consider the equation (4) for an arbitrary $q \in W_c^{\epsilon,p}(\mathbb{R}^2)$, the subset of $W^{\epsilon,p}(\mathbb{R}^2)$ consisting of functions with compact support. Note that when $\gamma \in W^{1,p+\epsilon}(\Omega)$ equals one near $\partial\Omega$, then it can be extended beyond Ω by $\gamma = 1$, and hence the potential q defined by (5) satisfies $q \in W_c^{\epsilon,p}(\mathbb{R}^2)$. The basic idea is to look for an exponentially growing solution $\Psi = (\Psi_1, \Psi_2)$ to (4) of the form (6). A simple calculation shows that m must satisfy

$$\begin{aligned} \overline{\partial} m_1(z,k) &= q(z) m_2(z,k), \\ (\partial + ik) m_2(z,k) &= \overline{q}(z) m_1(z,k). \end{aligned} \tag{8}$$

Note that the differential operators ∂ and $\overline{\partial}$ act with respect to z and $\overline{z}$ respectively; we will later consider the derivatives of m with respect to the parameter k.

To proceed we define the solid Cauchy integral operators $\partial^{-1}, \overline{\partial}^{-1}$ by

$$(\partial^{-1} f)(z) = \frac{1}{\pi} \int_{\mathbb{R}^2} \frac{f(\zeta)}{\overline{z} - \overline{\zeta}} d\mu(\zeta), \quad (\overline{\partial}^{-1} f)(z) = \frac{1}{\pi} \int_{\mathbb{R}^2} \frac{f(\zeta)}{z - \zeta} d\mu(\zeta), \tag{9}$$

where μ is the Lebesgue measure in the plane. The mapping properties of these operators are described next.

Proposition II.1. *Let T be either ∂^{-1} or $\overline{\partial}^{-1}$. Let $s \geq 0$, let $1 < p < 2$ and define $1/\tilde{p} = 1/p - 1/2$. Then*

$$T \in \mathcal{B}(W^{s,p}(\mathbb{R}^2), W^{s,\tilde{p}}(\mathbb{R}^2)). \tag{10}$$

Furthermore, if $r > 2$ and $f \in W_c^{s,r}(\mathbb{R}^2)$, then $Tf \in W^{1+s,r}(\mathbb{R}^2)$ and

$$\|Tf\|_{W^{1+s,r}(\mathbb{R}^2)} \leq C \|f\|_{W^{s,r}(\mathbb{R}^2)}, \tag{11}$$

where the constant C depends on the support of f. For $1 < p_1 < 2 < p_2 < \infty$ and $\alpha = 1 - 1/p_2$, we have

$$T \in \mathcal{B}(L^{p_1}(\mathbb{R}^2) \cap L^{p_2}(\mathbb{R}^2), C^{\alpha}(\mathbb{R}^2)). \tag{12}$$

Finally, the operator $(\partial + ik)^{-1} = e(z,-k)\partial^{-1}(e(z,k)\cdot)$ satisfies

$$\|(\partial + ik)^{-1} f\|_{W^{s-\delta,\tilde{p}}(\mathbb{R}^2)} \leq \frac{C}{|k|^{\delta}} \|f\|_{W^{s,p}(\mathbb{R}^2)} \tag{13}$$

for $k \in \mathbb{C} \setminus \{0\}$ and $0 \leq \delta \leq s$.

Proof. We will only consider the case $T = \partial^{-1}$; for $\overline{\partial}^{-1}$ the results follows by complex conjugation.

From the Hardy-Littlewood-Sobolev theorem of fractional integration [36, p. 119] it follows that $T \in \mathcal{B}(L^p(\mathbb{R}^2), L^{\tilde{p}}(\mathbb{R}^2))$, and since T commutes with differential operators we deduce that $T \in \mathcal{B}(W^{1,p}(\mathbb{R}^2), W^{1,\tilde{p}}(\mathbb{R}^2))$. The property (10) then follows for $0 < s < 1$ by interpolation.

To prove (11) we note that $W_c^{s,r}(\mathbb{R}^2) \subset W^{s,p}(\mathbb{R}^2)$ for $1 < p < 2$. Hence $f \in W_c^{s,r}(\mathbb{R}^2)$ implies by (10) that $g = Tf \in W^{s,\tilde{p}}(\mathbb{R}^2), 2 < \tilde{p} < \infty$. In particular $g \in W^{s,r}(\mathbb{R}^2)$. By taking derivatives we find that $\partial g = f \in W^{s,r}(\mathbb{R}^2)$, and that $\overline{\partial} g = \overline{\partial} T f = \overline{\partial}\partial^{-1} \in W^{s,r}(\mathbb{R}^2)$ due to the boundedness of the Beurling transform $\overline{\partial}\partial^{-1}$ on $W^{s,r}(\mathbb{R}^2), 1 < r < \infty, s \geq 0$ (see for instance [37]). This shows that $g \in W^{1+s,r}(\mathbb{R}^2)$ satisfies (11).

For a proof of the Hölder property (12) we refer to [38, Theorem 1.21].

To prove (13) we use the estimate

$$\|(\partial + ik)^{-1} f\|_{L^{\tilde{p}}(\mathbb{R}^2)} \leq \frac{C}{|k|} \|f\|_{W^{1,p}(\mathbb{R}^2)} \tag{14}$$

from [17, Lemma 2.1]. Since $|e(z,k)| = 1$ it follows from (10) that

$$\|(\partial + ik)^{-1} f\|_{L^{\tilde{p}}(\mathbb{R}^2)} \leq C_1 \|f\|_{L^p(\mathbb{R}^2)}, \tag{15}$$

and by using that also $(\partial + ik)$ commutes with differential operators we get

$$\|(\partial + ik)^{-1} f\|_{W^{s,\tilde{p}}(\mathbb{R}^2)} \leq C_2 \|f\|_{W^{s,p}(\mathbb{R}^2)} \tag{16}$$

for $s \geq 0$, $p > 2$. Note that both C_1 and C_2 are independent of k. Interpolating (15) and (14) with interpolation parameter s shows that

$$\|(\partial + ik)^{-1} f\|_{L^{\tilde{p}}(\mathbb{R}^2)} \leq \frac{C}{|k|^s} \|f\|_{W^{s,p}(\mathbb{R}^2)}, \tag{17}$$

and interpolating (16) and (17) with interpolation parameter δ/s gives the result. □

In the sequel we will need the following result concerning solvability of a certain $\overline{\partial}$-equation.

