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# Travelling waves of auto-catalytic chemical reaction of general order—An elliptic approach

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ABSTRACT

In this paper we study the existence and non-existence of travelling wave to parabolic system of the form $a_t = a_{xx} - af(b)$, $b_t = Db_{xx} + af(b)$, with $f$ a degenerate nonlinearity. In the context of an auto-catalytic chemical reaction, $a$ is the density of a chemical species called reactant $A$, $b$ that of another chemical species $B$ called auto-catalyst, and $D = D_B/D_A > 0$ is the ratio of diffusion coefficients, $D_B$ of $B$ and $D_A$ of $A$, respectively. Such a system also arises from isothermal combustion. The nonlinearity is called degenerate, since $f(0) = f'(0) = 0$. One case of interest in this article is the propagating wave fronts in an isothermal auto-catalytic chemical reaction of order $n: A + nB \to (n+1)B$ with $1 < n < 2$, and $D \neq 1$ due to different molecular weights and/or sizes of $A$ and $B$. The resulting nonlinearity is $f(b) = b^n$. Explicit bounds $v_*$ and $v^*$ that depend on $D$ are derived such that there is a unique travelling wave of every speed $v \geqslant v^*$ and there does not exist any travelling wave of speed $v < v_*$. New to the literature, it is shown that $v_* \propto v^* \propto D$ when $D < 1$. Furthermore, when $D > 1$, it is shown rigorously that there exists a $v_{\min}$ such that there is a travelling wave of speed $v$ if and only if $v \geqslant v_{\min}$. Estimates on $v_{\min}$ improve significantly that of early works. Another case in which two different orders of isothermal auto-catalytic chemical reactions are involved is also studied with interesting new results proved.



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## 1. Introduction

In this paper we study the existence and non-existence of travelling wave to parabolic system of the form

$$\text{(I)}\qquad \begin{cases} a_t = a_{xx} - af(b), \\ b_t = Db_{xx} + af(b), \end{cases}$$

where $D > 0$ is a constant, $f$ is a $C^1$ function and positive when $b > 0$ with $f(0) = 0$. The important additional assumption is $f'(0) = 0$, which distinguishes our case from the KPP type of equations studied by many authors in the literature [5,17,20,21,26], where substantially different phenomena and rich theory for KPP type equations are developed. We say the nonlinearity $f$ is degenerate, as will become clear in what follows, because the resulting ODE system for travelling wave solution has one-dimensional center manifold at one of the equilibrium points, whereas KPP type of nonlinearity having the property $f'(0) = k$ with $k > 0$, always yields a stable equilibrium point.

The purpose of the present study is two-fold: (i) to derive sharp results on travelling wave solutions such as existence, optimal range of travelling wave speed in relation to $D$ and nonlinearity and the existence of minimum speed, using a new approach, of particular cases which arise from many important applications, and (ii) to lay a solid foundation for a unified theory of general case with degenerate nonlinearity.

A positive solution of (I) is called a travelling wave if $a(x,t) = a(z)$, $b(x,t) = b(z)$ with $z = x - vt$, and $v$ a positive constant, satisfying

$$\lim_{z\to-\infty}(a,b) = (0, b_0), \qquad \lim_{z\to\infty}(a,b) = (a_0, 0), \tag{1.1}$$

where $a_0 > 0$, $b_0 > 0$. It is easy to see that existence implies $a_0 = b_0$.

System of type (I) arises from many applications. For instance, an isothermal auto-catalytic chemical reaction of the order $n + 1$,

$$A + nB \to (n+1)B \quad \text{with rate } kab^n,$$

yields, after a simple scaling, a system

$$\text{(II)}\qquad \begin{cases} a_t = b_{xx} - ab^n, \\ b_t = Db_{xx} + ab^n, \end{cases}$$

where $n \geqslant 1$, $k > 0$ is the reaction rate, and $a$ and $b$ are the concentrations of reactant $A$ and auto-catalyst $B$, respectively, with $D = D_B/D_A$ being the ratio of diffusion rate of auto-catalyst $B$ to that of reactant $A$. Two important special cases of (II) are $n = 1$ and $n = 2$.

Well-documented in the literature, the cubic reaction relation $n = 2$ has appeared in several important models of real chemical reactions, e.g. almost isothermal flames in the carbon-sulphide–oxygen reaction (Voronkov and Semenov [33]), iodate–arsenous acid reactions (Saul and Showalter [29]), hydroxylamine–nitrate reactions (Gowland and Stedman [13]), as well as other applications (Aris et al. [1] and Sel'kov [30]). The cases of quadratic $n = 1$ appears in Belousov–Zhabotinskii reaction and also gas-phase, radical chain branching and oxidation reactions, such as the carbon-monoxide–oxygen, and hydrogen–oxygen systems (see Gray et al. [14]; Merkin et al. [22]).

Experimental observations demonstrate the existence of propagating chemical wave fronts in chemical systems for which cubic-catalysis forms a key step [16,35]. These wavefronts, or travelling waves, arise due to the interaction of reaction and diffusion. Quite often when a quantity of auto-catalyst is added locally into an expanse of reactant, which is initially at uniform concentration, the ensuing reaction is observed to generate wavefronts which propagate outward from the initial reaction zone, consuming fresh reactant ahead of the wavefront as it propagates. This is the phenomenon



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to be addressed in this paper. For recent experimental study of travelling waves in cubic auto-catalysis with a drifting term added to the system (I), see [32].

