Tomographic reconstruction of vector fields in variable background media

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Abstract. This paper considers the problem of determination of a planar vector field when its Doppler data is modified by the presence of an unknown scalar field. Such a problem occurs in the investigation of velocity distribution in a flow through a media of variable sound speed. We show that the curl of the vector field can be stably recovered. In absorbing media the integrals are weighted to account for the attenuation along the path. When the boundary values of the vector field are known, its solenoidal part is determined from its curl. We use Bukhgeim's approach to the problem of tomography.

Keywords: attenuated Doppler transforms, attenuated Radon transform, A-analytic functions

1. Introduction

The attenuated Doppler transform of a vector field is the collection of weighted integrals of its projections along lines. When occurring, the weight accounts for the loss of intensity along the paths due to the absorption property of the medium. The inverse problem consists in the recovery of the vector field from its Doppler data. Such a problem occurs in the investigation of velocity distribution in a flow, or in the investigation of the stress distribution in metals, by ultrasonic time-of-flight measurements, see Braun and Hauk [2] and Norton [13]. For non-absorbing media the inversion problem has been investigated by Sharafutdinov [15] in the general framework of Riemannian geometry, and by Sparr et al. [16] in the Euclidean setting. The problem does not have a unique solution: the superposition of any compactly supported gradient field is indistinguishable from the data. However, the solenoidal part of the field is uniquely determined by the traces of the field on the boundary. For the absorbing media, the problem was first consider by Strahlen [17] in the case of constant attenuation. Surprisingly enough, in planar domains the full vector field can be recovered in the regions of positive absorption as shown by Kazantsev and Bukhgeim [9], and by Bal [3]. The former use A. L. Bukhgeim's theory of A-analytic functions [6], while the latter uses the inverse scattering method of Novikov [14].

The derivation in [2] considers a background of constant sound speed much larger then the magnitude of velocity. If one allows for an inhomogeneous background, we are lead to the problem of determination of a vector field in the presence of an unknown scalar field, which is the reciprocal of the background sound speed. In this case, the classical Doppler data is altered by the superposition of the attenuated Radon transform of the scalar field . We show that the *curl* of the vector field can be stably reconstructed independently of the background sound speed. We use Bukhgeim's approach to the inversion of the attenuated Radon transform problem [1].

In our considerations, Ω is a strictly convex planar domain; in particular it has a C^2 smooth boundary. The attenuation $a \in C_0^2(\overline{\Omega})$ is assumed to be known. The unknowns are the real vector field $\mathbf{F} = (F_1, F_2) \in C_0^2(\overline{\Omega}; \mathbb{R}^2)$ and the real map $f_0 \in C_0^2(\overline{\Omega})$. Outside the domain we extend a, $\mathbf{F}$ and f_0 to be 0.

For each $x \in \Omega$ and $\theta = (\cos\varphi, \sin\varphi) \in S^1$ let us consider the divergence beam transform:

$$Da(x, \theta) = \int_0^\infty a(x + s\theta)ds.$$

The measured data

$$m(x, \theta) = \int_{-\infty}^\infty (f_0 + \theta \cdot \mathbf{F})(x + t\theta)e^{-Da(x+t\theta)}dt \tag{1}$$

are given by the superposition of the attenuated Doppler transform of $\mathbf{F}$ with the attenuated Radon transform of f_0 for all $(x, \theta) \in \Omega \times S^1$. Since m is invariant to translations of x in the direction of θ, it is a function only on the tangent bundle of the circle S^1. We show that m stably determines

$$curl\mathbf{F} := \mathbf{e_z} \cdot \nabla \times (\mathbf{F_1}, \mathbf{F_2}, \mathbf{0}) = \partial_{\mathbf{x}} \mathbf{F_2} - \partial_{\mathbf{y}} \mathbf{F_1}$$

in Ω.

By Hodge decomposition, any vector field $\mathbf{F} \in H^1(\Omega; \mathbb{R}^2)$ decomposes into a gradient field and a divergence-free (solenoidal) field:

$$\mathbf{F} = \nabla u + \mathbf{F}^s, \tag{2}$$

where $u \in H_0^2(\Omega)$ and $\mathbf{F}^s = (F_1^s, F_2^s) \in H^1_{div}(\Omega; \mathbb{R}^2)$ with $\mathbf{F}^s|_{\partial\Omega} = \mathbf{F}|_{\partial\Omega}$. If $\mathbf{F} \in H^2(\Omega; \mathbb{R}^2)$ and $\mathbf{F}|_{\partial\Omega}$ is known, then $curl\mathbf{F}$ determines the solenoidal part $\mathbf{F}^s$ (see the section 3).