Proposition II.2. *Let $1 < p < 2$ and assume that $a \in L^p(\mathbb{R}^2) \cap L^2(\mathbb{R}^2)$. Then the equation*

$$\overline{\partial} m = a\overline{m} \tag{18}$$

has a unique solution m with $m - 1 \in L^{\tilde{p}}(\mathbb{R}^2)$.

Proof. The equation for $m-1$ is

$$\overline{\partial}(m-1) = a\overline{(m-1)} + a$$

or equivalently

$$(I - \overline{\partial}^{-1}(aC)(m-1) = \overline{\partial}^{-1} a, \tag{19}$$

where the operator aC is defined by $aC\colon f \mapsto a\overline{f}$. Note that $a \in L^p(\mathbb{R}^2)$ implies $\overline{\partial}^{-1} a \in L^{\tilde{p}}(\mathbb{R}^2)$ by (10), and note further that when $a \in L^2(\mathbb{R}^2)$, it follows from the Hölder inequality and (10) that the operator $\overline{\partial}^{-1}(aC)$ is real-linear and bounded in $L^r(\mathbb{R}^2),\ r > 2$. Hence it makes sense to consider (19) in $L^{\tilde{p}}(\mathbb{R}^2)$.

Since the operator $\overline{\partial}^{-1}(aC)$ is compact in $L^{\tilde{p}}(\mathbb{R}^2)$ by [37, Lemma 4.2], invertibility of $(I - \overline{\partial}^{-1}(aC))$ in $L^{\tilde{p}}(\mathbb{R}^2)$ is by the Fredholm alternative a consequence of uniqueness of a solution to the homogeneous equation. However, a solution $h \in L^{\tilde{p}}(\mathbb{R}^2)$ to the homogeneous equation would satisfy $\overline{\partial} h = a\overline{h}$, and hence $h = 0$ by the generalized Liouville theorem (see for instance [22]). This proves the result. □

Given our assumptions on q we have now the following result concerning existence and uniqueness of exponentially growing solutions.

Proposition II.3. *Let $q \in W_c^{\epsilon,p}(\mathbb{R}^2), p > 2, \epsilon > 0$. Then for $k \in \mathbb{C}$ there is a unique solution $m(z,k)$ to (8) with $(m_1(\cdot,k) - 1), m_2(\cdot,k) \in W^{\epsilon,r}(\mathbb{R}^2) \cap C^{\beta}(\mathbb{R}^2),\ r > 2, \beta = 1 - 1/p$.*

Proof. Consider the linear combinations

$$m_{\pm}(z,k) = m_1(z,k) \pm e(z,-k)\overline{m_2(z,k)}.$$

and note that (8) and the asymptotic condition in (6) implies

$$\overline{\partial} m_{\pm}(z,k) = \pm q(z) e(z,-k)\overline{m_{\pm}(z,k)}, \quad m_{\pm} \sim 1. \tag{20}$$

By Proposition II.2 this equation is uniquely solvable in $L^r(\mathbb{R}^2)$ for any $r > 2$. The additional Sobolev and Hölder regularity follows from (10) and (12). $\square$

Proposition II.3 shows that $m \sim (1,0)$ for $|z| \to \infty$. The next result shows that a similar asymptotic condition holds when $|k| \to \infty$.

Proposition II.4. *Let $k \in \mathbb{C} \setminus \{0\}$. Then for $2 < r < \infty$ and $0 \leq \delta \leq \epsilon$ we have the estimates*

$$\|m(\cdot,k) - (1,0)\|_{W^{\epsilon-\delta,r}(\mathbb{R}^2)} \leq \frac{C}{\langle k\rangle^\delta}, \tag{21}$$

$$\|m_1(\cdot,k) - 1\|_{W^{1+\epsilon-\delta,r}(\mathbb{R}^2)} \leq \frac{C}{\langle k\rangle^\delta}. \tag{22}$$

where $\langle k\rangle = (1+|k|^2)^{1/2}$.

Proof. Note first that the map $k \mapsto m(\cdot,k)$ is continuous, so it suffices to consider $|k| \geq 1$. Note further that $\overline{\partial}^{-1}(e(\cdot,-k)q) = e(\cdot,-k)(\overline{\partial} - i\overline{k})^{-1}(q)$. Since $|e(z,k)| = 1$, (13) implies

$$\|\overline{\partial}^{-1}(e(\cdot,-k)q)\|_{W^{\epsilon-\delta,r}(\mathbb{R}^2)} \leq \frac{C}{\langle k\rangle^\delta}\|q\|_{W^{\epsilon,p}(\mathbb{R}^2)}.$$

Hence (21) follows from the integral equation

$$(m_\pm(\cdot,k) - 1) \mp \overline{\partial}^{-1}(qe(\cdot,-k)\overline{(m_\pm(\cdot,k)-1)}) = \pm\overline{\partial}^{-1}(qe(\cdot,-k)) \tag{23}$$

inversion of (23), since the operator $\phi \mapsto \overline{\partial}^{-1}(e(\cdot,-k)q\phi) \in \mathcal{B}(W^{\epsilon-\delta,r}(\mathbb{R}^2))$ is uniformly bounded in k.

To prove (22) we use (8) to write $(m_1 - 1) = \overline{\partial}^{-1}(qm_2)$, and then the result follows from (21) and (11). $\square$

The estimates (21) and (22) replace the similar Hölder estimates in [24, Lemma 2.8], which are due to Brown and Uhlmann. This allows us to go beyond the class of Hölder continuous conductivities treated in [24]. Note further that for $\delta < \epsilon$ and $p > 2/(\epsilon - \delta)$, the Sobolev embedding $W^{\epsilon-\delta,p}(\mathbb{R}^2) \subset C^\alpha(\mathbb{R}^2)$, $\alpha = \epsilon - \delta - 2/p$, implies by (21) the pointwise estimate

$$|m(z,k) - (1,0)| \le \frac{C}{\langle k \rangle^{\delta}}$$

for almost every $z \in \mathbb{C}$.

