A symmetric symbol for triples of modular forms

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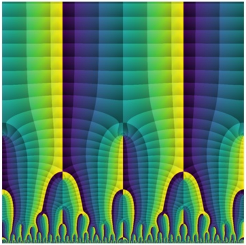

Wissam Ghantous
wissam.ghantous@ucf.edu
University of Central Florida

Outline

- Modular Forms
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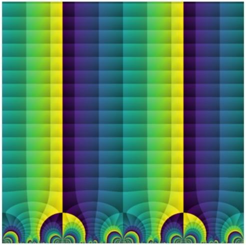
- A new p-adic symbol: $(f, g, h)_p$
- Symmetry and antisymmetry relations for $(f, \cdot, \cdot)_p$
- Main result: Symmetry relations for $(\cdot, \cdot, \cdot)_p$
- Computational aspects
- Surprising application: Computing Poincaré pairings
- A concrete example

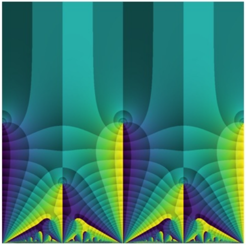
Modular forms

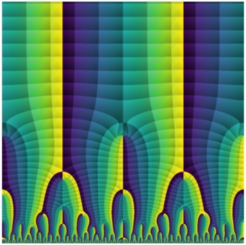
Modular forms play a crucial role in Number theory.
A modular form f is a holomorphic function $f : \mathfrak{H} \to \mathbb{C}$ such that

$$f\left(\frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^k f(\tau), \qquad \forall \tau \in \mathfrak{H}, \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \Gamma \subseteq \mathsf{SL}_2(\mathbb{Z}).$$

Moreover, we require that $(c\tau+d)^{-k} f(\frac{a\tau+b}{c\tau+d})$ has a Fourier expansion. The coefficients appearing in the q-expansions have algebraic/geometric meanings.







A curious phenomenon (setup)

Let f, g, h be three cuspidal eigenforms in $M_2(N, \mathbb{Q})$. Fix a prime $p \geq 5$, $p \nmid N$. Let α_f, β_f be the roots of the *Hecke polynomial*

$$x^2 - a_p(f)x + p. \tag{1}$$

Assume as well that: $\mathrm{val}_p(\alpha_f) = 0$, $\alpha_f \neq \beta_f$.
$\rightsquigarrow$ f is *ordinary* and *regular* at p.
Define the *p-stabilizations* of f:

$$f_\alpha(q) := f(q) - \beta_f f(q^p); \qquad f_\beta(q) := f(q) - \alpha_f f(q^p).$$

They have level pN, and are eigenforms for the U_p operator with respective eigenvalues α_f and β_f.
Define the following Euler factors:

$$\begin{gathered}\mathcal{E}(f,g,h) := (1-\beta_f\alpha_g\alpha_h p^{-2})(1-\beta_f\alpha_g\beta_h p^{-2})(1-\beta_f\beta_g\alpha_h p^{-2})(1-\beta_f\beta_g\beta_h p^{-2});\\ \tilde{\mathcal{E}}(f,g,h) := (1-\alpha_f\alpha_g\alpha_h p^{-2})(1-\alpha_f\alpha_g\beta_h p^{-2})(1-\alpha_f\beta_g\alpha_h p^{-2})(1-\alpha_f\beta_g\beta_h p^{-2});\\ \mathcal{E}_0(f) := 1-\beta_f^2(p)p^{-1}; \qquad \tilde{\mathcal{E}}_0(f) := 1-\alpha_f^2(p)p^{-1};\\ \mathcal{E}_1(f) := 1-\beta_f^2(p)p^{-2}; \qquad \tilde{\mathcal{E}}_1(f) := 1-\alpha_f^2(p)p^{-2}.\end{gathered} \tag{2}$$

A curious phenomenon (setup)