2. The transport model

The problem can be reformulated as an inverse boundary value problem associated with the transport equation as follows. Let us consider the family (indexed in $\theta \in S^1$) of problems

$$\begin{cases} \theta \cdot \nabla u(x,\theta) + a(x) u(x,\theta) = f_0(x) + \theta \cdot \mathbf{F}(x), \ (x,\theta) \in \Omega \times S^1, \\ u(x,\theta) = 0, \ (x,\theta) \in \Gamma_-, \end{cases} \tag{3}$$

where $\Gamma_\pm = \{(x,\theta) \in \partial\Omega \times S^1 : \pm n(x) \cdot \theta > 0\}$ denotes the incoming (-), respectively outgoing (+) boundary and $n(x)$ denotes the outer normal at some boundary point x. Given $a(x)$ for $x \in \Omega$ and $u(x,\theta)$ for $(x,\theta) \in \partial\Omega \times S^1$ find $\mathbf{F}(x)$ for $x \in \Omega$. Integration of the equation in (3) along the lines in the direction of θ shows the equivalence with our initial problem. Let $\tau_+(x,\theta)$ denote the length of the vector in the direction of θ originating at $x \in \Omega$ until the boundary is met; so that $(x + \tau_+(x,\theta)\theta, \theta) \in \Gamma_+$ and

$$m(x,\theta) = u(x + \tau_+(x,\theta)\theta, \theta), \qquad (x,\theta) \in \Omega \times S^1.$$

For strictly convex domains, the map $\tau_+(x,\cdot) \in C^2(S^1)$. This regularity is needed later to insure the appropriate decay in (9).

Following the simplified approach of Finch [7] the equation (3) are multiplied by the integrating factor $e^{b-Da}(x,\theta)$, where

$$b(x,\theta) = \frac{1}{2}(I - iH)Pa(\theta, x \cdot \theta^\perp).$$

In the formula above, $Pa(\theta,s) = \int_{\mathbb{R}} a(s\theta^\perp + t\theta)dt$ is the x-ray transform of the attenuation. The Hilbert transform $Hf(s) = \frac{1}{\pi}\int_{\mathbb{R}} \frac{f(t)}{s-t}dt$ is taken in the second variable and evaluated at $s = x \cdot \theta^\perp$. Since $b(\cdot,\theta)$ is constant in the direction of θ we have that

$$v(x,\theta) = e^{b(x,\theta) - Da(x,\theta)} u(x,\theta) \tag{4}$$

solves the differential equation

$$\theta \cdot \nabla v(x,\theta) = (f_0(x) + \theta \cdot \mathbf{F}(x)) e^{b(x,\theta) - Da(x,\theta)}. \tag{5}$$

Throughout the paper, the angular notation $\theta = (\cos\varphi, \sin\varphi)$ has been used. It was first noticed by Natterer [12] that all the negative or even index coefficients in the

Fourier expansion (in the angular variable) of $b - Da$ are zero; see also [7, 4] for an elegant argument. Therefore e^{-b+Da} has only nonnegative index modes:

$$e^{-b(x,\theta)+Da(x,\theta)} = \sum_{n=0}^{\infty} \alpha_n(x) e^{in\varphi}. \tag{6}$$

Provided that $e^{-b+Da}(x,\cdot) \in L^2(S^1)$, for every $x \in \Omega$ we have the Parseval identity

$$\left\| e^{-b+Da}(x,\cdot) \right\|^2_{L^2(S^1)} = \sum_{k=0}^{\infty} |\alpha_k(x)|^2. \tag{7}$$

Consider now the Fourier expansions

$$u(x,\theta) = \sum_{n=-\infty}^{\infty} u_n(x) e^{in\varphi},\ v(x,\theta) = \sum_{n=-\infty}^{\infty} v_n(x) e^{in\varphi}, \tag{8}$$

and let $\mathbf{u} = (u_{-1}, u_{-2}, ...)$, respectively $\mathbf{v} = (v_{-1}, v_{-2}, ...)$, denote the corresponding sequence of their negative index coefficients.