Consider now the non-physical scattering transform S of the potential q defined by (7). The map $q \mapsto S$ can be seen as a non-linear Fourier transform. The non-linearity of the transform is caused by the non-linear dependency of the integrand in (7) on q, and the relation to the Fourier transform comes from the fact that the asymptotic behavior $m_1 \sim 1$ implies the asymptotics $S \sim -\hat{\overline{q}}(-2k_1, 2k_2)$. Moreover, there is a Parseval identity, i.e. $q \in \mathcal{S}(\mathbb{R}^2)$ implies $S \in \mathcal{S}(\mathbb{R}^2)$ with the norm equality $\|q\|_{L^2(\mathbb{R}^2)} = \|S\|_{L^2(\mathbb{R}^2)}$, see [29].

The inverse scattering problem concerns the inversion of the map $q \mapsto S$. The fundamental idea is that for fixed $z \in \mathbb{C}$ the map $k \mapsto m(z,k)$ is differentiable. Denote by $\overline{\partial}_k = \partial/\partial\overline{k}$ the operator taking derivative with respect to the complex variable $\overline{k}$. Then we have

Lemma II.5. *For any $z \in \mathbb{C}$*

$$\begin{aligned} \overline{\partial}_k m_1(z,k) &= S(k)e(z,-k)\overline{m_2(z,k)}, \\ \overline{\partial}_k m_2(z,k) &= S(k)e(z,-k)\overline{m_1(z,k)}. \end{aligned} \tag{24}$$

Proof. The proof of this result relies on the fact that the map $k \mapsto (\partial + ik)^{-1}$ is Frechét differentiable on $\mathbb{C}$ in the strong operator topology of $\mathcal{B}(L^p_\alpha(\mathbb{R}^2), L^{\tilde{p}}_\beta(\mathbb{R}^2))$ for $\alpha > 2/p', \beta > 2/\tilde{p}$, and has

$$\overline{\partial}_k(\partial + ik)^{-1} f = -2e(z,-k)(\mathcal{F}f)(-2k_1, 2k_2),$$

see [17, Lemma 2.2]. □

When comparing (24) and (8) we see that in some sense S plays the same role for the inverse problem as q does for the direct problem.

To solve (24) (constructively) a certain amount of decay in the scattering transform S is needed. In [22] it was proved that for compactly supported $q \in L^p(\mathbb{R}^2)$, $p > 2$, the scattering transform $S \in L^2(\mathbb{R}^2)$. We will need the fact that further smoothness in q implies faster decay of S:

Lemma II.6. *Let* $q \in W_c^{\epsilon,p}(\mathbb{R}^2), 2 < p, \epsilon > 0$. *Then* $S \in L^s(\mathbb{R}^2)$ *for any* $s > 2/(\epsilon + 1)$.

Proof. We follow the ideas in the proof of [24, Proposition 3.3]: write

$$S(k) = -2i\mathcal{F}(\overline{q})(-2k_1, 2k_2) + T(k),$$

where

$$T(k) = -\frac{i}{\pi}\int_{\mathbb{R}^2} e(z,k)\overline{q}(z)(m_1(z,k) - 1)d\mu(z).$$

The fact that $\mathcal{F}(\overline{q})(-2k_1, 2k_2) \in L^s(\mathbb{R}^2)$ for any $s > 2/(1+\epsilon)$ follows since $q \in W^{\epsilon,r}(\mathbb{R}^2)$ for $r, 1 \le r \le 2$.

To treat the second term we consider for $0 < \delta < \epsilon$ the pseudodifferential operator

$$Mg(k) = -\frac{i}{\pi}\int_{\mathbb{R}^2} e^{2i\,\mathrm{Re}(zk)}\langle k\rangle^\delta (m_1(z,k) - 1)\phi(z)g(z)d\mu(z),$$

where ϕ is a smooth cut-off function with $\phi = 1$ on $\mathrm{supp}(q)$. A result due to Coifman and Meyer [39, p. 14] concerning the mapping property of a pseudodifferential operator with non-smooth symbol implies together with the estimate (22) that M is bounded on $L^2(\mathbb{R}^2)$, and hence

$$\|T\|_{L^s(\mathbb{R}^2)} = \|\langle k\rangle^{-\delta}M\overline{q}\|_{L^s(\mathbb{R}^2)} \le C\|\langle k\rangle^{-\delta}\|_{L^{(1/s-1/2)^{-1}}(\mathbb{R}^2)}\|q\|_{L^2(\mathbb{R}^2)} < C\|q\|_{L^2(\mathbb{R}^2)}$$

for $\delta(1/s - 1/2)^{-1} > 2$ or equivalently $s > 2/(\delta + 1) > 2/(\epsilon + 1)$. $\square$

The unique solvability of (24) is a direct consequence of Lemma (II.6).

Lemma II.7. *The function* $m(z,\cdot)$ *can be found as the unique solution to* (24) *with* $m_1(z,\cdot) - 1,\ m_2(z,\cdot) \in L^r(\mathbb{R}^2) \cap C^\alpha(\mathbb{R}^2)$ *for* $r > 2/\epsilon,\ \alpha < 1/2$.

Proof. Let $z \in \mathbb{C}$ be fixed and introduce

$$m^\pm(z,k) = m_1(z,k) \pm m_2(z,k). \tag{25}$$

Then for fixed $z \in \mathbb{C}$, (24) implies that

$$\overline{\partial}_k m^\pm(z,k) = \pm S(k)e(z,-k)\overline{m^\pm(z,k)}, \quad m^\pm \sim 1. \tag{26}$$

Since $S \in L^s(\mathbb{R}^2)$ for $s > 2/(\epsilon+1)$ by Lemma II.6, it follows by Proposition II.2 that (26) has a unique solution in $L^r(\mathbb{R}^2)$ for $r > 2/\epsilon$. The Hölder continuity is a consequence of (12). $\square$

Finally to reconstruct q we can for any fixed $r > 0$ use the formula

$$q(z) = \lim_{|k_0|\to\infty} \frac{1}{\mu(B_r(0))} \int_{\{k:\,|k-k_0|<r\}} (\partial + ik) m_2(z,k) d\mu(k). \tag{27}$$

This formula follows from (8) and (21); for a complete proof we refer to [22, Theorem 5.2]. This shows the invertibility of the map $q \mapsto S$.