Let

- ϕ be the Frobenius map,
- $\mathsf{d} := q\frac{\mathsf{d}}{\mathsf{d}q}$ be the Serre differential operator,
- $\langle \cdot, \cdot \rangle$ be the Poincaré pairing,
- $\mathsf{Proj}_{f_\gamma} : S_2^{\text{oc}}(N, \mathbb{Q}_p) \longrightarrow S_2^{\text{oc}}(N, \mathbb{Q}_p)_{f_\gamma}$ be the projection over f_γ. It is the unique linear functional that factors through the Hecke eigenspace associated to f_γ and is normalized to send f_γ to 1.

Consider the quantity:

$$\frac{\langle \omega_f, \phi(\omega_f) \rangle}{p} \left(\frac{\mathcal{E}_1(f)}{\mathcal{E}(f,g,h)} \beta_f \mathsf{Proj}_{f_\alpha} \left(\mathsf{d}^{-1}(g^{[p]}) \times h \right) + \frac{\tilde{\mathcal{E}}_1(f)}{\tilde{\mathcal{E}}(f,g,h)} \alpha_f \mathsf{Proj}_{f_\beta} \left(\mathsf{d}^{-1}(g^{[p]}) \times h \right) \right). \tag{3}$$

A curious phenomenon

Consider the quantity:

$$\frac{\langle \omega_f, \phi(\omega_f)\rangle}{p} \left(\frac{\mathcal{E}_1(f)}{\mathcal{E}(f,g,h)} \beta_f \mathsf{Proj}_{f_\alpha} \left(\mathsf{d}^{-1}(g^{[p]}) \times h \right) + \frac{\tilde{\mathcal{E}}_1(f)}{\tilde{\mathcal{E}}(f,g,h)} \alpha_f \mathsf{Proj}_{f_\beta} \left(\mathsf{d}^{-1}(g^{[p]}) \times h \right) \right). \tag{4}$$

It turns out that this quantity is independent – up to a sign – of the order of f, g and h.
This result is particularly surprising since it does not appear to be symbolically symmetric in f, g and h.

The above phenomenon can be generalized to modular forms of higher (balanced[1]) weights and any characters satisfying $\chi_f \chi_g \chi_h = 1$, modulo slight modifications.

[1] balanced weights: the largest one is strictly smaller than the sum of the other two, i.e. lengths of sides of a $\triangle$

Setup (most general form)

Fix three eigenforms

$$f \in S_k(N, \mathbb{Q}, \chi_f), \quad g \in S_\ell(N, \mathbb{Q}, \chi_g), \quad h \in S_m(N, \mathbb{Q}, \chi_h),$$

such that $\chi_f \chi_g \chi_h = 1$.
Fix a prime $p \geq 5$, $p \nmid N$.

Hecke polynomial: $x^2 - a_p(f)x + \chi_f(p)p^{k-1} = (x - \alpha_f)(x - \beta_f)$.
Assume that: $\mathrm{val}_p(\alpha_f) = 0$, $\alpha_f \neq \beta_f$.
The p-*stabilizations* of f:

$$f_\alpha(q) := f(q) - \beta_f f(q^p); \qquad f_\beta(q) := f(q) - \alpha_f f(q^p).$$

$\rightsquigarrow$ level pN, and are eigenforms for U_p with eigenvalues α_f and β_f.

Let $t := \frac{\ell+m-k-2}{2} \geq 0, c := \frac{k+\ell+m-2}{2}$. Define the Euler factors:

$$\mathcal{E}(f,g,h) := (1 - \beta_f \alpha_g \alpha_h p^{-c})(1 - \beta_f \alpha_g \beta_h p^{-c})(1 - \beta_f \beta_g \alpha_h p^{-c})(1 - \beta_f \beta_g \beta_h p^{-c});$$
$$\tilde{\mathcal{E}}(f,g,h) := (1 - \alpha_f \alpha_g \alpha_h p^{-c})(1 - \alpha_f \alpha_g \beta_h p^{-c})(1 - \alpha_f \beta_g \alpha_h p^{-c})(1 - \alpha_f \beta_g \beta_h p^{-c});$$

