Investigating the Dimensional Effects on Thomas Cyclic Systems

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Background

The Thomas Cyclic System, proposed by René Thomas, is a system of ordinary differential equations often used to describe the movement of a frictionally dampened particle in a 3D lattice. We employ both numerical and analytical methods to compare the behavior of the Thomas Strange Attractor to its variations. The main objective of this research is to explore the dynamical impact of dimensionality on the Thomas Cyclic System.

Modeling 3-D

Consider the Thomas Cyclic system:

$$\begin{aligned}\frac{dx}{dt} &= \sin(y) - bx\\ \frac{dy}{dt} &= \sin(z) - by\\ \frac{dz}{dt} &= \sin(x) - bz\end{aligned} \tag{3D}$$

where $b > 0$.

Behavior of the Thomas Strange Attractor

There are many challenges that arise with analysis of these systems. We list a few:

- Observe long-term behavior of equilibrium
- Conduct bifurcation analysis
- Compute numerical solutions
- Analyze chaotic behavior

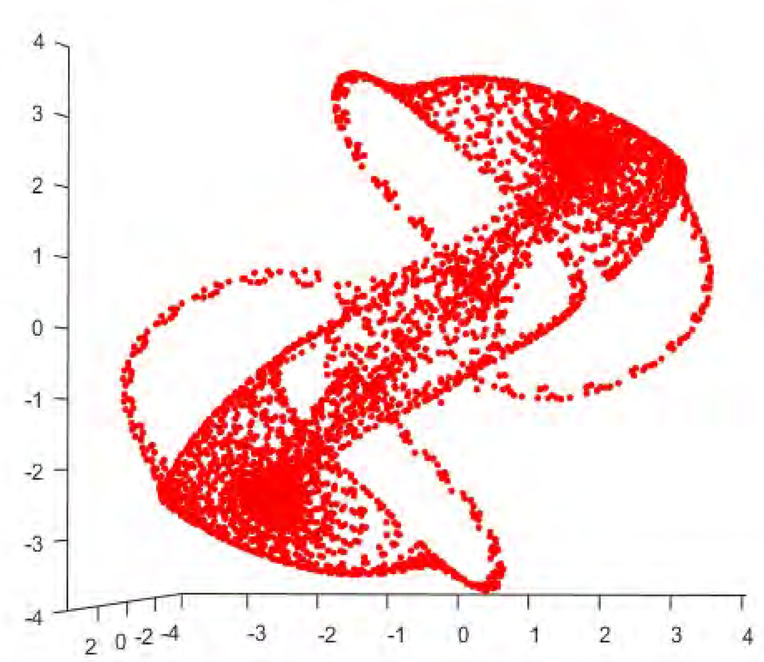
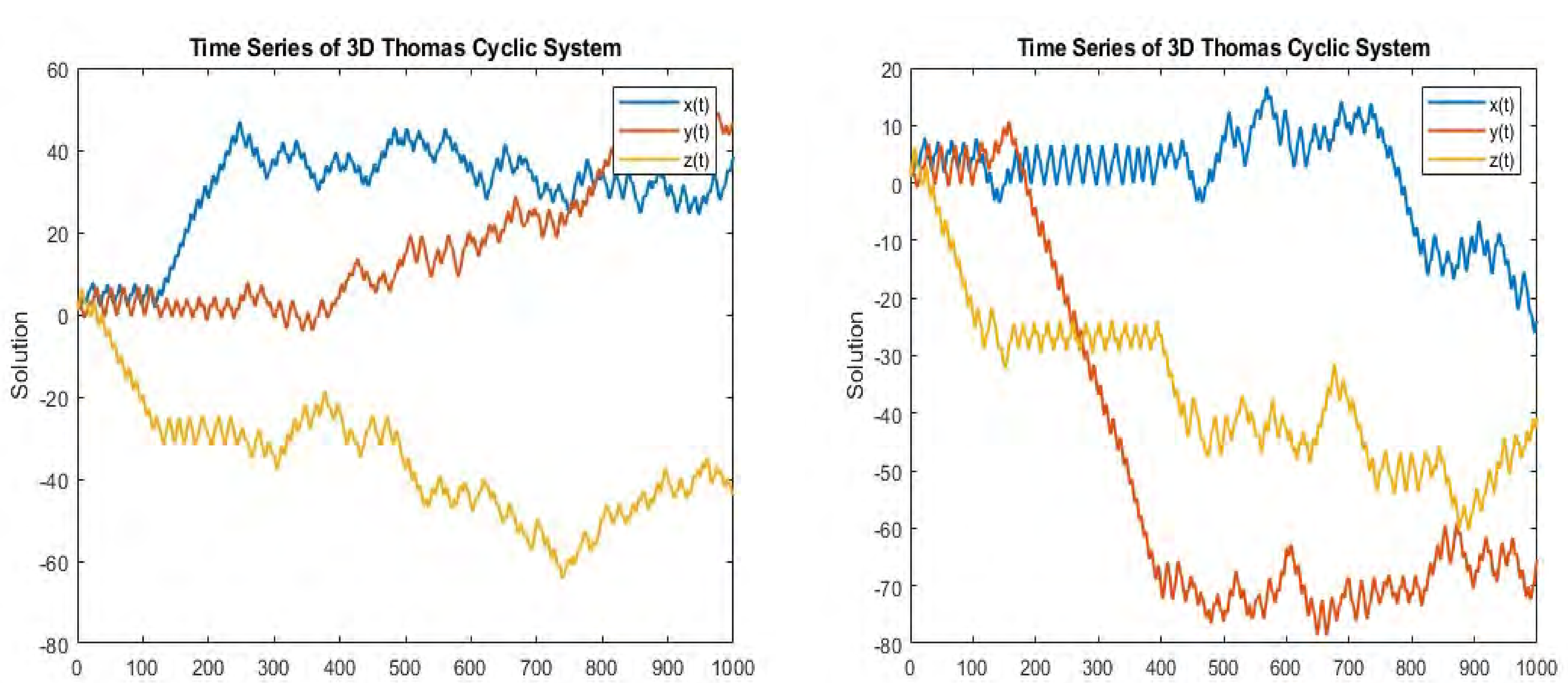