III. From Boundary data to Scattering data

Using that Ψ solves (4) and the fact that q is supported in Ω, an integration by parts in (7) shows that

$$S(k) = -\frac{i}{2\pi} \int_{\partial\Omega} e^{i\overline{zk}} \overline{\nu}(z) \Psi_2(z,k) d\sigma(z), \quad k \in \mathbb{C}, \tag{28}$$

where $d\sigma(z)$ is the surface measure on the boundary. This shows that to compute the scattering transform from the boundary data, we have to compute the trace on $\partial\Omega$ of the exponentially growing solutions.

In this section we give a complete characterization of $\Psi|_{\partial\Omega}$ in terms of the boundary data. The characterization is given by two neccesary conditions. The first observation is that when the potential q is defined from a conductivity by (5) then the Cauchy data $\mathcal{C}_q$ for (4) can be described by a certain equation on the boundary involving the Dirichlet-to-Neumann map. The second condition concerns the asymptotic behavior and gives again an equation on the boundary in terms of certain single layer potentials. We emphasize that the characterization is given by a set of equations on $\partial\Omega$ and is thus in essence constructive.

A. A boundary relation - characterization of the Cauchy data

In this section we give an explicit characterization of the Cauchy data for the first order system (4), in case the potential q comes from a conductivity.

Assume $u \in C^{1+\beta}(\overline{\Omega})$ solves (1) for some $f \in C^{1+\beta}(\partial\Omega)$. Then (v, w) defined by (3) solves (4), and on the boundary one easily finds the relation

$$\begin{pmatrix} v \\ w \end{pmatrix}\bigg|_{\partial\Omega} = \frac{1}{2}\begin{pmatrix} \overline{\nu} & -i\overline{\nu} \\ \nu & i\nu \end{pmatrix}\begin{pmatrix} \Lambda_\gamma f \\ \partial_\tau f \end{pmatrix}, \tag{29}$$

where ∂_τ denotes the tangential derivative along $\partial\Omega$, and ν is the complex normal at the boundary, i.e., if $(\nu_1(z), \nu_2(z))$ denotes the outer unit normal at $z \in \partial\Omega$, then $\nu(z) = \nu_1(z) + i\nu_2(z)$ and $\overline{\nu} = \nu_1(z) - i\nu_2(z)$. Inverting (29) gives

$$\begin{pmatrix} \Lambda_\gamma f \\ \partial_\tau f \end{pmatrix} = \begin{pmatrix} \nu & \overline{\nu} \\ i\nu & -i\overline{\nu} \end{pmatrix}\begin{pmatrix} v \\ w \end{pmatrix}\bigg|_{\partial\Omega}. \tag{30}$$

Let $C_0^\beta(\partial\Omega) = \{\phi \in C^\beta(\partial\Omega)\colon \int_{\partial\Omega} \phi d\sigma = 0\}$ and let $s\colon [0, |\partial\Omega]| \to \partial\Omega$ be an arclength parameterization of $\partial\Omega$. Then define $\partial_\tau^{-1}\colon C_0^\beta(\partial\Omega) \to C^{1+\beta}(\partial\Omega)$ by

$$(\partial_\tau^{-1} h)(z) = \int_0^{t_0} \phi(s(t))dt$$

where $z = s(t_0)$. With this definition we note that informally

$$\partial_\tau \partial_\tau^{-1}\phi = \phi, \quad \partial_\tau^{-1}\partial_\tau f = f - f(s(0)).$$

Now since $f \in C^{1+\beta}(\partial\Omega)$ implies that $\partial_\tau f = (\nu v - \overline{\nu} w) \in C_0^\beta(\partial\Omega)$, it follows from (30) by eliminating f that

$$iH_\gamma(\nu v|_{\partial\Omega} - \overline{\nu} w|_{\partial\Omega}) = (\nu v|_{\partial\Omega} + \overline{\nu} w|_{\partial\Omega}),$$

where $H_\gamma = \Lambda_\gamma \partial_\tau^{-1}$. This relation motivates the definition of the set

$$\begin{aligned}\mathcal{BR} = \{(h_1, h_2) \in C^\beta(\partial\Omega) \times C^\beta(\partial\Omega)\colon & (\nu h_1 - \overline{\nu} h_2) \in C_0^\beta(\partial\Omega) \\ & iH_\gamma(\nu h_1 - \overline{\nu} h_2) = \nu h_1 + \overline{\nu} h_2\}.\end{aligned} \tag{31}$$

A pair of functions $(h_1, h_2) \in \mathcal{BR}$ is said to satisfy the boundary relation.

We just found that solutions to (4), defined through (3), satisfy the boundary relation, but as the following theorem shows, (31) is in fact a complete characterization of $\mathcal{C}_q$:

Lemma III.1. *If $q \in W^{\epsilon,p}(\overline{\Omega})$ is given by (5), then $\mathcal{C}_q = \mathcal{BR}$.*

Proof. First we show that $\mathcal{BR} \subset \mathcal{C}_q$. Let $(h_1, h_2) \in \mathcal{BR}$ and let $u \in C^{1+\beta}(\overline{\Omega})$ be the unique solution to the Dirichlet problem

$$\begin{cases} \nabla \cdot \gamma \nabla u = 0, & \text{in } \Omega, \\ u = i\partial_\tau^{-1}(\nu h_1 - \overline{\nu} h_2), & \text{on } \partial\Omega. \end{cases}$$

Define a solution (v, w) to (4) by the relation (3) with u from above. Then $(v, w)|_{\partial\Omega} = (h_1, h_2)$ by (29) and (31), i.e. $(h_1, h_2) \in \mathcal{C}_q$.