$$\begin{aligned} \mathcal{E}_0(f) &:= 1 - \beta_f^2 \chi_f^{-1}(p) p^{1-k}; & \tilde{\mathcal{E}}_0(f) &:= 1 - \alpha_f^2 \chi_f^{-1}(p) p^{1-k}; \\ \mathcal{E}_1(f) &:= 1 - \beta_f^2 \chi_f^{-1}(p) p^{-k}; & \tilde{\mathcal{E}}_1(f) &:= 1 - \alpha_f^2 \chi_f^{-1}(p) p^{-k}. \end{aligned} \tag{5}$$

A new p-adic symbol: $(f, g, h)_p$

Starting point: The Garrett-Rankin triple product p-adic L-function [DR14]. Let $\mathbf{f}, \mathbf{g}, \mathbf{h}$ be Hida families, interpolating f, g and h at the weights k, ℓ and m. The Garrett-Rankin triple product p-adic L-function is defined as:

$$\mathscr{L}_p(\mathbf{f}, \mathbf{g}, \mathbf{h})(x, y, z) := \mathsf{Proj}_{f^*_{x,\alpha}}(d^{-1-t} g_y^{[p]} \times h_z). \quad (6)$$

Definition

Let f, g and h be three cuspidal modular forms of level N and respective weights k, ℓ and m which are ordinary at p. We define the p-adic triple symbol $(f, g, h)_p$ by

$$(-1)^t t! \frac{\langle \omega_f, \phi(\omega_f) \rangle}{p^{k-1}} \left(\frac{\mathcal{E}_1(f)}{\mathcal{E}(f, g, h)} \; \beta_{f^*} \; \mathsf{Proj}_{f^*_\alpha} \left(\mathsf{d}^{-1-t}(g^{[p]}) \times h \right) + \frac{\tilde{\mathcal{E}}_1(f)}{\tilde{\mathcal{E}}(f, g, h)} \; \alpha_{f^*} \mathsf{Proj}_{f^*_\beta} \left(\mathsf{d}^{-1-t}(g^{[p]}) \times h \right) \right).$$

Symmetry and antisymmetry relations for $(f, \cdot, \cdot)_p$

Theorem (Ghantous)

Let f, g, h be three cuspidal new forms of weights k, ℓ, m. Let $t := \frac{\ell+m-k-2}{2}$. We have the following symmetry relation:

$$(f, g, h)_p = (-1)^{t+1}(f, h, g)_p.$$

In other words, the parity of t determines the symmetry or anti-symmetry of $(f, \cdot, \cdot)_p$.
**This result also holds for the $\mathscr{L}_p(\mathbf{f}, \mathbf{g}, \mathbf{h})(x, y, z)$.*

Sketch of Proof

I essentially prove the theorem by showing that for $\gamma \in \{\alpha, \beta\}$, we have $\mathrm{Proj}_{f_\gamma}\left(\mathrm{d}^{-1-t}(g^{[p]}) \times h\right) = (-1)^{t+1}\mathrm{Proj}_{f_\gamma}\left(\mathrm{d}^{-1-t}(h^{[p]}) \times g\right)$, through an explicit calculation.

Symmetry and antisymmetry relations for $\mathscr{L}_p^{\mathcal{S}}(f,\cdot,\cdot)$

Proof.

Let $Y := (\mathsf{d}^{-1-t}g^{[p]})h^{[p]} - (-1)^{t+1}(\mathsf{d}^{-1-t}h^{[p]})g^{[p]}$.
Goal: $\mathsf{Proj}_{f_\alpha}(Y) = \mathsf{Proj}_{f_\beta}(Y) = 0$.