Throughout this paper, we agree that sequences are indexed with negative indices as above. In the followings l^2 denotes the usual space of square sumable sequences and we introduce $l^{2,1+}$ to be the subspace of l^2 defined by

$$l^{2,1+} := \{\mathbf{v} \in l^2 : \sum_{n=1}^{\infty}\sum_{k=0}^{\infty} |v_{-n-k}|^2 < \infty\}. \tag{9}$$

The following result specifies the connection between the regularity of the data and the decay properties needed later.

Proposition 2.1 *Let Ω be a strictly convex domain, $a, f_0 \in C_0^2(\overline{\Omega})$ and $\boldsymbol{F} \in C_0^2(\overline{\Omega};\mathbb{R}^2)$. Then $\mathbf{v}, \mathbf{u} \in C^2(\overline{\Omega}; l^{2,1+})$ and $\mathbf{u}$ is calculated from $\mathbf{v}$ by the convolution formula:*

$$u_n = \sum_{k=0}^{\infty} \alpha_k v_{n-k},\ n \le -1, \tag{10}$$

where α_n's are defined in (6).

Proof: The solution to (5) subject to $v|_{\Gamma_-} = 0$ is

$$v(x,\theta) = \int_{-\infty}^{0} (f_0 + \theta\cdot\mathbf{F})(x+t\theta) e^{(b-Da)(x+t\theta)} dt. \tag{11}$$

Since Ω is strictly convex $\tau_+ \in C^2(\overline{\Omega}\times S^1)$. Since $Da = \int_0^{\tau_+(x,\theta)} a(x+t\theta)dt$ and the Hilbert transform preserves the smoothness class, then both Da and b are also in $C^2(\overline{\Omega}\times S^1)$. From (11) and (4) we get $v, u \in C^2(\overline{\Omega}; C^2(S^1))$. Applying the Parseval's identity to the l^2 sequence $(n^2 v_{-n}(z))_{n\in\mathbb{N}}$, we obtain for any $k < 3/2$ and $z \in \overline{\Omega}$ that

$$\sum_{n=1}^{\infty} n^k |v_{-n}|(z) \le \left(\sum_{n=1}^{\infty} n^{2k-4}\right)^{1/2} \left(\sum_{n=1}^{\infty} n^4 |v_{-n}(z)|^2\right)^{1/2} < \infty.$$

Since the series above is finite we also have $|v_n(z)|^2 \le C|n|^{-2k}$ for all $n \le -1$. Following the fact that $\sum_{n=0}^{\infty}\sum_{j=1}^{\infty}(n+j)^{-2k} < \infty$ for $k > 1$, we conclude that $\mathbf{v} \in C(\overline{\Omega}, l^{2,1+})$. The reasoning above also applies to the first and second order partial derivatives of $\mathbf{v}$. The equation (10) represents the convolution in the angular variable and the series is in the l^2 sense. $\square$

3. Stable reconstruction of the *curl*F

In this section we make use of the following complex notations:

$$z = x_1 + ix_2,\ \overline{\partial} = (\partial_{x_1} + i\partial_{x_2})/2,\ \partial = (\partial_{x_1} - i\partial_{x_2})/2,\ f_1 = (F_1 + iF_2)/2.$$

By using the equivalence $\theta \cdot \nabla = e^{-i\varphi}\overline{\partial} + e^{i\varphi}\partial$ and by identifying modes in the Fourier series expansions, the equation (3) translates into the system:

$$\overline{\partial}u_1(z) + \partial u_{-1}(z) + a(z)u_0(z) \quad = f_0(z), \tag{12}$$

$$\overline{\partial}u_0(z) + \partial u_{-2}(z) + a(z)u_{-1}(z) \quad = f_1(z), \tag{13}$$

$$\overline{\partial}u_n(z) + \partial u_{n-2}(z) + a(z)u_{n-1}(z) = 0,\ n \le -1. \tag{14}$$

Notice that u_0 and f_0 are real valued, while f_1 is complex valued.