Next we see that $\mathcal{C}_q \subset \mathcal{BR}$. Let $(h_1, h_2) \in \mathcal{C}_q$ and let $(v, w) \in C^\beta(\overline{\Omega}) \times C^\beta(\overline{\Omega})$ be a solution to (4) with Cauchy data (h_1, h_2). Since q is of the form (5), we have the compatibility relation

$$\overline{\partial}(\gamma^{-1/2} v) = \partial(\gamma^{-1/2} w),$$

which ensures the existence of a $u \in C^{1+\beta}(\overline{\Omega})$ such that

$$\gamma^{-1/2} \begin{pmatrix} v \\ w \end{pmatrix} = \begin{pmatrix} \partial u \\ \overline{\partial} u \end{pmatrix}.$$

It is easy to check that u is a solution of the conductivity equation in the form $2\partial u \overline{\partial}\gamma + 2\overline{\partial} u \partial\gamma + 4\gamma\partial\overline{\partial} u = 0$. Now relation (29) with $f = u|_{\partial\Omega}$ shows that $(h_1, h_2) \in \mathcal{BR}$. □

B. Boundary characterization of asymptotic behavior

Consider the exponentially growing solution $\Psi = (\Psi_1, \Psi_2)$. Notice that the first element is analytic outside Ω, while the second element is anti-analytic outside Ω. Moreover, they have a prescribed behavior at infinity. We will see that these properties are completely described by an integral equation on $\partial\Omega$.

In the proof we will need a few tools from complex analysis. Let $D \subset \mathbb{C}$ be an open bounded domain with C^1 boundary, and let $f \in C^1(\overline{D})$. Define

$$\Phi(z) = \frac{1}{2\pi i}\int_{\partial D}\frac{\phi(\zeta)}{\zeta - z}d\zeta, \qquad z \notin \partial D.$$

Here and in the sequel the integral on the boundary ∂D is understood as a path integral taken on a positively oriented path describing ∂D. For $z \in \partial D$ the expression is not well-defined since the integral kernel is singular, but it is well known that Φ does have a continuous extension to ∂D from both inside and outside. To be more exact we introduce the operator

$$\mathcal{S}f(z) = \frac{1}{2\pi i}p.v.\int_{\partial D}\frac{f(\zeta)}{\zeta - z}d\zeta, \qquad z \in \partial D,$$

which is known to be bounded in $C^\alpha(\partial D),\ 0 < \alpha < 1$. The behavior of $\Phi(z)$ when z tends to ∂D is now described by the Sokhotski-Plemelj formula

$$\lim_{z^\pm \to \partial\Omega}\Phi(z) = \mp\frac{1}{2}f(z) + \mathcal{S}f(z), \qquad z \in \partial D. \tag{32}$$

We refer to [40] for the proofs.

The following lemma gives a necessary condition for a pair of functions defined in $\mathbb{C}\setminus\overline{\Omega} = \mathbb{C}\setminus\overline{\Omega}$ to be analytic and anti-analytic respectively and have a special exponential decay at infinity. Introduce for $z \in \mathbb{C}\setminus\{0\}$

$$g_k(z) = \frac{1}{\pi}\frac{e^{-ikz}}{z},$$

a Green's kernel for $\overline{\partial}$, which takes into account exponential growth at infinity. Let $z \in \partial\Omega$ and define the single layer potentials $\mathcal{S}_k,\ \overline{\mathcal{S}}_k \in \mathcal{B}(C^\alpha(\partial\Omega))$ by

$$\mathcal{S}_kf(z) = p.v.\int_{\partial\Omega} f(\zeta)g_k(\zeta - z)d\zeta, \quad \overline{\mathcal{S}}_kf(z) = p.v.\int_{\partial\Omega} f(\zeta)\overline{g_k(\zeta - z)}d\overline{\zeta}.$$

The result is then:

Lemma III.2. *Let v be analytic and w be anti-analytic in $\mathbb{C}\setminus\overline{\Omega}$ and assume that $v|_{\partial\Omega}, w|_{\partial\Omega} \in C^\alpha(\partial\Omega)$ and that $(ve^{-izk} - 1) \in L^r(\mathbb{C}\setminus\overline{\Omega})$*

and $we^{-izk} \in L^r(\mathbb{C} \setminus \overline{\Omega})$ *for some* r, α *satisfying* $0 < \alpha < 1,\ 1 \le r < \infty$. *Then the trace* $(h_1, h_2) = (v, w)|_{\partial\Omega}$ *satisfies*

$$\begin{aligned}(I - i\mathcal{S}_k)h_1 &= 2e^{izk} \\ (I + i\overline{\mathcal{S}}_k)h_2 &= 0.\end{aligned} \tag{33}$$

Proof. Let $B_R = B(0, R)$ be the ball centered at zero with radius R and assume that $\overline{\Omega} \subset B_R$. Since v is analytic in $\mathbb{C} \setminus \overline{\Omega}$, the Cauchy integral formula implies that

$$v(z)e^{-ikz} - 1 = \frac{1}{2\pi i}\int_{\partial B_R} \frac{v(\zeta)e^{-ik\zeta} - 1}{\zeta - z} d\zeta - \frac{1}{2\pi i}\int_{\partial\Omega} \frac{v(\zeta)e^{-ik\zeta} - 1}{\zeta - z} d\zeta$$

for $z \in B_R \setminus \overline{\Omega}$. Now, since $ve^{-izk} - 1 \in L^r(\mathbb{R}^2)$, The integral on ∂B_R converges to zero as $R \to \infty$ at least on a sequence of increasing radii. Moreover, since $z \in \mathbb{C} \setminus \overline{\Omega}$, Cauchy's formula gives

$$\frac{1}{2\pi i}\int_{\partial\Omega} \frac{1}{\zeta - z} d\zeta = 0.$$

Therefore

$$v(z) + \frac{1}{2i}\int_{\partial\Omega} v(\zeta) g_k(\zeta - z) d\zeta = e^{izk},\ z \in \mathbb{C} \setminus \overline{\Omega}. \tag{34}$$

Now let $z \in \mathbb{C} \setminus \overline{\Omega}$ approach some point on the boundary and apply (32) to get the equation for v.