We also note that the study of fronts in thermal-diffusive flows with advection has a close relation to the travelling wave solutions to system (I), where a typical system is as follows:

$$\begin{cases} T_t + u \cdot \nabla T = \kappa \Delta T + g(T)\eta, \\ \eta_t + u \cdot \nabla \eta = \frac{\kappa}{Le} \Delta \eta - g(T)\eta, \end{cases}$$

where $\kappa$ and $Le$ are constants, $\eta$ is the concentration of a chemical reactant, $T$ the temperature, see [5,17,20,21,26].

The existence of travelling wave and the estimate of minimum travelling wave speed were studied by Billingham and Needham [6,7] and the present authors [10] for (II). To normalize the situation, it was chosen in [6,10] that $a_0 = b_0 = 1$. This will be the working assumption for various systems studied in this paper. In [6], it was proved that when $n = 1$, travelling wave solution exists if and only if $v \geqslant 2\sqrt{D}$. Whereas, when $n = 2$, it was proved in [10] that

(a) travelling wave solution exists if

$$v \geqslant \begin{cases} \sqrt{\frac{D}{1+D^{-1}}} & \text{if } D > 1, \\ \frac{4D}{\sqrt{1+4D}} & \text{if } D < 1; \end{cases}$$

(b) travelling wave solution does not exist if

$$v < \begin{cases} \sqrt{\frac{D}{2}} & \text{if } D > 1, \\ \frac{D}{\sqrt{2}} & \text{if } D < 1. \end{cases}$$

It can be seen from the above that the case of $n = 2$ is substantially different from that of $n = 1$. A convenient way to summarize the difference is that the minimum speed $v_{\min}$ is of order $\sqrt{D}$ if $D > 1$, but of the order $D$ if $D < 1$ when $n = 2$. As a matter of fact, the same characterization holds if $n > 2$. But, the arguments used in [10] does not cover the case of $n \in (1, 2)$. An interesting question, in light of the very different behaviours of $n = 1$ versus $n = 2$ is: What is the situation when $n \in (1, 2)$? In particular how the transition occurs from $n = 1$ to $n = 2$? We want to study these features in detail in this paper.

Another closely related case is the following system:

$$\text{(III)} \quad \begin{cases} a_t = a_{xx} - ab^m - kab^n, \\ b_t = Db_{xx} + ab^m + kab^n, \end{cases}$$

where the exponents $n > m \geqslant 1$ and the constant $k > 0$. This can be regarded as an auto-catalytic chemical reaction of mixed order, involving both

$$A + nB \to (n+1)B \quad \text{and} \quad A + mB \to (m+1)B,$$

see [12] and [18], with $k$ measuring the relative strength of $n$th order to that of $m$th order auto-catalytic chemical reaction. The system (III), when $n = 1$ and $m = 2$ was studied in [12] where, by using an improved invariant region method, as against a simple one in [6], existence of travelling wave solution and estimation of minimum travelling wave speed were proved. But, it seems hard to the present authors how such technique can be applied effectively to the more general case we would like to study in this paper. Instead, we shall use a new approach initiated in [10] which takes advantage of the monotonicity of $a$ and $b$, enabling us to reduce the resulting ODE system for travelling wave from third order to second order.

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It is interesting to note that despite many deep and interesting results for single equation case, such as Aronson and Weinberger [2], Chen and Guo [9], Fife and McLeod [11], Sattinger [28] and others in the excellent review paper by Xin [34], results on systems are relative few and far in between.

The organization of the paper is as follows. In Section 2, we study the system (II), and in Section 3 we analyze the system (III).

## 2. The case of auto-catalysis of order *n*

The equations which govern travelling wave solutions of (II) are obtained by looking for a solution of (II) in the form of $\alpha = a(x - vt)$, $\beta = b(x - vt)$. They take the form, with $z = x - vt$,

$$\begin{aligned} &\alpha_{zz} + v\alpha_z - \alpha\beta^n = 0, \\ &\beta_{zz} + v\beta_z + \alpha\beta^n = 0, \end{aligned} \tag{2.1}$$

where $v > 0$ is the constant speed of propagation. Finding a travelling wave solution is to solve

$$\begin{cases} \alpha_{zz} + v\alpha_z = \alpha\beta^n, \quad \alpha \geqslant 0, \ \forall z \in \mathbb{R}, \\ D\beta_{zz} + v\beta_z = -\alpha\beta^n, \quad \beta \geqslant 0, \ \forall z \in \mathbb{R}, \\ \lim_{z\to\infty} \big(\alpha(z), \beta(z)\big) = (1, 0), \\ \lim_{z\to-\infty} \big(\alpha(z), \beta(z)\big) = (0, 1). \end{cases} \tag{2.2}$$

### 2.1. Basic properties of travelling waves

Suppose $(v, \alpha, \beta)$ solves (2.2). Then $[\alpha_z + v\alpha + D\beta_z + v\beta]_z = 0$, by adding up the two equations in (2.2), so that $\alpha_z + D\beta_z + v(\alpha + \beta)$ is a constant function. Using the boundary conditions, we find that

$$\alpha_z + D\beta_z + v(\alpha + \beta - 1) = 0 \quad \text{on } \mathbb{R}.$$

With the new variable $w = \beta_z$, (2.2) is equivalent to the following third order ODE system

$$\begin{cases} \alpha_z = v(1 - \alpha - \beta) - Dw, \\ \beta_z = w, \\ w_z = -D^{-1}\big(\alpha\beta^n + vw\big), \\ \lim_{z\to\infty} \big(\alpha(z), \beta(z), w(z)\big) = (1, 0, 0), \\ \lim_{z\to-\infty} \big(\alpha(z), \beta(z), w(z)\big) = (0, 1, 0). \end{cases} \tag{2.3}$$

It is clear that in the $(\alpha, \beta, w)$ phase space, there are two equilibrium points: $(0, 1, 0)$ and $(1, 0, 0)$. The following is a few basic properties of travelling wave solutions.

