Forward Scattering for Thin Coated Domains: Numerical Solution Using Generalized Impedance Boundary Conditions

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Introduction

This project seeks to model the wave scattering off an object with a thin coating. Since directly calculating the wave field in the thin coating (the transmission problem) is complex and computationally expensive, we instead used a generalized impedance boundary condition (GIBC) on the domain boundary. We present numerical experiments that demonstrate the effectiveness of our GIBC conditions in simplifying the transmission problem.

Problem to be Solved

Visual of Wave Scattering:

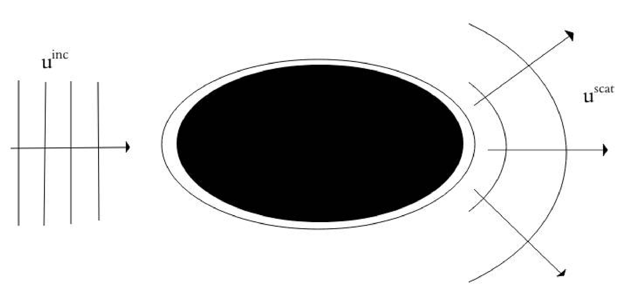

Figure 1. Plane Wave Diagram

Forward Problem:
$\Delta u + k^2 u = 0$
$\mathcal{B}u = 0$
Sommerfeld Radiation Condition for u^s:
$u = u^s + u^i$

Practical Applications:

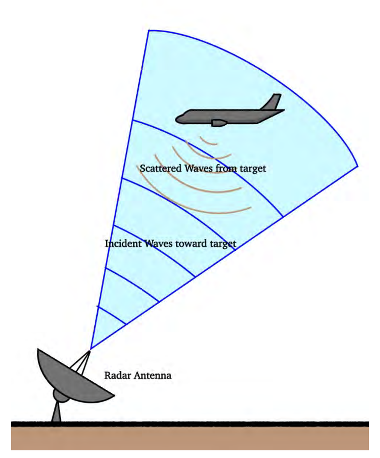

Figure 2. Radar Detection

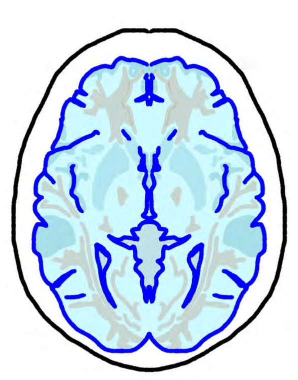

Figure 3. Medical Imaging

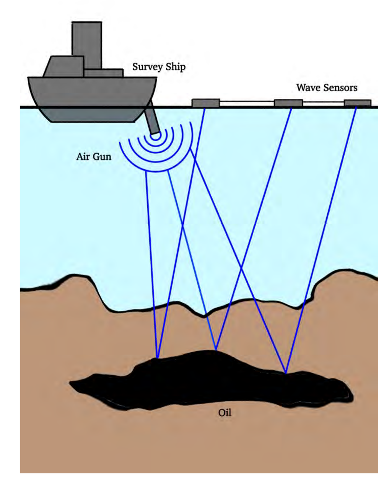

Figure 4. Oil Prospecting

Approximation Using Impedance [1]

- Dirichlet Boundary Condition: $u^i + u^s = u^{tot} = 0$
- Generalized Impedance Boundary Condition: $u^\delta + D^\delta \frac{\partial u^\delta}{\partial n}$
- Series Expansion of Total Field:

$$u_+^\delta(x) = \sum_{j=0}^{\infty} \delta_0^j u_+^j(s, \frac{\nu}{\delta_0}) = \tilde{u}_+^\delta(s,\xi) \quad \text{in } \Omega_+^\delta$$

$$u_-^\delta(x) = \sum_{j=0}^{\infty} \delta_0^j u_-^j(x) = \tilde{u}_-^\delta(x) \quad \text{in } \Omega_-$$

- Substitution into Boundary Value Problems (BVPs). Once the BVP for $j = 1$ (1st Order) is solved:

$$\frac{\partial^2 u_+^1}{\partial \xi^2} = -\left(3\xi c \frac{\partial^2}{\partial \xi^2} + c\frac{\partial}{\partial \xi}\right) u_+^0$$

$$u_+^1(s,0) = 0$$

$$\frac{\partial u_+^1}{\partial \xi}(s,0) = \varphi_0$$

Approximation Using Impedance [2]