To prove the result for w, we note that $\overline{w}$ is analytic in $\mathbb{C} \setminus \overline{\Omega}$ and $we^{-izk}e(z, k) \in L^r(\mathbb{R}^2)$ implies that $\overline{w}e^{-izk} \in L^r(\mathbb{R}^2)$. As above we then get

$$\overline{w}(z) + \frac{1}{2i}\int_{\partial\Omega} \overline{w}(\zeta) g_k(\zeta - z) d\zeta = 0,\ z \in \mathbb{C} \setminus \overline{\Omega}, \tag{35}$$

and the result again follows by invoking (32) and then taking complex conjugate. ☐

C. Complete characterization of $\Psi|_{\partial\Omega}$

We know that the exponentially growing solution $(\Psi_1(\cdot, k), \Psi_2(\cdot, k))|_{\partial\Omega} \in \mathcal{C}_q$ must satisfy (33). The following result shows that this condition is a complete characterization.

Theorem III.3. *Let $q \in W^{\epsilon,p}(\overline{\Omega})$. Then the* (33) *has in $\mathcal{C}_q$ the unique solution $\Phi = (\Psi_1, \Psi_2)|_{\partial\Omega}$.*

Proof. Since $(\Psi_1, \Psi_2)|_{\partial\Omega} \in \mathcal{C}_q$ and satisfies the assumptions of Lemma III.2, it is a solution to (33).

Let $h \in \mathcal{C}_q$ be another solution to (33). Extend h inside Ω to a solution (v, w) to the system (4), and guided by (34) and (35) extend h in $\mathbb{C} \setminus \overline{\Omega}$ to (v, w) by

$$v(z) = -\frac{1}{2i}\int_{\partial\Omega} h_1(\zeta) g_k(\zeta - z) d\zeta + e^{izk},$$
$$w(z) = -\frac{1}{2i}\int_{\partial\Omega} h_2(\zeta) \overline{g_k(\zeta - z)} d\overline{\zeta}.$$

We will prove that defined this way, (v, w) is a solution to (4) in $\mathbb{R}^2$ and that $ve^{-izk} - 1, we^{-izk} \in L^r(\mathbb{R}^2),\ r > 2$. Then the result follows from the uniqueness in Proposition II.3.

By construction (v, w) solves (4) in Ω and in $\mathbb{C} \setminus \overline{\Omega}$, so we have to show that the equation holds as we cross $\partial\Omega$. Since the support of q is strictly inside Ω, v is analytic and w is anti-analytic in a neighborhood of the boundary. Hence for the equation to hold it suffices to have continuity across the boundary and then invoke Morera's theorem (see for instance [41, p. 98]). We will show the continuity of v, for w similar reasoning works. Let z approach some point $z_0 \in \partial\Omega$ from outside. Then using (32) we get

$$\lim_{z^+ \to z_0} v(z) = -\left(-\frac{h_1(z_0)}{2} + \frac{1}{2i}\mathcal{S}_k h_1(z_0)\right) + e^{iz_0 k}.$$

Now use (33) to conclude $\lim_{z \to z_0} v(z) = h_1(z_0)$. The continuity of v from inside follows by the construction.

That $ve^{-izk} - 1 \in L^r(\mathbb{R}^2),\ r > 2$, follows since

$$v(z)e^{-izk} - 1 = -\frac{1}{2\pi i}\int_{\partial\Omega} \frac{h_1(\zeta)e^{-ik\zeta}}{\zeta - z} d\zeta = O\left(\frac{1}{|z|}\right), \text{ as } |z| \to \infty.$$

Again a similar argument shows $we^{-izk} \in L^r(\mathbb{R}^2)$. This proves the theorem. $\square$

This theorem shows by (28) that the non-physical scattering transform S is uniquely determined by the Cauchy-data $\mathcal{C}_q$. When the potential is defined from a conductivity by (5) we have by Lemma III.1 an explicit description of $\mathcal{C}_q$ given by the boundary relation. Hence on $\partial\Omega$ we have the system of equations

$$\begin{pmatrix} (I - i\mathcal{S}_k) & 0 \\ 0 & (I + i\overline{\mathcal{S}_k}) \\ (iH_\gamma - I)\nu & -(iH_\gamma + I)\overline{\nu} \end{pmatrix} \begin{pmatrix} \Psi_1 \\ \Psi_2 \end{pmatrix} = \begin{pmatrix} 2e^{izk} \\ 0 \\ 0 \end{pmatrix},$$

from which $\Psi|_{\partial\Omega}$ can be found. This gives by (28) a constructive way of obtaining S from Λ_γ.

IV. From S to γ

In this section we will prove how to reconstruct γ from $S(k)$. This could be done by computing q from S by solving (24) and applying (27), and then finding γ by solving (5) with the asymptotic condition $\gamma^{1/2} \sim 1$. Since this approach requires taking the parameter k to infinity and solving a differential equation, we will take a different route. As observed by Barceló, Barceló and Ruiz, γ can be reconstructed directly from m at $k = 0$. More precisely let $\widetilde{m}$ be the unique solution given by Proposition II.3 to (4) with q substituted by $\tilde{q} = -\overline{q}$. Then as a direct consequence of the definition (5) and the uniqueness in Proposition II.2, we derive the formula ([24, Proposition 4.2])

$$\gamma(z)^{1/2} = \widetilde{m}_1(z, 0) + \overline{\widetilde{m}_2(z, 0)}. \tag{36}$$

We will prove a result concerning the relation between the scattering transform of q and $\tilde{q}$. Such a result was given by Beals and Coifman in [29] (see also [24, Proposition 4.3]). Inspired by their brief argument we can show the following result.

Lemma IV.1. *Let $\widetilde{S}$ be the scattering transform of $\tilde{q} = -\overline{q}$. Then*

$$\widetilde{S}(k) = \overline{S(-k)} \tag{37}$$

Proof. From (8) it follows that

$$
\begin{aligned}
\bar\partial(m_1 - 1) &= q m_2 \\
&= q(\partial + ik)^{-1}(\overline{q} m_1) \\
&= q e(\cdot, -k)\partial^{-1}(e(\cdot, k)\overline{q} m_1),
\end{aligned}
$$

and hence

$$
(m_1 - 1) = \bar\partial^{-1} q e(\cdot, -k)\partial^{-1}(e(\cdot, k)\overline{q} m_1). \tag{38}
$$