**Proposition 2.1.** *The systems* (2.2) *and* (2.3) *are equivalent. Any solution* $(\alpha, \beta)$ *to* (2.2) *or* $(\alpha, \beta, w)$ *to* (2.3) *has the following properties*:

1. $\alpha_z > 0 > \beta_z$ *on* $\mathbb{R}$;
2. $\alpha + \beta < 1$ *on* $\mathbb{R}$ *if* $D < 1$, $\alpha + \beta \equiv 1$ *if* $D = 1$, *and* $\alpha + \beta > 1$ *if* $D > 1$;
3. $v = \int_{-\infty}^{\infty} \alpha(z)\beta^n(z)\,dz > 0$;
4. *The equilibrium point* $(0, 1, 0)$ *of* (2.3) *is a saddle with a two-dimensional stable manifold and a one-dimensional unstable manifold. The eigenvalues and associated eigenvectors are*:



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$$\lambda_1 = -\nu D^{-1}, \qquad \mathbf{e}_{\lambda_1} = (0, -1, -\lambda_1)^T,$$
$$\lambda_2 = -\frac{1}{2}\left(\sqrt{\nu^2+4}+\nu\right), \qquad \mathbf{e}_{\lambda_2} = \left(\lambda_2(D\lambda_2+\nu), -1, -\lambda_2\right)^T,$$
$$\lambda_3 = \frac{1}{2}\left(\sqrt{\nu^2+4}-\nu\right), \qquad \mathbf{e}_{\lambda_3} = \left(\lambda_3(D\lambda_3+\nu), -1, -\lambda_3\right)^T;$$

5. *When $n > 1$, the equilibrium point $(1, 0, 0)$ is degenerate; it has a two-dimensional stable manifold and a one-dimensional center manifold. The eigenvalues and associated eigenvectors are:*

$$\mu_1 = -\nu, \qquad \mathbf{e}_{\nu_1} = (1, 0, 0)^T,$$
$$\mu_2 = -\nu D^{-1}, \qquad \mathbf{e}_{\nu_2} = \left(0, 1, -\nu D^{-1}\right)^T,$$
$$\mu_3 = 0, \qquad \mathbf{e}_{\nu_3} = (1, -1, 0)^T.$$

All items except (3) were proven in [6]. The equation in (3) is obtained by integrating the first equation in (2.2) with the boundary conditions $\alpha(\infty) = 1$ and $\alpha(-\infty) = 0$.

The third property in the proposition demonstrates that $\nu > 0$. The fourth property clearly tells us that the travelling wave we are looking for is indeed the one-dimensional unstable manifold associated with the equilibrium $(0, 1, 0)$. Hence, given $\nu > 0$, a travelling wave of speed $\nu$, if it exists, is unique up to a translation.

### 2.2. New setting—A non-autonomous $2 \times 2$ system

Different from earlier work in [6,27], here we shall use a transformation turning the third order autonomous system (2.3) into a second order non-autonomous system, using $u := 1 - \beta$ as the independent variable. This is allowed since for the solution of interest, $\beta_z < 0$, so $z \to 1 - \beta(z)$ has an inverse. To make the resulting system as simple as possible, we also scale the variables. Hence, we introduce

$$u = 1 - \beta, \qquad A = \frac{D\alpha}{\nu^2}, \qquad y = \frac{\nu z}{D}, \qquad \kappa := \frac{D}{\nu}.$$

The system of differential equations (2.2) becomes

$$\begin{cases} u_{yy} + u_y = A(1-u)^n & \text{on } \mathbb{R}, \\ A_y = \kappa^2(u + u_y) - DA & \text{on } \mathbb{R}. \end{cases}$$

Since $u_y > 0$ for the solution of interest, we can use $u$ as the independent variable. Introducing $P(u) = u_y$, we have an equivalent system of second order non-autonomous (singular) ODEs

$$\begin{cases} PP' = A[1-u]^n - P & \forall u \in [0,1], \\ PA' = \kappa^2[P+u] - DA & \forall u \in [0,1], \\ P(u) > 0, \quad A(u) > 0, & \forall u \in (0,1), \\ P(0) = 0, \quad A(0) = 0. \end{cases} \tag{2.4}$$

**Lemma 2.1.** *For every $D > 0$ and $\kappa > 0$, (2.4) admits a unique solution. In addition,*

$$P(u) = \lambda u + O\left(u^2\right), \qquad A(u) = \lambda(1+\lambda)u + O\left(u^2\right) \quad \text{as } u \searrow 0, \tag{2.5}$$

*where*

$$\lambda := \frac{1}{2}\left(\sqrt{4\kappa^2 + D^2} - D\right) \quad \left(\text{the only positive root to } \lambda(\lambda + D) = \kappa^2\right).$$