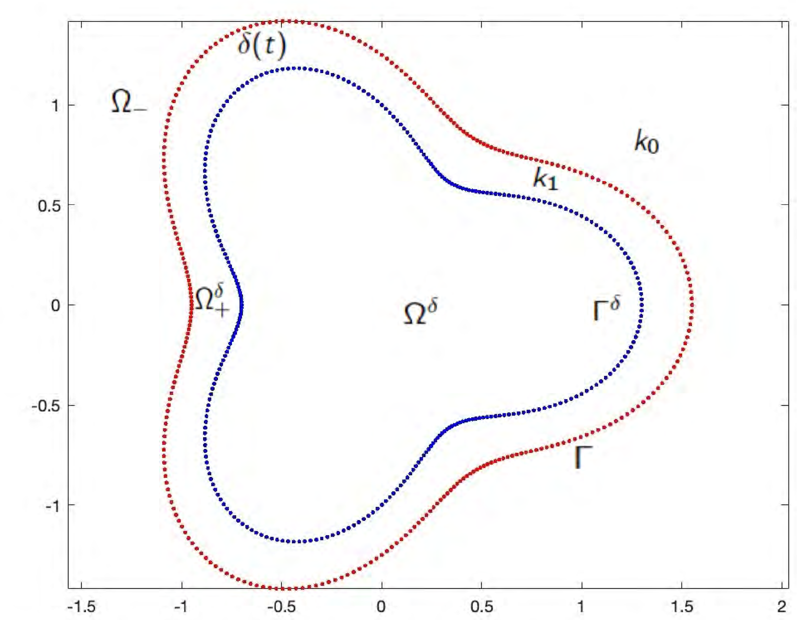

Figure 5. Geometry of an object with thin coating

Ω_+^δ	The thin, penetrable film around the domain.
Ω^δ	Coated impenetrable object (domain).
Ω_-	Exterior medium in a vacuum.
Γ	Boundary of the coating/thin film.
Γ^δ	Boundary of the impenetrable object/domain, Ω^δ
u^δ	Total field (made up of u_-^δ & u_+^δ)
u_-^δ	Restriction of u^δ into Ω_-. Equivalent to $u_{inc} + u_s^\delta$
u_+^δ	Restriction of u^δ into Ω_+^δ
$\delta(t)$	Thickness of the coating (small amount)

we get:

$$u_+^1(s,\xi) = (\xi - f)\vartheta_0$$

- Substituting the expanded total field, continuity, and boundary condition equations into the Dirichlet boundary condition:

$$\sum_{j=0}^k \delta^j u_+^j(s,0) = -D_{\delta,k}\left(\sum_{j=0}^k \delta^j \varphi_j\right)$$

- Finally, substituting the BVP solutions into this general form produces the GIBCs for different orders of approximation:

0) Ignores thin coating: $D^{\delta,0} = 0$
1) Avoids detailed wave equation for thin coating: $D^{\delta,1} = \delta(s)$
2) Avoid solving a more complex wave equation: $D^{\delta,2} = \delta(s)\left(1 - \frac{1}{2}\delta(s)c(s)\right)$

Numerical Solution

- Potential Theory: In potential theory, the scattered field is represented using boundary integral operators D and S.

$$u^s = (D + i\eta S)\phi$$

- Boundary Integral Equation: Formulated and solved using numerical methods. Specifically, the Nyström method is employed for discretization.
- This approximates the integral equation by converting it into a system of linear equations Ax = b.
- We solve Ax = b for the density ϕ, where:

$$A = \frac{I}{2} + D + i\eta S + D(T + i\eta(-\frac{I}{2} + Sp))$$

$$x = \phi$$

$$b = -u^{inc} - D\frac{\partial u}{\partial n}$$

Numerical Example

Our results in comparing our impedance approximation to the direct transmission solver for scattered wave data points.