Let $X := \sum_{i=0}^{t}(-1)^i\mathsf{d}^{-1-t+i}g^{[p]}\mathsf{d}^{-1-i}h^{[p]}$. Then, $Y = \mathsf{d}X$ is exact.

• So Y is in the kernel of e_{ord} (which is a multiple of Proj_{f_α}):
$e_{\mathsf{ord}}\left(\mathsf{d}^{-1-t}\left(g^{[p]}\right)\times h^{[p]}\right) = (-1)^{t+1}e_{\mathsf{ord}}\left(h^{[p]}\times \mathsf{d}^{-1-t}\left(g^{[p]}\right)\right)$.
Finally, we can drop the $^{[p]}$ from first $h^{[p]}$ and second $g^{[p]}$ (they differ by an element in $\mathsf{Ker}(U_p)$.
Thus, $e_{\mathsf{ord}}\left(\mathsf{d}^{-1-t}\left(g^{[p]}\right)\times h\right) = (-1)^{t+1}e_{\mathsf{ord}}\left(h\times \mathsf{d}^{-1-t}\left(g^{[p]}\right)\right)$.

• Y is also trivial is cohomology.
So $\langle\omega_f, Y\rangle = 0$, and $\left\langle\omega_f, \mathsf{Proj}_{f^*}(Y)\right\rangle = 0$. So,
$\left\langle\omega_f, \mathsf{Proj}_{f^*}(\mathsf{d}^{-1-t}\left(g^{[p]}\right)\times h^{[p]})\right\rangle = (-1)^{t+1}\left\langle\omega_f, \mathsf{Proj}_{f^*}(h^{[p]}\times\mathsf{d}^{-1-t}\left(g^{[p]}\right))\right\rangle$.
Again, we can drop the $^{[p]}$ from first $h^{[p]}$ and second $g^{[p]}$. □

Main result

Theorem (Ghantous)

Let f, g, h be three cuspidal newforms of weights k, ℓ, m. Then,

$$(f, g, h)_p = (-1)^k (g, h, f)_p = (-1)^m (h, f, g)_p.$$

*Key step for the proof: expressing $(f, g, h)_p$ using the Abel-Jacobi map.

In [DR14], the authors construct a generalized Gross-Kudla-Schoen diagonal cycle $\Delta := \Delta_{k,\ell,m}$ and show that

$$\mathscr{L}_p(\mathbf{f}, \mathbf{g}, \mathbf{h})(x, y, z) = \frac{(-1)^{t+1}}{t!} \frac{\mathcal{E}(f_x, g_y, h_z)}{\mathcal{E}_0(f_x)\mathcal{E}_1(f_x)} \mathsf{AJ}_p(\Delta)(\eta^{\mathsf{u\text{-}r}}_{f_x} \otimes \omega_{g_y} \otimes \omega_{h_z}).$$

We consider the change $(\eta_f^{\mathsf{u\text{-}r}} \otimes \omega_g \otimes \omega_h) \rightsquigarrow (\omega_f \otimes \omega_g \otimes \omega_h)$, and prove that

$$(f, g, h)_p = AJ_p(\Delta_{k,\ell,m})(\omega_f \otimes \omega_g \otimes \omega_h).$$

Main result

Proof.

Let $s : (W := \mathcal{E}^{k-2} \times \mathcal{E}^{\ell-2} \times \mathcal{E}^{m-2}) \longrightarrow (W' := \mathcal{E}^{\ell-2} \times \mathcal{E}^{k-2} \times \mathcal{E}^{m-2})$ be the map that permutes the first and second terms. The functoriality properties of the p-adic Abel Jacobi map give us ($r := (k+\ell+m-6)/2$)