*Furthermore, $A'(u) > 0$ for all $u \in [0, 1)$ and there are only two possible cases*:

(a) $P(1) > 0$; *there does not exist any travelling wave solution to* (2.2).
(b) $P(1) = 0$; *there exists a travelling wave solution to* (2.2), *unique up to translation.*

**Proof.** The proof is essentially the same as Lemma 2.2 in [10], and we omit it. □

### 2.3. A scalar equation

Next we review the existence of travelling wave of unit speed to the equation

$$u_{zz} + u_z = ku(1-u)^n, \quad 0 \leqslant u \leqslant 1 \text{ on } \mathbb{R}, \qquad u(-\infty) = 0, \qquad u(\infty) = 1. \tag{2.6}$$

Here $n \geqslant 1$ is a parameter and $k$ is a positive constant. We seek upper bounds on $k$ for the existence of a solution. Since a solution, if it exists, satisfies $u_z > 0$ on $\mathbb{R}$, we can write $u_z = Q(u)$ and work on the $(u, Q)$ phase plane. The resulting equation on the phase plane is

$$\begin{cases} QQ' + Q = ku(1-u)^n & \forall u \in [0,1], \\ Q(0) = 0, \quad Q > 0 \text{ on } (0,1). \end{cases} \tag{2.7}$$

There is a one-to-one correspondence between solutions to (2.6) and solutions to (2.7) satisfying the additional requirement $Q(1) = 0$.

**Lemma 2.2.** *For each $n \geqslant 1$ and $k > 0$, there exists a unique solution $Q = Q(n,k;\cdot)$ to* (2.7). *In addition, there exists a positive constant $K(n)$ such that $Q(n,k;1) = 0$ if $k \in (0, K(n)]$ and $Q(n,K;1) > 0$ if $k \in (K(n), \infty)$. Consequently,* (2.7) *admits a solution if and only if $k \in (0, K(n)]$. In addition, $K(n)$ is a strictly increasing function of $n$ and $K(1) = \frac{1}{4}$, $K(2) = 2$.*

**Proof.** The existence of $Q$ and $K$ follows from the comparison principle. The exact value of $K(1)$ is calculated by a known fact that the function $K(1)u(1-u)$ is concave, so the minimum wave speed $v = 1$ satisfies $1 = 2\sqrt{K(1)}$; hence $K(1) = 1/4$. In the case $n = 2$, the exact solution is given by $Q = u(1-u)$, so $K(2) = 2$. We omit details, because it is a standard argument. □

### 2.4. The case of $D \geqslant 1$

The following result shows the existence of $v_{\min}$ and provides sharp bounds.

**Theorem 2.1.** *Suppose $D \geqslant 1$ and $n \geqslant 1$. There exists a positive constant $v_{\min}$ such that* (2.2) *admits a travelling wave if and only if $v \geqslant v_{\min}$. In addition, $v_{\min}$ is bounded by*

$$\sqrt{\frac{D}{K(n)}} \leqslant v_{\min} \leqslant \sqrt{\frac{D}{K(n)}} \frac{1}{\sqrt{1 - (1 - \frac{1}{D})\frac{\sqrt{4K(n)+1}-1}{\sqrt{4K(n)+1}+1}}},$$

*where $K(n)$ is the same constant as in Lemma* 2.2.

**Lemma 2.3.** *Suppose $D \geqslant 1$. Then $DA(u) \geqslant \kappa^2 u$ for all $u \in [0,1]$. Consequently, there is no travelling wave solution to* (2.2) *when $\kappa^2 > DK(n)$, i.e., when $v < \sqrt{D/K(n)}$.*

**Proof.** If $D = 1$, $A(u) = \kappa^2 u$ for all $u \in [0,1]$. When $D > 1$, for every $u \in (0,1)$,

$$P\big[DA - \kappa^2 u\big]' = -D\big[DA - \kappa^2 u\big] + (D-1)\kappa^2 P > -D\big[DA - \kappa^2 u\big].$$

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In addition, when $u$ is sufficiently small, $DA(u) = D(1+\lambda)\lambda u + O(u^2) > [D+\lambda]\lambda u = \kappa^2 u$. Applying the Gronwall's inequality, we derive that $DA > \kappa^2 u$ on $(0, 1)$.

Now suppose $\kappa^2 > DK(n)$. Let $\hat{k} \in (K(n), k^2/D)$. Then $A(u) \geqslant \hat{k}u$ on $[0, 1]$ so that

$$PP' + P = A(1-u)^n \geqslant \hat{k}u(1-u)^n \quad \forall u \in [0, 1].$$

We compare $P(u)$ and the solution $Q(n, \hat{k}; u)$ given in Lemma 2.2. Using Taylor expansion we can show that $P(u) > Q(n, \hat{k}; u)$ for all $u \in (0, \epsilon]$ for some $\epsilon > 0$. In the interval $[\epsilon, 1]$ we can use the regular comparison principle to show that $P(u) > Q(n, \hat{k}; u)$ for all $u \in [\epsilon, 1)$. In particular, $P(1) \geqslant Q(n, \hat{k}; 1) > 0$, so that there is no travelling wave solution to (2.2). Since $\kappa = D/v$, the condition $\kappa^2 > DK(n)$ is the same as $v < \sqrt{D/K(n)}$. □

