Optimal design problems with cost function: regularity, uniqueness, and free boundary analysis

Saja Gherri [1] Andrew Hale [2]

[1]University of Michigan, Ann Arbor , [2]University of Minnesota, Twin Cities

Abstract

The volume problem in optimal design concerns the best way of insulating a heated body with a fixed volume of insulating material to minimize the total amount of heat flow. We introduce a generalization of the volume problem, known as the cost problem. In this case, the cost of placing insulation around the body is governed by a cost function, and the total cost of the design must remain under a fixed budget. We prove the existence of minimizers for the cost problem, via a penalization technique, and establish their regularity and that of their free boundaries. In particular we obtain a sharp geometric control of minimizers along their free boundaries. Furthermore, we hope to develop sufficient conditions for uniqueness and stability of solutions under a small change of the cost function. These results will provide a better understanding of the cost problem and can lead to solutions in certain cases.

Volume Cost Problem

The volume problem concerns the optimal method of insulating a heated body under a volume constraint. The cost problem is a generalization of this, and assigns a variable "cost" to occupy certain regions.

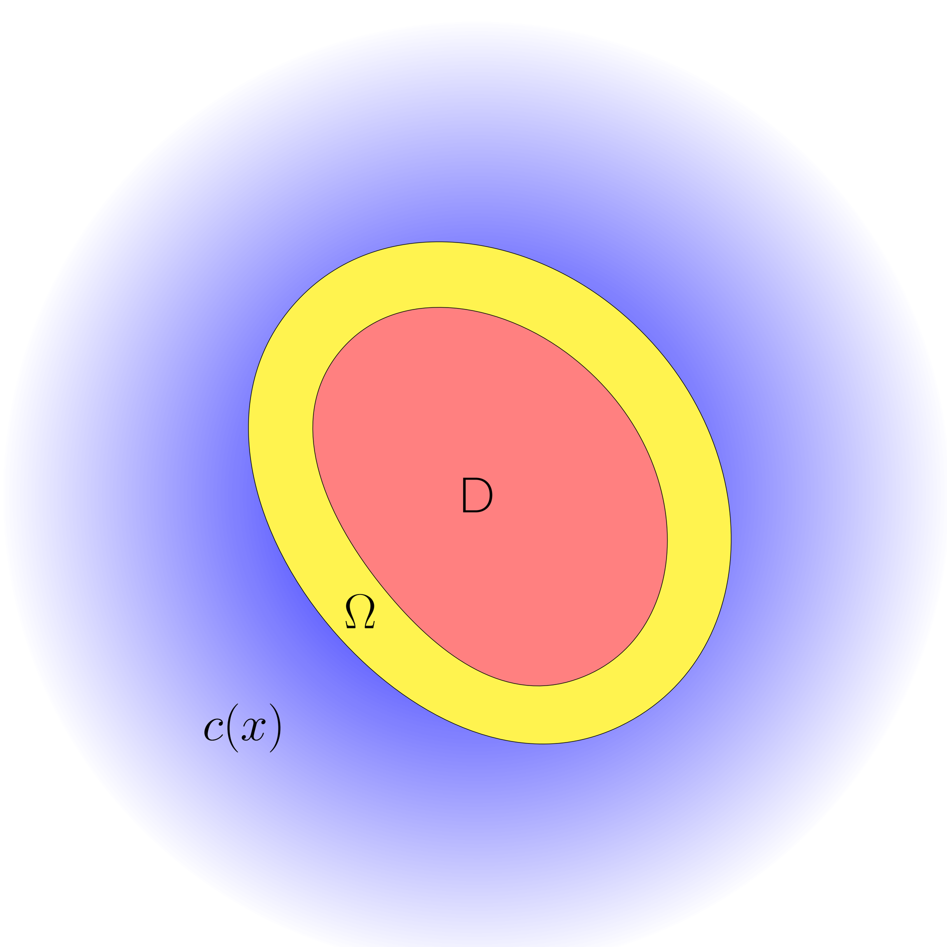

Figure 1. A heated body D is insulated by Ω.

$$min\left\{J(u)=\int_{D^C}|\nabla u|^2 \,\middle|\, \int_{\{u>0\}} c(x) \leq B\right\}$$

Solutions to the cost problem minimize heat flow while obeying the budget constraint. We demonstrate the existence of solutions using a penalization technique, where configurations that exceed the budget are penalized. As the penalty increases, we can show that solutions remain within the budget constraint.