Similarly, by identifying modes in the Fourier series expansion , the equation (5) implies

$$\overline{\partial}v_n(z) + \partial v_{n-2}(z) = 0, \quad n \le -1. \tag{15}$$

The sequence valued map $z \to \mathbf{v}(z) = (v_{-1}(z), v_{-2}(z), ...)$ which solves (15) is $\mathcal{L}^2$-analytic in the sense of Bukhgeim [6]. In here $\mathcal{L}^2$ denotes the double left translation operator:

$$\mathcal{L}^2(v_{-1}, v_{-2}, v_{-3}...) = (v_{-3}, v_{-4}, ...). \tag{16}$$

If $\mathbf{v} \in C^1(\Omega; l^2) \bigcap C(\overline{\Omega}; l^2)$ is $\mathcal{L}^2$-analytic then its values inside Ω can be computed via a Cauchy-type integral formula from their boundary values as shown in [6]. In [7], Finch proves

$$\begin{aligned} v_n(z) = & \frac{1}{2\pi i}\int_{\partial\Omega} \frac{v_n(\zeta)}{\zeta - z} d\zeta \\ & + \frac{1}{2\pi i}\sum_{j=1}^{\infty}\int_{\partial\Omega} v_{n-2j}(\zeta)\left(\frac{\overline{\zeta} - \overline{z}}{\zeta - z}\right)^j \left\{\frac{d\zeta}{\zeta - z} - \frac{d\overline{\zeta}}{\overline{\zeta} - \overline{z}}\right\}, \end{aligned} \tag{17}$$

for $\quad n \le -1, \quad x \in \Omega$, without appealing to $\mathcal{L}^2$-analyticity. The Proposition 2.1 allows us to interpret the series in (17) in the l^2 sense.

Theorem 3.1 (Reconstruction) *Let Ω be a strictly convex domain, $a, f_0 \in C_0^2(\overline{\Omega})$ and $\boldsymbol{F} \in C_0^2(\overline{\Omega}; \mathbb{R}^2)$. Then the Doppler data $m(x, \theta)$ defined in (1) uniquely determine curl$\boldsymbol{F}$ in Ω.*

Proof: The trace $v|_{\partial\Omega}$ given in (4) is determined by the data:

$$v(x, \theta) = \begin{cases} m(x, \theta)e^{b(x,\theta)}, & \theta \cdot n(x) > 0, \\ 0, & \theta \cdot n(x) \le 0, \end{cases} \tag{18}$$

where $n(x)$ denotes the outer unit normal at some point $x \in \partial\Omega$. The Proposition 2.1 gives that $\mathbf{v}|_{\partial\Omega} \in C^2(\partial\Omega, l^{2,1+})$. In the reconstruction procedure we only need $\mathbf{v}|_{\partial\Omega} \in C(\partial\Omega, l^2)$, however for the stability result the extra decay is needed. Using the Cauchy integral formula (17) we calculate $\mathbf{v}$ inside Ω and then $\mathbf{u}$ by the convolution formula (10). The series in (17) are l^2 convergent. Since $u_0, u_{-1}, u_{-2} \in C^2(\Omega)$, we can take ∂ on both sides of (13) to get

$$\frac{1}{4}\Delta u_0 + \partial^2 u_{-2} + \partial(a u_{-1}) = \partial f_1. \tag{19}$$

Moreover, since u_0 is real valued by equating the imaginary part in (19) we get

$$curl\mathbf{F} = 4\mathbb{I}m(\partial f_1) = 4\mathbb{I}m\{\partial^2 u_{-2} + \partial(a u_{-1})\}. \tag{20}$$

□

Next we prove that the reconstruction method in the proof of the Theorem (3.1) is in fact stable. The result is based on the stability of the Cauchy integral formula [19, 5]. To simplify notation, let $\langle \cdot, \cdot \rangle$ denote the l^2-inner product on the space of sequences, and $||\cdot||$ be its corresponding norm. The following result is the Green's formula for sequence valued maps.

Lemma 3.2 *Let Ω be a planar domain with C^1 boundary. For $\mathbf{v} \in C^1(\overline{\Omega}; l^2)$ we have*

$$2\int_\Omega ||\partial \mathbf{v}||^2 dx = 2\int_\Omega ||\overline{\partial}\mathbf{v}||^2 dx + i\int_{\partial\Omega} \langle \mathbf{v}, \partial_s \mathbf{v}\rangle ds, \tag{21}$$

where ∂_s is the tangential derivative at the boundary.