**Lemma 2.4.** *Suppose $D > 1$. Then,*

$$A(u) < \lambda(1+\lambda)u, \qquad P(u) < \lambda u \quad \forall u \in (0, 1).$$

*Consequently, there exists a travelling wave solution to (2.2) when $\lambda(\lambda+1) \leqslant K(n)$, i.e. when*

$$v \geqslant \sqrt{\frac{D}{K(n)}} \frac{1}{\sqrt{1 - (1 - \frac{1}{D}) \frac{\sqrt{4K(n)+1}-1}{\sqrt{4K(n)+1}+1}}}.$$

**Proof.** A higher order Taylor expansion near $u = 0$ shows that $A < \lambda(\lambda+1)u$ and $P < \lambda u$ for all sufficient small positive $u$. Set

$$\hat{B} = \sup\{b \in (0, 1) \mid P(u) < \lambda u,\ A(u) < \lambda(1+\lambda)u\ \forall u \in (0, b)\}.$$

We show that $\hat{B} = 1$. Suppose on the contrary that $\hat{B} < 1$. Then either $P(\hat{B}) - \lambda\hat{B} = 0$ or $A(\hat{B}) - \lambda(1+\lambda)\hat{B} = 0$. In $(0, \hat{B}]$,

$$\begin{aligned} P[A - \lambda(1+\lambda)u]' &= \kappa^2(P+u) - DA - \lambda(1+\lambda)P \\ &= \lambda(D+\lambda)(P+u) - DA - \lambda(1+\lambda)P \\ &= -D[A - \lambda(1+\lambda)u] + \lambda(D-1)(P - \lambda u) \\ &\leqslant -D[A - \lambda(1+\lambda)u]. \end{aligned}$$

Gronwall's inequality then implies that $A < \lambda(\lambda+1)u$ on $(0, \hat{B}]$. Similarly, for all $u \in (0, \hat{B}]$,

$$\begin{aligned} P[P - \lambda u]' &= -(1+\lambda)P + A(1-u)^n \\ &= -(1+\lambda)(P - \lambda u) - \lambda(1+\lambda)u + A(1-u)^n \\ &< -(1+\lambda)(P - \lambda u). \end{aligned}$$

The Gronwall's inequality shows that $P < \lambda u$ on $(0, \hat{B}]$. We reach a contradiction. This proves that $\hat{B} = 1$; i.e. $P(u) < \lambda u$ and $A(u) < \lambda(1+\lambda)u$ for all $u \in (0, 1)$.

Suppose $\lambda(1+\lambda) \leqslant K(n)$. We can use comparison argument to show that $P(u) \leqslant Q(n, K(n); u)$ for all $u \in [0, 1]$, so that $P(1) = 0$. Namely, there exists a travelling wave solution to (2.2). □

**Proof of Theorem 2.1.** The estimate of $v_{\min}$, when it exists, follows from the above two lemmas.

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We notice that the set of admissible speed is a closed set. Indeed, if there is no travelling wave of speed $\hat{v} > 0$, then the solution $(P, A)$ to (2.4) with $v = \hat{v}$ has the property that $P(1) > 0$. It then follows by continuous dependence that for any $v$ sufficiently close to $\hat{v}$, the solution to (2.4) also satisfies $P(1) > 0$. This implies that there is no travelling wave of speed $v$ for any $v$ sufficiently close to $\hat{v}$. Thus the complement of the set of admissible speed is open; that is the set of admissible speed is closed.

Hence, to show the existence of $v_{\min}$, it suffices to show that if $v_1 > v_0$ and there exists a travelling wave of speed $v_0$, then there also exists a travelling wave of speed $v_1$. For this, we denote $\kappa_i = D/v_i$ and $(P_i, A_i)$ the solution to (2.4) with $\kappa_i = D/v_i$, $i = 0, 1$. The existence of a travelling wave of speed $v_0$ implies that $P_0(1) = 0$. To show that there exists a travelling wave of speed $v_1$, it is necessary and sufficient to show that $P_1(1) = 0$. For this, it suffices to show that $P_1 < P_0$ in $(0, 1)$.

Notice that $\kappa_1 < \kappa_0$. Denote by $\lambda_i$ the positive root to $\lambda_i(\lambda_i + D) = \kappa_i^2$. Then $\lambda_1 < \lambda_0$. The asymptotic expansion for $(P, A)$ then implies that there exists $\epsilon > 0$ such that for $u \in (0, \epsilon]$, $P_1(u) < P_0(u)$ and $A_1(u) < A_0(u)$. In addition, for small $u$, the functions $\alpha_i := DA_i/\kappa_i^2$ satisfy

$$\begin{aligned}\alpha_0 - \alpha_1 &= \left\{\frac{D\lambda_0(\lambda_0 + 1)}{\kappa_0^2} - \frac{D\lambda_1(\lambda_1 + 1)}{\kappa_1^2}\right\}u + O\big(u^2\big)\\ &= D\left\{\frac{\lambda_0 + 1}{\lambda_0 + D} - \frac{\lambda_1 + 1}{\lambda_1 + D}\right\}u + O\big(u^2\big)\\ &= \frac{D(D-1)(\lambda_0 - \lambda_1)}{(\lambda_0 + D)(\lambda_1 + D)}u + O\big(u^2\big) > 0\end{aligned}$$

since $D > 1$ and $\lambda_0 > \lambda_1$. Now let

$$\hat{B} = \sup\big\{b \in (0, 1) \bigm| P_1(u) < P_0(u)\ \forall u \in (0, b)\big\}.$$

We claim that $\hat{B} = 1$. Suppose the contrary, $\hat{B} < 1$. Then $P_0(\hat{B}) = P_1(\hat{B}) > 0$.