Regularity

Under different assumptions for the cost function, we recover differing levels of regularity.

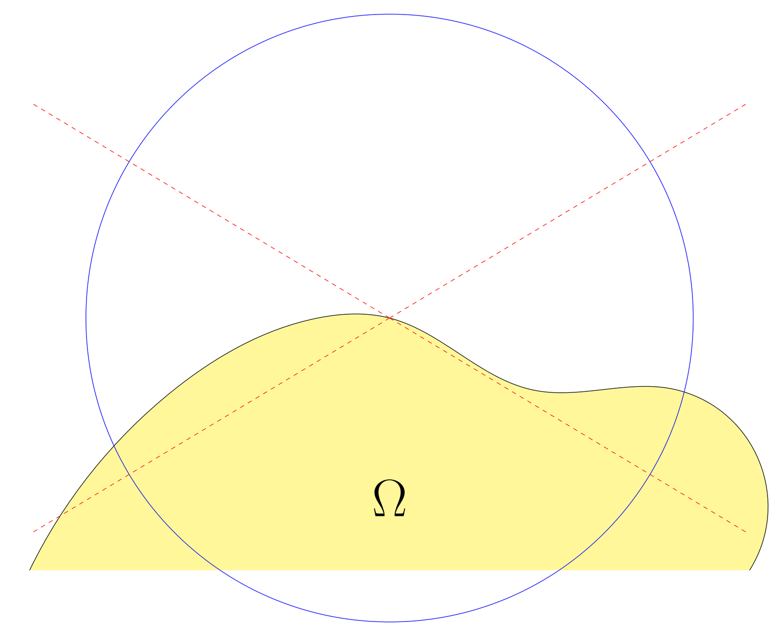

Figure 2. Lipschitz regularity of the free boundary. Within the blue circle, the black boundary must remain within the red lines.

For $c(x) \in L^p$ with $p > n/2$, we demonstrate that u is in $C^{0,\alpha}$ for $\alpha = 1-(n/2p)$, where n is the dimension. For $p=\infty$, we establish Lipschitz continuity. We use this to demonstrate non-degeneracy near the free boundary and uniform positive density of the positive phase in the case of a bounded cost function.

Uniqueness

Under certain assumptions,

the minimizer of the cost problem is unique.

This means there is only one best way to insulate a heated body under the following conditions:

1. The heated body D is starlike with respect to the origin.
2. The function $r^2c(r\vec{x})$ is increasing with respect to r.