Proof: For two functions f, g rewrite the Green's formula in complex notations

$$\begin{aligned} 2\int_\Omega \overline{\partial} f \overline{g} dx &= \int_{\partial\Omega} \nu f \overline{g} ds - 2\int_\Omega f \overline{\partial}\overline{g} dx, \\ 2\int_\Omega \partial f \overline{g} dx &= \int_{\partial\Omega} \overline{\nu} f \overline{g} ds - 2\int_\Omega f \partial \overline{g} dx, \end{aligned} \tag{22}$$

where $\nu = n_1 + i n_2$ is the complexified outer normal $n = (n_1, n_2)$ on the boundary. In the first equation let f range over each component of $\mathbf{v}$ and g range over the corresponding component of $\overline{\partial}\mathbf{v}$. In the second equation, let f range over the components of $\mathbf{v}$ and g range over the corresponding $\partial\mathbf{v}$. Sum each of the equations over the components, then subtract them to get

$$2\int_\Omega ||\partial\mathbf{v}||^2 dx = 2\int_\Omega ||\overline{\partial}\mathbf{v}||^2 dx + \int_{\partial\Omega} \langle \mathbf{v}, (\nu\partial - \overline{\nu}\overline{\partial})\mathbf{v}\rangle ds.$$

Since $(\nu\partial - \overline{\nu}\overline{\partial}) = -i\partial_s$ the lemma is proved. □

To simplify notation, for any integer n let P_n denote the projection onto the n-th tail:

$$P_n(v_{-1}, ...v_{-n}, v_{-n-1}, ...) = (\underbrace{0, ..., 0}_{n \ times}, v_{-n-1}, ...).$$

The identities below are special cases of Bukhgeim's identity in [19, 5].

Lemma 3.3 *Let $\mathbf{v} \in C^1(\overline{\Omega}; l^2) \bigcap C^1(\partial\Omega; l^{2,1+})$ be a solution of (15). Then*

$$2\int_\Omega ||\partial\mathbf{v}||^2 dx = i\sum_{j=0}^{\infty} \int_{\partial\Omega} \langle \mathbf{v}, P_{2j}\partial_s\mathbf{v}\rangle ds, \tag{23}$$

$$2\int_\Omega ||\overline{\partial}\mathbf{v}||^2 dx = i\sum_{j=1}^{\infty} \int_{\partial\Omega} \langle \mathbf{v}, P_{2j}\partial_s\mathbf{v}\rangle ds. \tag{24}$$

Proof: We remark first that each term in the right hand side of (23) is purely imaginary since for each $n \leq -1$:

$$2\mathbb{R}e\int_{\partial\Omega} v_n \partial_s \bar{v}_n ds = \int_{\partial\Omega} v_n \partial_s \bar{v}_n ds + \int_{\partial\Omega} \bar{v}_n \partial_s v_n ds = \int_{\partial\Omega} \partial_s |v_n|^2 ds = 0.$$

From Green's identity (21) and the $\mathcal{L}^2$-analyticity equation (15) we get

$$2\int_{\Omega} ||\partial \mathbf{v}||^2 dx = 2\int_{\Omega} ||\mathcal{L}^2 \partial \mathbf{v}||^2 dx + i\int_{\partial\Omega} \langle \mathbf{v}, \partial_s \mathbf{v}\rangle ds \tag{25}$$

For $j = 1, ..., k$ successively replace in the identity above $\mathbf{v}$ by $\mathcal{L}^{2j}\mathbf{v}$ and add them together:

$$2\int_{\Omega} ||\partial \mathbf{v}||^2 dx = 2\int_{\Omega} ||\mathcal{L}^{2k} \partial \mathbf{v}||^2 dx + i\sum_{j=0}^{k}\int_{\partial\Omega} \langle \mathbf{v}, P_{2j}\partial_s \mathbf{v}\rangle ds. \tag{26}$$

Now use $\lim_{k\to\infty} \int_{\Omega} ||\mathcal{L}^{2k}\partial \mathbf{v}||^2 dx = 0$ to conclude the proof of (23). The equation (24) follows now from the Lemma 3.2. □

Corollary 3.4 *Let $\mathbf{v} \in C^1(\overline{\Omega}; l^2) \bigcap C^1(\partial\Omega; l^{2,1+})$ be a solution of (15). Then*

$$\int_{\Omega} ||\mathbf{v}||^2 dx \leq C\left(\int_{\partial\Omega} ||\mathbf{v}||^2 ds + i\sum_{j=0}^{n}\int_{\partial\Omega} \langle \mathbf{v}, P_{2j}\partial_s \mathbf{v}\rangle ds\right), \tag{27}$$

where $C > 0$ depends only on the diameter of Ω.