First we claim that $A_0 > A_1$ on $(0, \hat{B}]$. Suppose it is not true, then there is a $u_1 \in (0, \hat{B}]$ at which $A_0(u_1) = A_1(u_1)$. Since $\kappa_0 > \kappa_1$, there exists $u_2 \in (0, u_1)$ such that $\alpha_0(u_2) = \alpha_1(u_2)$ and $\alpha_0(u_2)' \leqslant \alpha_1(u_2)'$. But, at $u = u_2$,

$$[\alpha_0 - \alpha_1]' = \frac{D(u - \alpha_0)}{P_0} - \frac{D(u - \alpha_1)}{P_1} = \frac{D(\alpha_0 - u)(P_0 - P_1)}{P_0 P_1} > 0$$

since $\alpha_0 - u = [DA_0 - \kappa_0^2 u]/\kappa_0^2 > 0$ by Lemma 2.3 and $P_0 > P_1$ in $(0, \hat{B}) \ni u_2$. Thus, we must have $A_0 > A_1$ in $[0, \hat{B}]$. Consequently, we obtain from the equation for $P_i$ that

$$\frac{1}{2}\big[P_1^2 - P_0^2\big]' = [P_0 - P_1] + (A_1 - A_0)[1 - u]^n < [P_0 - P_1] = \frac{P_0^2 - P_1^2}{P_0 + P_1}.$$

The Gronwall's inequality on $[\epsilon, \hat{B}]$ then given $P_1^2 - P_0^2 < 0$ on $[\epsilon, \hat{B}]$, contradicting $P_0(\hat{B}) = P_1(\hat{B})$. Hence, $\hat{B} = 1$ and $P_1 < P_0$ on $(0, 1)$. This completes the proof of Theorem 2.1. □

### *2.5. The case of $D < 1$*

The main result for $D < 1$ case is the following theorem.

**Theorem 2.2.** *Suppose $D < 1$ and $1 < n < 2$. For the travelling wave problem* (2.1),

(i) *there exists a unique* (*up to translation*) *solution if* $v \geqslant \dfrac{2D}{(-D^2 + v^2)^{1/2}}$,

(ii) $$\text{there does not exist any solution if } v < \sqrt{\frac{D}{K(n)}} \frac{1}{\sqrt{1-(1-\frac{1}{D})\frac{\sqrt{4K(n)+1}-1}{\sqrt{4K(n)+1}+1}}},$$

*where*

$$v = \frac{n-1+\sqrt{(n-1)^2+8(3-n)D+16D^2}}{4}.$$

Clearly the above result provides a very satisfactory bound on the range of wave speeds. In particular, it shows that $v_{\min}(D) \propto D$ for small $D$. It is also clear that when $n \to 1$, the existence range for $v$ becomes $v \geqslant 2\sqrt{D}$, the standard result, and when $n \to 2$ it is $\lambda \leqslant 1/4$, a result proved in [10].

The main difference of present case from $D > 1$ case is that $A$ and $P$ behaviour differently when $D < 1$. In particular, it will be shown in Lemmas 2.6 and 2.8 that $P < \lambda u$, while $A > \lambda(1+\lambda)u$ in $(0,1)$. In addition, to show existence if $v$ is bigger than a constant multiple of $D$, we cannot rely on the obvious bound of $A < \kappa^2(u+P)/D$, nor does the bound (see Lemma 2.5) derived for $n \geqslant 2$ in [10] is sufficient. Instead, we need a better estimate which comes out only after we sort out the complex relation between $P$ and $A$, and their dependence on $u$.

**Lemma 2.5.** *Suppose $D < 1$. Then $A(u)(1-u)^{n/2} \leqslant \lambda[P+u]$ $\forall u \in [0,1)$.*

**Proof.** When $u = 0$, the two sides are equal. Computation shows, in $(0,1]$,

$$\begin{aligned}
&P\big[(1-u)^{n/2}A - \lambda(P+u)\big]' \\
&\quad = (1-u)^{n/2}\big[\kappa^2(P+u) - DA\big] - \frac{1}{2}nPA(1-u)^{n/2-1} - \lambda A(1-u)^n \\
&\quad \leqslant -\big[D+\lambda(1-u)^{n/2}\big]\big[A(1-u)^{n/2} - \lambda(P+u)\big] + (P+u)\big[\big(\kappa^2-\lambda^2\big)(1-u)^{n/2} - \lambda D\big] \\
&\quad = -\big[D+\lambda(1-u)^{n/2}\big]\big[A(1-u)^{n/2} - \lambda(P+u)\big] - \lambda D(P+u)\big[1-(1-u)^{n/2}\big] \\
&\quad \leqslant -\big[D+\lambda(1-u)^{n/2}\big]\big[A(1-u)^{n/2} - \lambda(P+u)\big].
\end{aligned}$$

Here we have dropped the term $\frac{1}{2}nP(1-u)^{n/2-1}$ in the first inequality and used $\kappa^2 = \lambda(\lambda + D)$ in the second inequality. The assertion of the lemma thus follows from the Gronwall's inequality. □

**Lemma 2.6.** *Suppose $D < 1$. Then $A \geqslant \lambda(P+u)$ and $\lambda u(1-u)^n \leqslant P \leqslant \lambda u$ in* $[0,1]$.