Applying the operator ρ_Ω of restriction to Ω gives the equation

$$
(I - \overline{A_{q,k}} A_{q,k})\rho_\Omega m_1 = \rho_\Omega 1
$$

for $\rho_\Omega m_1 \in L^2(\Omega)$, where $A_{q,k} \in \mathcal{B}(L^2(\Omega))$ is defined by

$$
A_{q,k}\phi = \rho_\Omega \partial^{-1}(e(\cdot, k)\overline{q}\phi).
$$

Using an argument similar to the arguments in the proofs of Proposition II.2 and Proposition II.3 shows that $(I - \overline{A_{q,k}} A_{q,k})$ is invertible in $L^2(\Omega)$ and hence

$$
m_1(\cdot, k) = (I - \overline{A_{q,k}} A_{q,k})^{-1}(1). \tag{39}
$$

Moreover, multiplying by $qe(\cdot, k)$ in (38) we find the equation

$$
\overline{q} e(\cdot, k) m_1(\cdot, k) = (I - (A_{-\overline{q},-k})^*(\overline{A_{-\overline{q},-k}})^*)^{-1}(\overline{q} e(\cdot, k)), \tag{40}
$$

where $A_{q,k}^* = -qe(\cdot, -k)\overline{\partial}^{-1}$ is the adjoint with respect to the usual inner product $(\cdot, \cdot)$ in $L^2(\Omega)$. Hence by using (40) and (39) we now find that

$$
\begin{aligned}
S(k) &= -\frac{i}{\pi}\big(1, \overline{q} e(\cdot, k) m_1(\cdot, k)\big) \\
&= -\frac{i}{\pi}\big(1, (I - (A_{-\overline{q},-k})^*(\overline{A_{-\overline{q},-k}})^*)^{-1}(\overline{q} e(\cdot, k))\big) \\
&= -\frac{i}{\pi}\big((I - \overline{A_{-\overline{q},-k}} A_{-\overline{q},-k})^{-1}(1), \overline{q} e(\cdot, k)\big) \\
&= -\frac{i}{\pi}\big(\widetilde{m}_1(\cdot, -k), \overline{q} e(\cdot, k)\big) \\
&= \overline{-\frac{i}{\pi}\big(1, -qe(\cdot, -k)\widetilde{m}_1(\cdot, -k)\big)} \\
&= \overline{\widetilde{S}(-k)}.
\end{aligned}
$$

□

From (26) and (37) it follows that the $\overline{\partial}_k$-equation for $\widetilde{m}$ is

$$\overline{\partial}_k \widetilde{m}^+(z,k) = \overline{S(-k)} e(z,-k) \overline{\widetilde{m}^+(z,k)},$$

which together with the asymptotic condition $\widetilde{m}^+ \sim 1$ allows the computation of $\widetilde{m}^+$.

Since γ is real, we can rewrite the formula (36) using (25)

$$\begin{aligned} \gamma^{1/2}(z) &= \widetilde{m}_1(z,0) + \overline{\widetilde{m}_2(z,0)} \\ &= \operatorname{Re} \widetilde{m}_1(z,0) + \operatorname{Re} \overline{\widetilde{m}_2(z,0)} \\ &= \operatorname{Re} \widetilde{m}^+(z,0). \end{aligned}$$

This ends the reconstruction.

V. Reduction to the case $\gamma = 1$ near $\partial\Omega$

It is well known that for sufficiently regular conductivities, the boundary value of the conductivity as well as the normal derivatives at the boundary can be computed from the Dirichlet-to-Neumann map. In this section we will show that by knowing only the trace of the conductivity, it can be extended outside the domain in a way such that the regularity is preserved and the extended conductivity is constant outside some larger compact set. Moreover, we will see that the Dirichlet-to-Neumann map corresponding to the extended conductivity can be computed. Results along this line are well known for sufficiently regular conductivities [17], but care has to been taking when we work with less regular conductivities.

When $\gamma \in W^{1+\epsilon,p}(\Omega)$, $p > 2, \epsilon > 0$, a method for the reconstruction of $\gamma|_{\partial\Omega}$ is given in [17]. We would like to use this trace to construct a function γ_e in $\mathbb{R}^n$, which is known explicitly in $\mathbb{R}^n \setminus \Omega$ and has the properties that $\gamma_e|_\Omega = \gamma$ and $(\gamma_e - 1) \in W^{1+\epsilon,p}(\mathbb{R}^2)$ is compactly supported inside a larger domain $\Omega_e \supset \Omega$. For this extension to preserve regularity we will have to assume that $0 < \epsilon < 1/p$.

The construction of an extension relies on the following lemma:

Lemma V.1. *Assume* $1 \leq p$ *and* $1/p - 1 < s < 1/p$. *Let* $f \in W^{1+s,p}(\Omega)$ *and* $g \in W^{1+s,p}(\mathbb{R}^n \setminus \overline{\Omega})$ *with* $f|_{\partial\Omega} = g|_{\partial\Omega}$. *Then the function*

$$h = \begin{cases} f & \text{in } \Omega, \\ g & \text{in } \mathbb{R}^n \setminus \overline{\Omega} \end{cases}$$

satisfies $h \in W^{1+s,p}(\mathbb{R}^n)$.

Proof. Since $\partial\Omega$ is smooth we can without loss of generality assume that $\Omega = \mathbb{R}^n_+ = \{(x', x_n) \in \mathbb{R}^n \mid x_n > 0\}$ and hence $\mathbb{R}^n \setminus \overline{\Omega} = \mathbb{R}^n_-$. Let $E_\pm$ denote operators extending functions defined on $\mathbb{R}^n_\pm$ to $\mathbb{R}^n$. It is well known that such operators exist and that $E_\pm \in \mathcal{B}(W^{r,p}(\mathbb{R}^n_\pm), W^{r,p}(\mathbb{R}^n)), r \in \mathbb{R}$ (see for instance [42, section 2.9]). Then $h = E_-g + \chi_{x_n>0}(E_+f - E_-g)$, where $\chi_{x_n>0}$ is the characteristic function of $\mathbb{R}^n_+$.