Figure 1. Thomas Strange Attractor

3D Simulation Results



(a) $b = 0.0001, I.C. = (1, 2, 1)$

(b) $b = 0.0001, I.C. = (0.99, 1.99, 0.99)$

Figure 2. Chaos Indicated by Sensitivity to I.C.

Modeling 1-D

Consider a one dimensional variation of the Thomas Cyclic System:

$$\frac{dx}{dt} = \sin(x) - bx \tag{1D}$$

where $b > 0$.

Theorem 1 Let $f(x) = \sin(x)$, $g(x) = bx$, and $h(x) = f(x) - g(x)$. Define a sequence $(p_k)_{k=1}^{\infty}$ such that $p_1 = 0 < p_2 < p_3 < ... < p_k < ...$ and $p_k = \tan(p_k)$ for all odd $k \in \mathbb{N}$. Let $j \in \mathbb{N}$. Define the interval $I_j = (cos(p_{k+2}), cos(p_k))$ for j odd and $I_j = cos(p_{k+2})$ for j even. If $b \in I_j$, then there are exactly $2j + 1$ equilibrium of $h(x)$, denoted as $-\bar{x}_j < ... < \bar{x}_0 = 0 < ... < \bar{x}_j$.

Theorem 2 Let $f(x) = \sin(x)$, $g(x) = bx$, and $h(x) = f(x) - g(x)$. Define a sequence $(p_k)_{k=1}^{\infty}$ such that $p_1 = 0 < p_2 < p_3 < ... < p_k < ...$ and $p_k = \tan(p_k)$ for all odd $k \in \mathbb{N}$. Let $j \in \mathbb{N}$. Define the interval $I_j = (cos(p_{k+2}), cos(p_k))$ for j odd and $I_j = cos(p_{k+2})$ for j even. We observe the following:

1. If j is odd:
   $\begin{cases}\text{If } \bar{x}_j \in \left((j-1)\pi, (j-1)\pi + \frac{\pi}{2}\right), & \text{then } h'(\bar{x}_j) > 0 \text{ and the fixed point is unstable.}\\ \text{If } \bar{x}_j \in \left((j-1)\pi + \frac{\pi}{2}, j\pi\right), & \text{then } h'(\bar{x}_j) < 0 \text{ and the fixed point is stable.}\end{cases}$
2. If j is even, the result above holds except for $\bar{x}_j \in (j\pi, (j+1)\pi)$. Instead, $h'(\bar{x}_j) = 0$ and the fixed point is semi-stable.

We also find that the stability of $-\bar{x}_j$ is identical to that of $\bar{x}_j$.