**Proof.** It is easy to show that

$$\begin{aligned}
P\big[A - \lambda(P+u)\big]' &= \kappa^2(P+u) - DA - \lambda A(1-u)^n \\
&= -\big[D+\lambda(1-u)^n\big]\big[A-\lambda(P+u)\big] + (P+u)\big[\kappa^2 - \lambda^2(1-u)^n - \lambda D\big] \\
&> -\big[D+\lambda(1-u)^n\big]\big[A-\lambda(P+u)\big] \quad \text{in } (0,1).
\end{aligned}$$

When $u > 0$ but very small, Taylor expansion at $u = 0$ shows $A > \lambda(P+u)$. It follows from Gronwall's inequality that $A \geqslant \lambda(P+u)$ in $[0,1]$. To show the results concern $P$, we calculate that, with $\eta$ and $\delta$ two positive constants,

$$\big[P - \eta u(1-u)^\delta\big]' = -1 + \frac{A(1-u)^n}{P} - \eta(1-u)^\delta + \eta\delta u(1-u)^{\delta-1}.$$

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Let $\delta = 0$, easy computation shows, by letting $A = \lambda(1+\lambda)u + \mu_1 u^2$, $P = \lambda u + \mu_2 u^2$ around $u = 0$, that when $D < 1$, $\mu_1 > 0$ and $-n\lambda < \mu_2 < 0$. Hence, $P - \lambda u < 0$ when $0 < u \ll 1$. If $P - \lambda u = 0$ at some positive value $u = u_0$, then $(P - \lambda u)' \geqslant 0$ at the same point. But, using the result of Lemma 2.5, we have, at $u = u_0$,

$$[P - \lambda u]' \leqslant -1 + \lambda(1-u)^{n/2} + \frac{\lambda u(1-u)^{n/2}}{P} - \lambda \leqslant -1 + \lambda(1-u)^{n/2} + (1-u)^{n/2} - \lambda < 0.$$

A contradiction! This proves $P \leqslant \lambda u$ in $[0,1]$.

A similar argument shows $P \geqslant \lambda u(1-u)^n$ in $[0,1]$. □

**Lemma 2.7.** *Suppose $D < 1$. Then $A > \lambda(1+\lambda)u$ on $(0,1)$. Consequently, when $\lambda(1+\lambda) > K(n)$, i.e.*

$$v < \sqrt{\frac{D}{K(n)}} \frac{1}{\sqrt{1 - (1 - \frac{1}{D}) \frac{\sqrt{4K(n)+1} - 1}{\sqrt{4K(n)+1} + 1}}},$$

*there is no travelling wave solution to* (2.2).

**Proof.** Direct calculation gives that

$$\begin{aligned} P\big[A - \lambda(1+\lambda)u\big]' &= -D\big[A - \lambda(1+\lambda)u\big] + \frac{D-1}{D+\lambda}[P - \lambda u] \\ &> -D\big[A - \lambda(1+\lambda)u\big] \quad \forall u \in (0,1) \end{aligned}$$

since by Lemma 2.6, $P < \lambda u$. In addition, $A = \lambda(1+\lambda)u + O(u^2) > \lambda(1+\lambda)u$, for all sufficiently small positive $u$, Gronwall's inequality gives $A > \lambda(1+\lambda)u$ on $[0,1)$.

One can show that $P(u) > Q(n, k^2; u)$ for all $u \in (0,1)$ by first using an asymptotic expansion at $u = 0$ for $0 < u \leqslant \epsilon$ and then a comparison principle for the differential equation in $(\epsilon, 1)$.

It then follows from Lemma 2.2 that when $\lambda(1+\lambda) > K(n)$, we must have $P(1) \geqslant Q(n, k^2; 1) > 0$, i.e., there does not exist any solution to the travelling wave problem. □

For the existence part of Theorem 2.2, Lemma 2.5, which is sufficient in case $n \geqslant 2$ is not good enough. We need to derive better estimates.

**Lemma 2.8.** *Suppose $D < 1$ and $1 < n < 2$. Then, $A(1-u)^{n-1} \leqslant \mu(P+u)$ in $[0,1]$ if*

$$\mu \geqslant \frac{\kappa^2}{\lambda(n-1) + D} > \lambda.$$

**Proof.** It is easy to show, using $P > \lambda u(1-u)^n$ in $(0,1)$,

$$\begin{aligned} P\big[(1-u)^{n-1}A - \mu(P+u)\big]' &= (1-u)^{n-1}\big[\kappa^2(P+u) - DA\big] - (n-1)(1-u)^{n-2}PA - \mu A(1-u)^n \\ &\leqslant (1-u)^{n-1}\big[\kappa^2(P+u) - DA\big] - (n-1)\lambda u(1-u)^{2n-2}A - \mu A(1-u)^n \\ &= -\big[D + \mu(1-u) + (n-1)\lambda u(1-u)^{n-1}\big]\big[(1-u)^{n-1}A - \mu(P+u)\big] \\ &\quad + (P+u)\big[\kappa^2(1-u)^{n-1} - \mu^2(1-u) - \mu D - (n-1)\lambda\mu u(1-u)^{n-1}\big]. \end{aligned}$$