Proof: For each $n \leq -1$, the Poincaré inequality

$$\int_{\Omega} |v_n(x)|^2 dx \leq C\left(\int_{\Omega} |\nabla v_n(x)|^2 dx + \int_{\partial\Omega} |v_n(x)|^2 dx\right)$$

holds with a constant $C > 0$ dependent on the diameter of Ω only. Sum all the terms in $n \leq -1$, use the trivial bound $|\nabla v_n|^2 \leq 4|\partial v_n|^2 + 4|\overline{\partial} v_n|^2$, and the identities (23) and (24) to conclude the result. □

We are now able to prove that the reconstruction method is stable.

Theorem 3.5 (Stability) *Let Ω be a strictly convex domain, $a, f_0 \in C_0^1(\overline{\Omega})$ and $\boldsymbol{F} \in C_0^1(\overline{\Omega}; \mathbb{R}^2)$. Then*

$$\int_{\Omega} |curl \boldsymbol{F}|^2 dx \leq C\left(\int_{\partial\Omega} ||\mathbf{v}||^2 ds + i\sum_{j=0}^{\infty}\int_{\partial\Omega} \langle \mathbf{v}, P_{2j}\partial_s \mathbf{v}\rangle ds \right. \tag{28}$$
$$\left. + i\sum_{j=0}^{\infty}\int_{\partial\Omega} \langle \partial\mathbf{v}, P_{2j}\partial_s \partial\mathbf{v}\rangle ds\right),$$

where the constant C depends on the diameter of the domain and the $C^2(\overline{\Omega})$-sup-norm of a.

Proof: Following the equation (20) we estimate: $|curl\mathbf{F}| \leq 2(|(\partial a)u_{-1}| + |a\partial u_{-1}| + |\partial^2 u_{-2}|)$ or

$$|curl\mathbf{F}|^2 \leq 12\left(|(\partial a)u_{-1}|^2 + |a\partial u_{-1}|^2 + |\partial^2 u_{-2}|^2\right). \tag{29}$$

Integrate in Ω to get

$$\frac{1}{12}\int_\Omega |curl\mathbf{F}|^2 dx \leq \sup_{\overline{\Omega}} |\nabla a| \int_\Omega |u_{-1}|^2 dx \tag{30}$$

$$+ \sup_{\overline{\Omega}} |a| \int_\Omega |\partial u_{-1}|^2 dx + \int_\Omega |\partial^2 u_{-2}|^2 dx. \tag{31}$$

We estimate each of the integrals in the right hand side above in terms of $\mathbf{v}$. From (10) we have for each $z \in \overline{\Omega}$:

$$|u_{-1}(z)|^2 = \left|\sum_{k=0}^{\infty} \alpha_k(z) v_{-k-1}(z)\right|^2 \leq \left(\sum_{k=0}^{\infty} |\alpha_k(z)|^2\right) ||\mathbf{v}(z)||^2.$$

Upon integration in Ω and making use of (7) we conclude the first estimate:

$$\int_\Omega |u_{-1}|^2 dx \leq \sup_{z\in\overline{\Omega}} \|e^{-b+Da}(z,\cdot)\|^2_{L^2(S^1)} \int_\Omega ||\mathbf{v}||^2 dx. \tag{32}$$

Similarly,

$$\int_\Omega |\partial u_{-1}|^2 dx \leq \sup_{z\in\overline{\Omega}} \|\partial\left(e^{-b+Da}\right)(z,\cdot)\|^2_{L^2(S^1)} \int_\Omega ||\mathbf{v}||^2 dx \tag{33}$$

$$+ \sup_{z\in\overline{\Omega}} \|e^{-b+Da}(z,\cdot)\|^2_{L^2(S^1)} \int_\Omega ||\partial\mathbf{v}||^2 dx.$$

Next we estimate the last term of (30). By taking ∂^2 in (10) we get

$$\partial^2 u_{-2} = \sum_{k=0}^{\infty}(\partial^2\alpha_k)v_{-k-2} + 2\sum_{k=0}^{\infty}(\partial\alpha_k)v_{-k-2} + \sum_{k=0}^{\infty}\alpha_k(\partial^2 v_{-k-2}).$$