1D Simulation Results

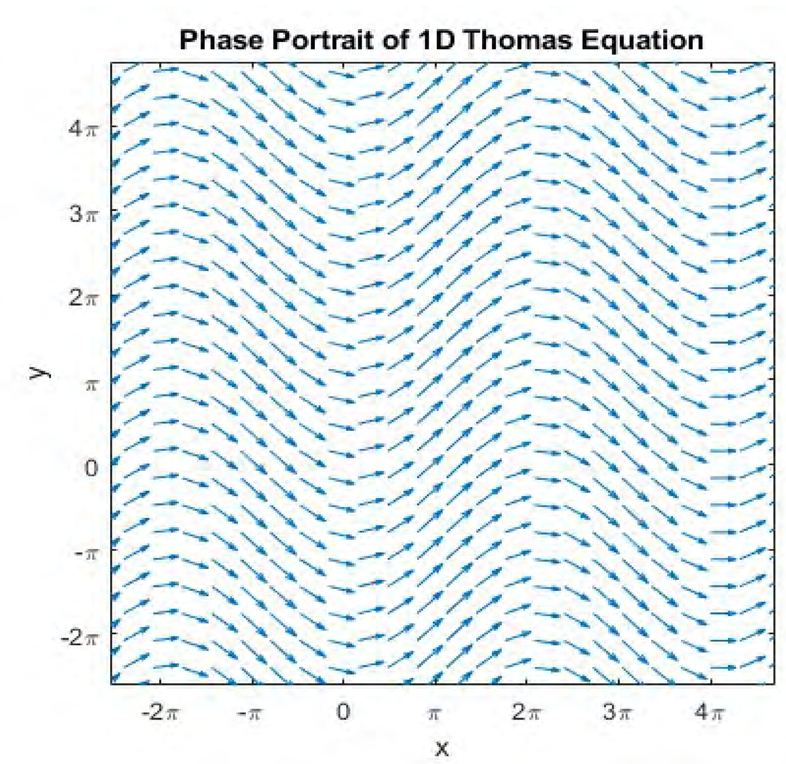


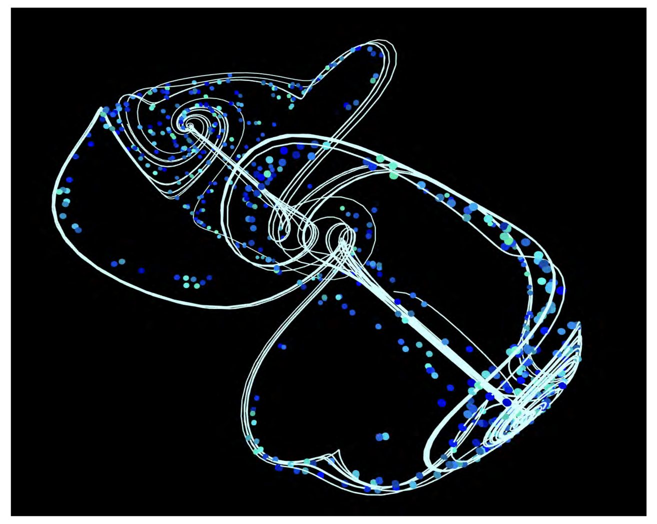
Figure 3. $b = 0.0001$

Phase portraits can be useful for analyzing the stability of those equilibrium points. By observing the trajectories, we can determine where x converges over time.

Expected Results

- 1-D results: Countably infinite equilibrium points emerge as $b \to 0$
- 2-D results: Limit Cycle for $b \to 0$
- 3-D results: Chaos for $b \to 0$

Interactive Simulation



Modeling 2-D

Consider a two dimensional variation of the Thomas Cyclic system:

$$\begin{aligned}\frac{dx}{dt} &= \sin(y) - bx\\ \frac{dy}{dt} &= \sin(x) - by\end{aligned} \tag{2D}$$

where $b > 0$.