It is clear that $h \in L^p(\mathbb{R}^n)$ and $\partial_{x_j} h \in W^{s,p}(\mathbb{R}^n)$ for $j = 1, 2, \cdots, n-1$, so to prove the result we need to show that $\partial_{x_n} h \in W^{s,p}(\mathbb{R}^n)$ or equivalently $\chi_{x_n>0}(E_+f - E_-g) \in W^{s,p}(\mathbb{R}^n)$. Since $(E_+f - E_-g)|_{x_n=0} = 0$ by assumption, it follows that $\partial_{x_n}\chi_{x_n>0}(E_+f - E_-g) = \chi_{x_n>0}\partial_{x_n}(E_+f - E_-g)$. We then reach the conclusion by using the fact that multiplication by a characteristic function is a bounded operator in $W^{s,p}(\mathbb{R}^n)$ for $1/p - 1 < s < 1/p$ ([42, section 2.8.7]) and that $\partial_{x_n}(E_+f - E_-g) \in W^{s,p}(\mathbb{R}^n)$. □

Let now $\gamma|_{\partial\Omega} \in W^{1+\epsilon-1/p,p}(\partial\Omega)$ be given and define $\tilde{\gamma} = E(\gamma|_{\partial\Omega}) \in W^{1+\epsilon,p}(\mathbb{R}^n)$, where the extension operator $E \in \mathcal{B}(W^{1+\epsilon-1/p,p}(\partial\Omega), W^{1+\epsilon,p}(\mathbb{R}^n))$ is any right inverse of the trace operator [42, section 2.7.2]. Since $\gamma|_{\partial\Omega} \geq c > 0$ it follows that $\tilde{\gamma} \geq \tilde{c} > 0$ in a neighborhood of $\partial\Omega$, and hence by adding and multiplying smooth functions, we can construct a new strictly positive extension γ' of $\gamma|_{\partial\Omega}$ such that $\gamma' - 1 \in W^{1+\epsilon,p}(\mathbb{R}^n)$ is compactly supported. The function

$$\gamma_e = \begin{cases} \gamma & \text{on } \Omega \\ \gamma' & \text{on } \mathbb{R}^n \setminus \overline{\Omega} \end{cases}$$

is by the Lemma V.1 in $W^{1+\epsilon,p}(\mathbb{R}^n)$ for $0 < \epsilon < 1/p$ and hence a suitable extension.

Introduce the extended domain $\Omega_e \subset \mathbb{R}^2$, which is smooth, bounded and satisfies $\text{supp}(\gamma_e - 1) \subset \Omega_e$. We will see next that the Dirichlet-to-Neumann map corresponding to γ_e on $\partial\Omega_e$ can be calculated from the Dirichlet-to-Neumann map for the domain Ω given the known value of the conductivity in $\Omega_e \setminus \overline{\Omega}$:

Lemma V.2. *Let $\Omega_e, \Omega \subset \mathbb{R}^n$ be smooth and bounded domains such that $\Omega \subset \Omega_e$. Let $\gamma_e \in L^\infty(\Omega_e)$, let Λ_γ be the Dirichlet-to-Neumann map on $\partial\Omega$ corresponding to $\gamma_e|_\Omega$ and let Λ_{γ_e} be the Dirichlet-to-Neumann map on $\partial\Omega_e$ corresponding to γ_e. Then Λ_{γ_e} can be recovered from Λ_γ and $\gamma_e|_{\Omega_e\setminus\overline{\Omega}}$.*

Proof. From the definition of the Dirichlet-to-Neumann map we have for any $f, g \in H^{1/2}(\partial\Omega_e)$, that

$$\begin{aligned}\langle \Lambda_{\gamma_e} f, g\rangle &= \int_{\Omega_e} \gamma \nabla u \cdot \nabla v \\ &= \int_{\Omega_e\setminus\overline{\Omega}} \gamma_e \nabla u \cdot \nabla v + \langle \Lambda_\gamma(u|_{\partial\Omega}), v|_{\partial\Omega}\rangle,\end{aligned}$$

where $u \in H^1(\Omega_e)$ denotes the unique solution to

$$\begin{aligned}\nabla \cdot \gamma_e \nabla u &= 0 \text{ in } \Omega_e, \\ u &= f, \text{ on } \partial\Omega_e,\end{aligned} \tag{41}$$

and $v \in H^1(\Omega_e)$ is any function with $v|_{\partial\Omega_e} = g$. Hence we see that Λ_{γ_e} can be found from $\gamma_e|_{\Omega_e\setminus\overline{\Omega}}$ and Λ_γ without explicit knowledge of γ in Ω provided that the solution u to (41) can be found in $\Omega_e \setminus \overline{\Omega}$.

We claim that u in $\Omega_e \setminus \overline{\Omega}$ can be found as the unique solution to boundary value problem

$$\begin{aligned}\nabla \cdot \gamma_e \nabla u &= 0 \text{ in } \Omega_e \setminus \overline{\Omega}, \\ u &= f \text{ on } \partial\Omega_e, \\ \gamma_e(\partial_\nu u)|_{\partial\Omega} &= \Lambda_\gamma(u|_{\partial\Omega}) \text{ on } \partial\Omega.\end{aligned} \tag{42}$$

Here $\gamma_e(\partial_\nu u)|_{\partial\Omega} \in H^{-1/2}(\partial\Omega)$ is the normal derivative on $\partial\Omega$ defined in the weak sense for $g \in H^{1/2}(\partial\Omega)$ as

$$\langle \gamma_e(\partial_\nu u)|_{\partial\Omega}, g\rangle = \int_{\Omega\setminus\overline{\Omega}} \gamma_e \nabla u \cdot \nabla v, \tag{43}$$

with $v \in H^1(\Omega \setminus \overline{\Omega}), v|_{\partial\Omega} = g, v|_{\partial\Omega_e} = 0$.

That $u|_{\Omega_e \setminus \overline{\Omega}}$ solves (42) is trivial. For the uniqueness we assume that $u_0 \in H^1(\Omega_e \setminus \overline{\Omega})$ solves (42) with $f = 0$. Extend u_0 into Ω as the solution to

$$\nabla \cdot \gamma_e \nabla u = 0 \text{ in } \Omega, \qquad u = u_0 \text{ on } \partial\Omega.$$

Since the extended u_0 is in $H^1(\Omega_e)$ and solves (41) with $f = 0$, we conclude that $u_0 = 0$ in Ω_e. □

We emphasize that the solution to (42) can be constructed explicitly using for instance pseudodifferential calculus, see [43].

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