$$\begin{array}{ccc} \mathsf{CH}^{r+2}(W)_0 & \xrightarrow{\mathsf{AJ}_p} & \mathsf{Fil}^{r+2}\mathsf{H}^{2r+3}_{\mathsf{dR}}(W)^\vee \\ \downarrow s_* & & \downarrow s^{*,\vee} \\ \mathsf{CH}^{r+2}(W')_0 & \xrightarrow{\mathsf{AJ}_p} & \mathsf{Fil}^{r+2}\mathsf{H}^{2r+3}_{\mathsf{dR}}(W')^\vee . \end{array}$$

i.e., $\mathsf{AJ}_p s_* = s^{*,\vee}\mathsf{AJ}_p$. So given $Z \in \mathsf{CH}^{r+2}(W)_0$, $\omega \in \mathsf{Fil}^{r+2}\mathsf{H}^{2r+3}_{\mathsf{dR}}(W')$:

$$\mathsf{AJ}_p(s_*Z)(\omega) = (s^{*,\vee}\mathsf{AJ}_p(Z))(\omega) = \mathsf{AJ}_p(Z)(s^*\omega).$$

But $s_*\Delta_{k,\ell,m} = (-1)^{\sharp\sharp\sharp}\Delta_{\ell,k,m}$ and $s^*(\omega_{g,f,h}) = (-1)^{\sharp\sharp\sharp}\omega_{f,g,h}$. Hence,

$$(f,g,h)_p = (-1)^{(k+\ell-m)/2}(g,f,h)_p.$$

□

Computational aspects

Algorithms used:

1. Ordinary projections of overconvergent modular forms;
2. Ordinary projections of nearly overconvergent modular forms;
3. Projections of overconvergent modular forms over the slope σ space.

Methods they rely on:

1. Approximate OCMFs using a finite Katz basis, write U_p as matrix A in that basis, then $e_{\mathsf{ord}} \approx A^{?}$.
2. $e_{\mathrm{ord}} = e_{\mathrm{ord}} \circ \pi_{\mathsf{oc}}$,
   $\pi_{\mathsf{oc}}((\mathsf{d}^s \phi_1) \times \phi_2) = \binom{\kappa_1+\kappa_2+2s-2}{s}^{-1} [\phi_1, \phi_2]_s$, $\forall s \geq 0$, (see [LSZ20]).
3. Put A in SNF, and use an algebra trick suggested by D. Loeffler.

Poincaré pairings (weight 2)

Goal: compute $\Omega_f := \langle \omega_f, \phi(\omega_f) \rangle$**.**

Let $E := E_f =$ elliptic curve associated to f.
The differential $\omega_f = \sum_n a_n(f) q^n \frac{\mathrm{d}q}{q}$ corresponds to $\omega_E := \frac{\mathrm{d}x}{y}$.

Let M represent the action of Frobenius, up to precision p^m, on $\omega_E = \frac{\mathrm{d}x}{y}$ and $\eta_E := x\frac{\mathrm{d}x}{y}$.
Then, $\langle \omega_E, M\omega_E \rangle = \langle \omega_E, m_{11}\ \omega_E + M_{21}\ \eta_E \rangle = M_{21}$.
Hence,

$$\Omega_f = M_{21} \mod p^m.$$

*The matrix M can be efficiently computed via Kedlaya's algorithm.
*Slight simplification: actually, one needs to include the modular degree m_E of E as a correction factor. So,

$$\boxed{\Omega_f = m_E\ M_{21} \mod p^m.} \tag{7}$$

Poincaré pairings (general weight)

Goal: compute $\Omega_f := \langle \omega_f, \phi(\omega_f) \rangle \mod p^m$, for a modular form f of weight $k > 2$, level N, and a precision level $m \in \mathbb{N}$.
Trick: exploit the symmetry of $(f, g, h)_p$.