2D Simulation Results

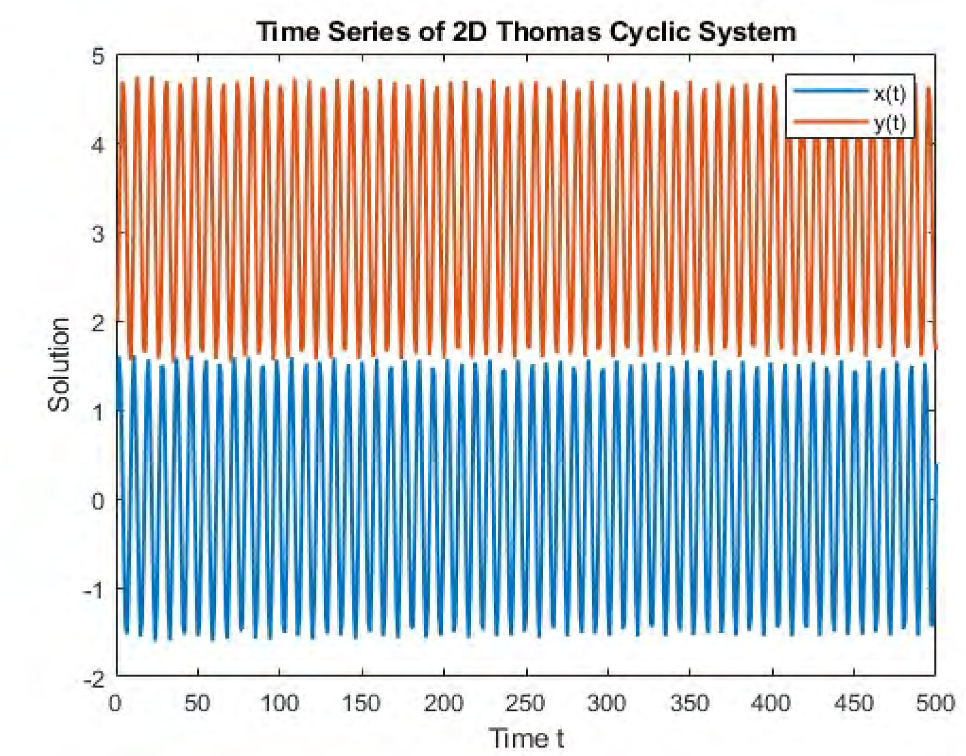


Figure 4. $b = 0.0001, I.C. = (1, 2)$

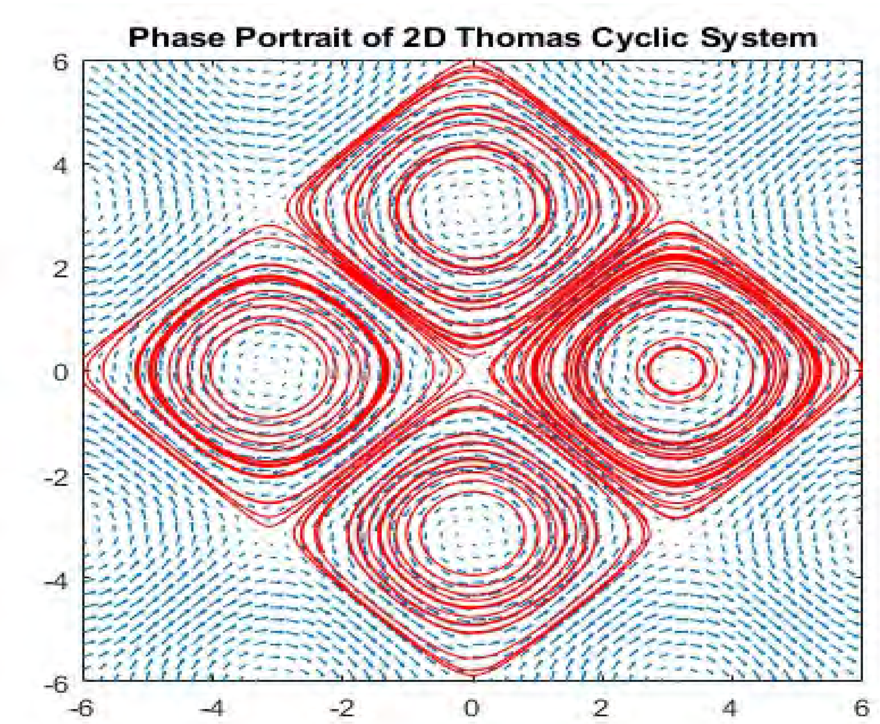


Figure 5. $b = 0.0001, I.C. = (1, 2)$

Evidence of Periodic Solution in Time Series and Phase Portrait

Future Works

- Extension to a 4-dimensional system and eventually to n-dimensions
- Applications to anomalous (random) diffusion equations, which are observed in biological cell processes
- In relation to Turing patterns, understanding how anomalous diffusion affects pattern formation in nature

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