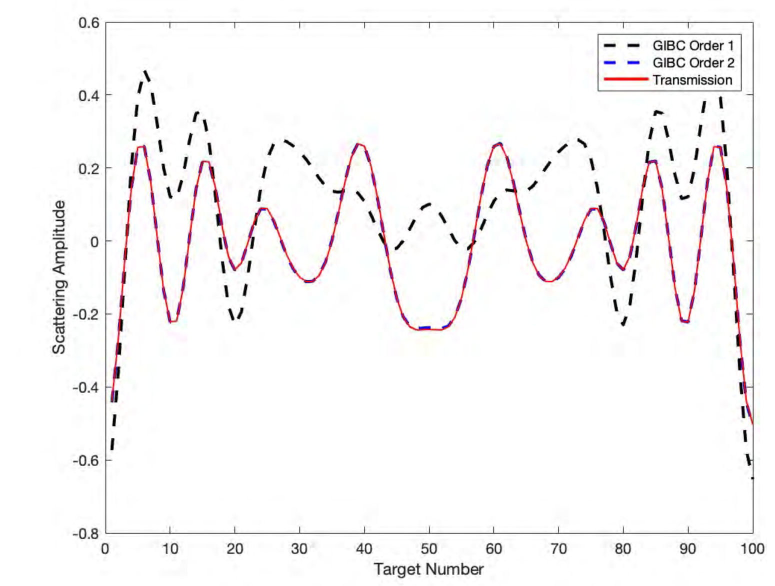

Figure 6. GIBC vs Transmission: Real

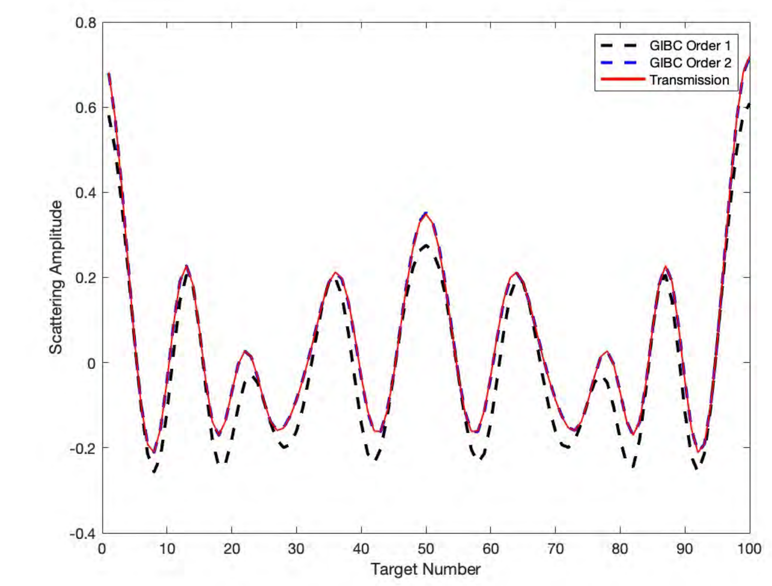

Figure 7. GIBC vs Transmission: Complex

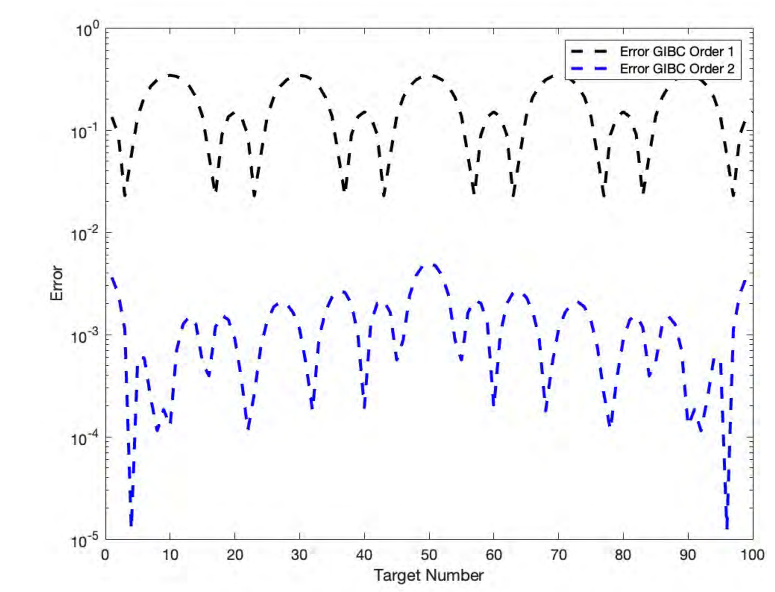

Figure 8. Error for Order 1 & Order 2: Real

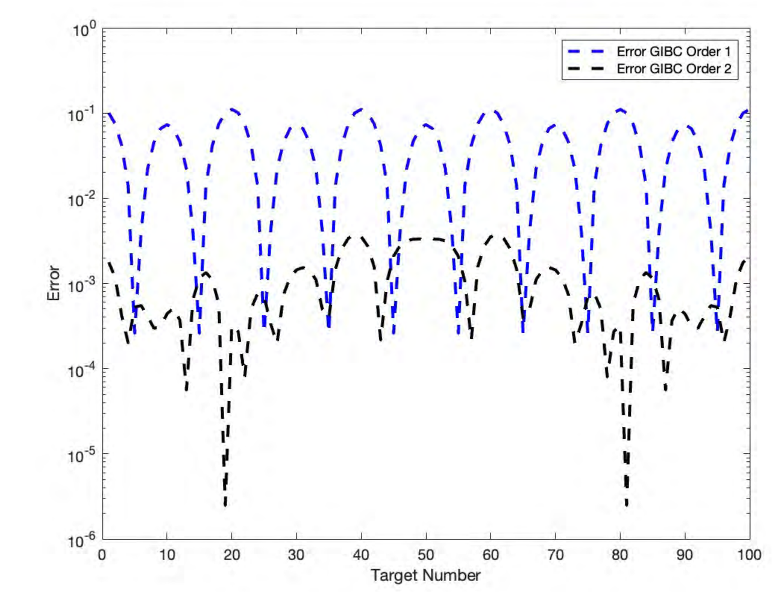

Figure 9. Error for Order 1 & Order 2: Complex

Conclusion & Future Work

In conclusion, we have been able to create a solver that approximates the wave scattered off an object with a thin coating with results fairly close to the direct solution, as seen above.

This can be used to change the material properties of the thin coatings of objects with special properties, e.g. cloaking devices.

Our project solves the forward scattering problem, which itself is only used as a step in solving the inverse problem, while the inverse scattering problem allows for more practical applications. Therefore this will be used in the future as a tool to simplify inverse problem modeling for objects with a thin coating.

Acknowledgements

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References