DISTANCE PROBLEMS IN $\mathbb{R}^n$

Maximo Fernandez[1], Jacob Shaw[2], Ngoc Tram Nguyen[3]

[1]University of Central Florida, [2]Clemson University, [3]University of California, Los Angeles

Abstract

The ancient Archimedean Principle states that the shortest distance between two point is a straight line. In this research we extend this principle by showing that straight lines are the only minimizers for energy functionals with strictly convex Lagrangians. Conversely, we show that if straight lines are always the shortest path between points, then the Lagrangian must be convex. We also investigate the shortest distance problem with road obstructions.

Background

Given points $p, q \in \mathbb{R}^n$, let

$$\mathcal{P} = \{\, \gamma : I \to \mathbb{R}^n \mid \gamma(0) = p, \gamma(1) = q, \gamma \in W^{1,2} \,\}$$

be the set of continuous paths between them. Results by Dr. Katiuscia Teixeira[1] show the arc-length functional

$$\ell(\gamma) = \int_0^1 \sqrt{1 + \dot\gamma(t)^2}\, dt$$

for planar $\gamma \in \mathcal{P}$ is minimized by straight lines. We generalize ℓ to **energy functionals** E of the form

$$E(\gamma) = \int_0^1 F(\dot\gamma(t))\, dt$$

where the **Lagrangian** of E is $F : \mathbb{R}^n \to \mathbb{R}$. We extend these ideas to paths constrained by an **obstruction** B. Below are examples of these notions.

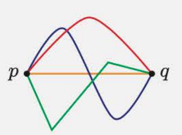

Figure 1: Paths in $\mathcal{P}$

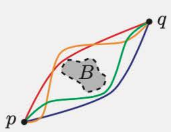

Figure 2: Paths in $\mathcal{P}_\beta$

For the road obstruction problem, we focus our results on graph functions in $\mathbb{R}^2$ with constraint $\gamma \geq \beta$, where β is a C^1 graph of the boundary ∂B. We seek to minimize E in the following family

$$\mathcal{P}_\beta = \{\, \gamma \in \mathcal{P} \mid \gamma \geq \beta \,\}.$$

Due to the generality of our results, the minimization problem is modeled in the **Sobolev spaces** $W^{1,2}$ and $W_0^{1,2}$ of weakly differentiable functions. We will also refer to $T_r(x)$ as the **linearization** of F at r.

Shortest Distance Problem

Theorem 1. *If E has a (strictly) convex Lagrangian F, then straight lines are (unique) minimizers to E in $\mathcal{P}$.*

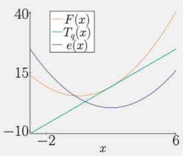

Figure 3: $F(x) = x^2 + 5$

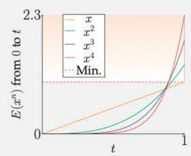

Figure 4: Sample energies

Sketch of proof. $E(\gamma)$ is uniquely determined by $\int_I e(\dot\gamma)$, where $e(x) = F(x) - T_r(x)$, $r = q - p$. Since e is convex and the minimum of $e(x)$ occurs at $x = r$, $\dot\gamma = r$ so γ is a straight line. □

Theorem 2 (Converse). *If E is minimized uniquely by straight lines, then F is convex.*

Sketch of proof. Let $v = q - p$. Let $\gamma(t) = vt$ be the minimizer, fix $\varphi \in W_0^{1,2}$; then $i(s) = E(\gamma + s\varphi)$ has a minimum at $s = 0$. Fix $w \neq 0 \in \mathbb{R}^n$, define φ as follows:

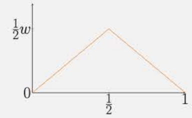

Figure 5: φ

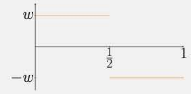

Figure 6: $\dot\varphi$

Since $i(0)$ is a minimum, $i(s) + i(-s) \geq 2i(0)$. Thus $F(v + sw) + F(v - sw) \geq 2F(v)$. □

Shortest Distance Problem with an Obstacle

Theorem 3. *There exists a minimizer γ to E in $\mathcal{P}_\beta$.*

Lemma 4. *If γ minimizes E, then $\ddot\gamma \leq 0$, and $\ddot\gamma = 0$ when $\gamma > \beta$. Furthermore, γ must touch β.*

Sketch of proof. If $\ddot\varphi > 0$ while $\varphi \geq \beta$, then by Theorem 1 there exists a straight γ line joining the endpoints of φ which is energy minimizing. If $\gamma = \beta$, then $\ddot\gamma < 0$ by the definition of β.