Algorithm 1
Pick φ of weight 2, and g of weight k, so $(2, k, k)$ is balanced.
Compute $(\varphi, f, g)_p$ and $(f, g, \varphi)_p / \Omega_f$.
*Computing $(\varphi, f, g)_p$ involves computing Ω_φ, which is doable as φ has weight 2.
*Computing $(f, g, \varphi)_p / \Omega_f$ doesn't involve any Poincaré pairings.
We obtain:

$$\Omega_f = \frac{(\varphi, f, g)_p}{(f, g, \varphi)_p / \Omega_f}. \tag{8}$$

Remark

This won't necessarily work on the first try, but we can vary the choice of φ and g.

Poincaré pairings (general weight)

Goal: compute $\Omega_f := \langle \omega_f, \phi(\omega_f) \rangle \mod p^m$, for a modular form f of weight $k > 2$, level N, and a precision level $m \in \mathbb{N}$.
Trick: exploit the symmetry of $(f, g, h)_p$.

Algorithm 2
Pick a new modular form f_0 of weight $k_0 > 2$ and level N. Apply Algorithm 1 to obtain Ω_{f_0}.
Pick a modular form g of any weight ℓ such that the triple (f, f_0, g) is balanced.
Compute $(f_0, f, g)_p$ and $(f, g, f_0)_p / \Omega_f$. We obtain:

$$\Omega_f = \frac{(f_0, f, g)_p}{(f, g, f_0)_p / \Omega_f}. \tag{9}$$

Remark

We can vary the choice of f_0 and g.

A concrete example

Take $N := 45$ and let $f, g, h, h_2, h_3 \in S_4(\mathbb{Q}, 45)$ be the cuspidal newforms:

$$\begin{aligned}
f &= q - q^2 - 7q^4 - 5q^5 - 24q^7 + 15q^8 + 5q^{10} - 52q^{11} ..., \\
g &= q - 3q^2 + q^4 + 5q^5 + 20q^7 + 21q^8 - 15q^{10} + 24q^{11} ..., \\
h &= q + 4q^2 + 8q^4 + 5q^5 + 6q^7 + 20q^{10} - 32q^{11} + ..., \\
h_2 &= q - 5q^2 + 17q^4 + 5q^5 - 30q^7 - 45q^8 - 25q^{10} - 50q^{11} + ..., \\
h_3 &= q + 5q^2 + 17q^4 - 5q^5 - 30q^7 + 45q^8 - 25q^{10} + 50q^{11} + ...
\end{aligned}$$

Let $f_0 \in S_2(\mathbb{Q}, 45)$ be the newform given by

$$f_0 = q + q^2 - q^4 - q^5 - 3q^8 - q^{10} + 4q^{11} + ...$$

A concrete example

We compute

$$(f_0, f, h_2)_p/\Omega_{f_0} = 16513223984800935050336063815246 \cdot 17^3 \bmod 17^{30},$$
$$(f, h_2, h_0)_p/\Omega_f = 13539421372161396100812664727177 \cdot 17 \bmod 17^{30},$$
$$(f_0, h_3, g)_p/\Omega_{f_0} = -33668845951010127545613025517222 \cdot 17^2 \bmod 17^{30}$$
$$(h_3, g, f_0)_p/\Omega_{h_3} = 933939362915231153601891365554 \bmod 17^{30}.$$

Using Kedlaya's algorithm, we compute

$$\Omega_{f_0} = \langle \omega_{f_0}, \phi(\omega_{f_0}) \rangle = 7374052221695942635874395263608211 \cdot 17 \bmod 17^{30}.$$

Thus, we deduce that we must have

$$\Omega_f = \Omega_{f_0} \frac{(f_0, f, h_2)_p/\Omega_{f_0}}{(f, h_2, f_0)_p/\Omega_f} = \text{-}886254611396421462835219595910 0 \cdot 17^3 \bmod 17^{27},$$
$$\Omega_{h_3} = \Omega_{f_0} \frac{(f_0, h_3, g)_p/\Omega_{f_0}}{(h_3, g, f_0)_p/\Omega_{h_2}} = \text{-}172883095677247429473582011622 6 \cdot 17^3 \bmod 17^{26}.$$

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