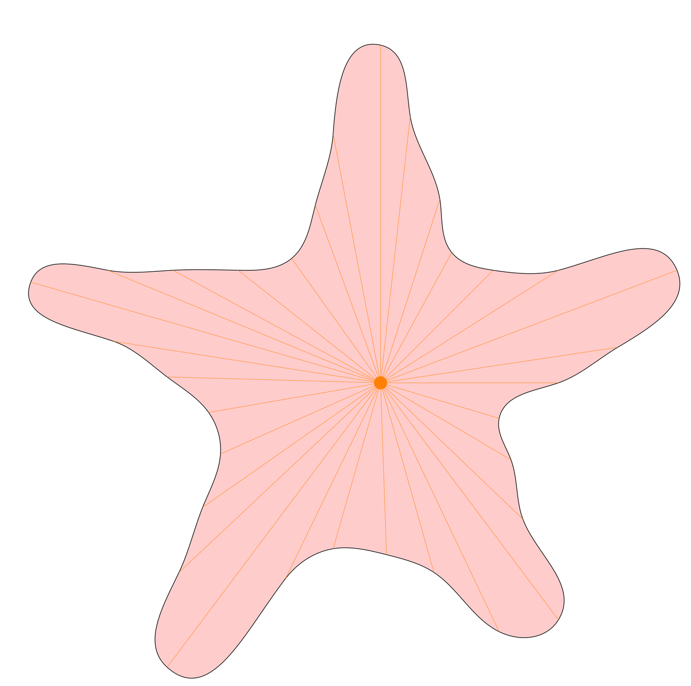

Figure 3. A starlike domain with respect to the central point.

Stability

Stability results describe how a slight change in the cost function corresponds to a small change in the solution.

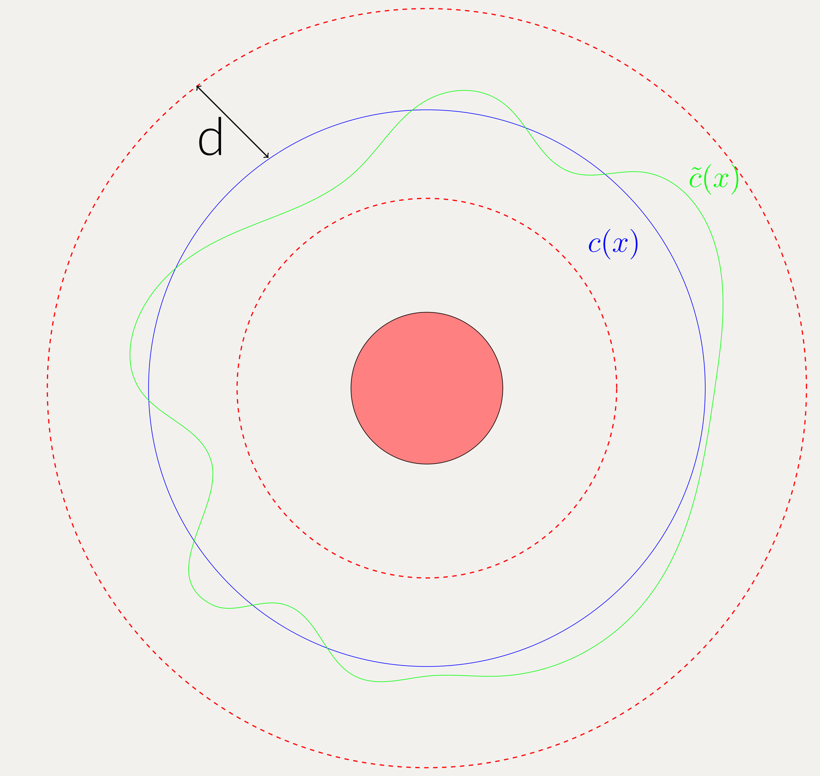

Figure 4. The new boundary, in green, must remain within distance d of the old boundary, in blue.

We consider two cost functions, $c(x)$ and $\tilde{c}(x)$, satisfying $|\sqrt{c(x)}-\sqrt{\tilde{c}(x)}| < \delta$. Suppose for a fixed starlike body D, these cost functions admit two free boundaries, Γ and $\tilde{\Gamma}$. Then,

$$\rho(\Gamma,\tilde{\Gamma}) \leq ln(1+C\delta) = d,$$

where ρ is a sense of distance and C is a constant depending on D.

Future Directions

- Suppose we have two insulated bodies; what is the **critical radius** within which the insulators become connected?
- Is uniqueness possible with only a cost function requirement? If so, what is the weakest cost function that guarantees uniqueness?
- Are **numerical approximations** possible? If so, they could provide insight into computational approaches to free boundary problems at large.

References

Acknowledgements: We thank the NSF, the UCF Math department, and Professor Eduardo Teixeira. This project was supported by NSF grant DMS-224377