Similarly, by Cauchy inequality followed by integration, we obtain

$$\int_\Omega |\partial^2 u_{-2}|^2 \leq 3\sup_{z\in\overline{\Omega}} \|\partial^2\left(e^{-b+Da}\right)(z,\cdot)\|^2_{L^2(S^1)} \int_\Omega ||\mathbf{v}||^2 dx \tag{34}$$

$$+ 6\sup_{z\in\overline{\Omega}} \|\partial\left(e^{-b+Da}\right)(z,\cdot)\|^2_{L^2(S^1)} \int_\Omega ||\partial\mathbf{v}||^2 dx$$

$$+ 3\sup_{z\in\overline{\Omega}} \|e^{-b+Da}(z,\cdot)\|^2_{L^2(S^1)} \int_\Omega ||\partial^2\mathbf{v}||^2 dx.$$

Due to the commutativity of differentiation and since $\mathbf{v} \in C^2(\Omega, l^2) \bigcap C^1(\overline{\Omega}, l^2)$ we have that $\partial\mathbf{v}$ is also $\mathcal{L}^2$-analytic and the Lemma 3.3 applies:

$$2\int_\Omega ||\partial^2\mathbf{v}||^2 dx = i\sum_{j=0}^{\infty}\int_{\partial\Omega} \langle\partial\mathbf{v}, P_{2j}\partial_s\partial\mathbf{v}\rangle ds. \tag{35}$$

Using the bounds of $\int_\Omega \|\mathbf{v}\| dx$, $\int_\Omega \|\partial\mathbf{v}\| dx$ and $\int_\Omega \|\partial^2\mathbf{v}\| dx$ in terms of their boundary values in (27), (23) and (35) we obtain the stability estimate 28. The constant C depends on the sup-norm over $\overline{\Omega}$ of e^{b-Da} and its derivatives up to the second order, which in turn are bounded by derivatives of a of the same order. $\square$

The solenoidal part of $\mathbf{F}$ in (2) can be stably recovered from the $curl\mathbf{F}$ as follows. By taking the *curl* in (2) and using the fact that $\mathbf{F}^s$ is divergence free, we obtain the system $\partial_{x_2}F_1^s - \partial_{x_1}F_2^s = curl\mathbf{F}$ and $\partial_{x_1}F_1^s + \partial_{x_2}F_2^s = 0$. One more differentiation implies that each component of $\mathbf{F}^s$ solves in Ω the Dirichlet problem:

$$\left\{\begin{array}{ll} \Delta F_1^s(x) & = -\partial_{x_2}(curl\mathbf{F}(x)), \\ F_1^s(x) & = F_1(x), \quad x \in \partial\Omega \end{array}\right. \text{ and } \left\{\begin{array}{ll} \Delta F_1^s(x) & = \partial_{x_2}(curl\mathbf{F}(x)), \\ F_1^s(x) & = F_1(x), \quad x \in \partial\Omega. \end{array}\right.$$

If the trace $\mathbf{F}|_{\partial\Omega}$ is also known, by solving the Dirichlet problems above, one can determine $\mathbf{F}^s$ in Ω.

4. Concluding Remarks

We considered the integral geometry problem of reconstructing a planar vector field when its doppler data is superposed with the Radon data of an unknown scalar field. Using Bukhgeim's approach to the tomography problem we showed that, in both the attenuated and non attenuated case, the solenoidal part of a vector field could be stably recovered from these altered data and from its traces on the boundary.

In the region $\{z \in \Omega : \ a(z) > 0\} \bigcap \{z \in \Omega : \ f_0(z) = 0\}$ the equation (12) gives $u_0 = -2\mathbb{R}e(\partial u_{-1})/a$ and, following (13), the entire vector field can be recovered via the formula

$$F_1 + iF_2 = 2\partial u_{-2} + 2au_{-1} - 4\overline{\partial}\left(\frac{\mathbb{R}e(\partial u_{-1})}{a}\right). \tag{36}$$

This is a bit more general than the reconstruction result in [9] since the data we consider are influenced by regions where f_0 is non-vanishing.

The stability result shows that, in principle, the reconstruction method can be numerically implemented. Caution is however in order. Although the formula (20) contains only u_{-1} and u_{-2}, they depend on all the coefficients $v_{-1}, v_{-2}, ...$ Preliminary numerical experiments show that merely truncating the Fourier series in (8) introduces numerical instability, even for noiseless data. The same problem occurs in the classical attenuated Doppler transform. In [9] the problem was successfully overcome by developing a new basis for symmetric 2-tensor fields based on bivariate Chebyshev's ridge polynomials. Similar ideas may work for the numerical implementation of this work.

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