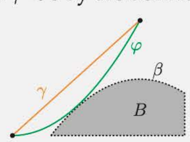

Figure 7: $\ddot\gamma = 0$

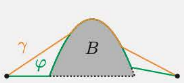

Figure 8: $\ddot\gamma < 0$

By Theorem 3, the infimum must exist and be attained in $\mathcal{P}_\beta$. If γ does not touch β, then we can create a path $\tilde\gamma$ with $E(\tilde\gamma) < E(\gamma)$, so the minimum must intersect β. □

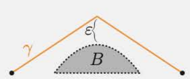

Figure 9: γ gap

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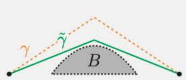

Figure 10: Shorter $\tilde\gamma$

Lemma 5. *If γ touches β, it does so at a tangent. That is, $\dot\beta = \dot\gamma$ at their intersection.*

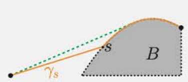

Figure 11: γ_s

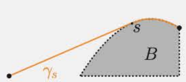

Figure 12: Minimizing $E(\gamma_s)$

Sketch of proof. Let $i(s) = E(\gamma_s)$. By letting $i'(s) = 0$, we find that $\sqrt{1 + \dot\gamma^2} = \sqrt{1 + \dot\beta^2}$, so $\dot\beta = \pm\dot\gamma$. We can confirm $\dot\beta = \dot\gamma$ is a minimum. □

Lemma 6. *If p, q lie on a convex portion of the graph of β, then the shortest path $\gamma \geq \beta$ connecting p, q is on β.*

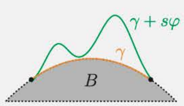

Figure 13: $\gamma = \beta$

Sketch of proof. Let $\varphi \in W_0^{1,2}$ be nonnegative, and let $s \geq 0$. We can vary $i(s) = E(\gamma + s\varphi)$ to find that $i'(s) > 0$ for all s. Thus, $\gamma = \beta$ must be a global minimum. □

Theorem 7 (Regularity). *If γ minimizes the distance between two points separated by an obstacle B with $\beta \in C^1$, then $\gamma \in C^1$*

Sketch of proof. By Theorem 3 and Lemmas 4, 5, and 6, γ must approach β along a tangent line passing through p and continue on ∂B until leaving along another tangent line to approach q. Hence, $\gamma \in C^1$. □

Corollary 8. *These results generalize to non-graph functions in $\mathbb{R}^2$. That is, if $\beta \in C^1$, then the minimizer to E in $\mathcal{P}_\beta$ is C^1.*

Results

As stated by Corollary 8, our results generalize to various obstacles in $\mathbb{R}^2$. Given the nature of the minimizer found through Theorem 7, we may also conclude that minimizers do not exist when our codomain $\mathbb{R}^2 - B$ is open. (See Lemma 4)

Our results allow the characterization of minimizers to E in $\mathcal{P}_\beta$ by studying B. Understanding the existence and regularity of solutions in this manner allows broad applications to physical systems.

Future Research

- Investigate maximizers to such energy functionals and their applications.
- Extend results of the shortest distance problem with an obstacle to higher dimensions ($\mathbb{R}^n$).
- Characterize energy extremizers for surfaces and volumes between paths; Generalize the shortest distance problem with obstacles to n dimensions.

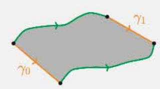

Figure 3: Joining paths

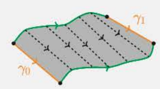

Figure 4: Homotopy trace

References

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[3] Katiuscia Teixeira. On the longest distance problem. *The Mathematical Intelligencer*, 12 2022.

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