Let $h(u) = \kappa^2 - \mu^2(1-u)^{2-n} - \mu D(1-u)^{1-n} - (n-1)\lambda\mu u$, then

$$P\big[(1-u)^{n-1}A - \mu(P+u)\big]' \leqslant -\big[D + \mu(1-u) + (n-1)u\lambda u(1-u)^{n-1}\big]\big[(1-u)^{n-1}A - \mu(P+u)\big]$$

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if $h(u) \leqslant 0$ in $[0,1]$. It is clear that $h(0) \leqslant 0$ if $\mu \geqslant \lambda$, and by elementary computation

$$h(u) \leqslant \kappa^2 - (n-1)\lambda\mu - \mu D(1-u)^{1-n} \leqslant 0 \quad \text{if } \mu \geqslant \frac{\kappa^2}{(n-1)\lambda + D}.$$

This completes the proof of the lemma. □

**Proof of Theorem 2.2.** The non-existence follows directly from Lemmas 2.7. We now prove the existence. Simple computation shows that

$$v \geqslant \frac{2D}{(-D+v^2)^{1/2}} \quad \Longleftrightarrow \quad \mu \equiv \kappa^2/\big[(n-1)\lambda + D\big] \leqslant 1/4.$$

We proceed to show that there exists $\eta > \mu$ such that $P - \eta u(1-u) \leqslant 0$ on $(0,1)$. It is easy to verify, using result of Lemma 2.8, that

$$\begin{aligned}\big[P - \eta u(1-u)\big]' &= -1 + \frac{A}{P}(1-u)^n - \eta(1-2u) \\ &\leqslant -1 + \mu(1-u) + \frac{\mu u(1-u)}{P} - \eta(1-2u).\end{aligned}$$

At the point where $P = \eta u(1-u)$, the right-hand side equals to

$$-1 + \mu(1-u) + \frac{\mu}{\eta} - \eta(1-2u). \tag{2.8}$$

Elementary computation shows that if $\mu \leqslant 1/4$, there exists an $\eta > \mu$ which makes the quantity in (2.8) negative for all $u \in (0,1)$. Hence, there exists $\eta > \mu$ such that $P - \eta u(1-u) \leqslant 0$ on $(0,1)$. In consequence, $P(1) = 0$. This proves the existence, and completes the proof of the theorem. □

## 3. The case of mixed order

In this section we study the system (III). We proceed in the same way as in the case of system (II) so that it is easier to track our proof and a lot of detailed demonstrations can be saved due to their similarity to the previous case.

The equations which govern travelling wave solutions are obtained by looking for a solution of (III) in the form of $\alpha = a(x - vt)$, $\beta = b(x - vt)$. They take the form, with $z = x - vt$,

$$\begin{aligned} &a_{zz} + va_z - \alpha\beta^m - k\alpha\beta^n = 0, \\ &\beta_{zz} + v\beta_z + \alpha\beta^m + k\alpha\beta^n = 0, \end{aligned} \tag{3.1}$$

where $v > 0$ is the constant speed of propagation. Here the two exponents $m$ and $n$ satisfy $m \geqslant 1$ and $n > m$. Finding a travelling wave solution is to solve

$$\begin{cases} \alpha_{zz} + v\alpha_z = \alpha\beta^m + k\alpha\beta^n, & \alpha \geqslant 0,\ \forall z \in \mathbb{R}, \\ D\beta_{zz} + v\beta_z = -\alpha\beta^m - k\alpha\beta^n, & \beta \geqslant 0,\ \forall z \in \mathbb{R}, \\ \lim\limits_{z\to\infty} \big(\alpha(z), \beta(z)\big) = (1,0), \\ \lim\limits_{z\to-\infty} \big(\alpha(z), \beta(z)\big) = (0,1). \end{cases} \tag{3.2}$$

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### 3.1. Some basic properties of travelling waves

Suppose $(v, \alpha, \beta)$ solves (3.2). Then $[\alpha_z + v\alpha + D\beta_z + v\beta]_z = 0$ so that $\alpha_z + D\beta_z + v(\alpha+\beta)$ is a constant function. Using the boundary conditions, we find that

$$\alpha_z + D\beta_z + v(\alpha+\beta-1) = 0 \quad \text{on } \mathbb{R}.$$

With the new variable $w = \beta_z$, (3.2) is equivalent to the following third order ODE system

$$\begin{cases} \alpha_z = v(1-\alpha-\beta) - Dw, \\ \beta_z = w, \\ w_z = -D^{-1}\big(\alpha\beta^m + k\alpha\beta^n + vw\big), \\ \lim\limits_{z\to\infty}\big(\alpha(z), \beta(z), w(z)\big) = (1,0,0), \\ \lim\limits_{z\to-\infty}\big(\alpha(z), \beta(z), w(z)\big) = (0,1,0). \end{cases} \tag{3.3}$$

It is clear that in the $(\alpha, \beta, w)$ phase space, there are two equilibrium points: $(0,1,0)$ and $(1,0,0)$. The following is a few basic properties of travelling wave solutions.

**Proposition 3.1.** *The systems* (3.2) *and* (3.3) *are equivalent. Any solution* $(\alpha, \beta)$ *to* (3.2) *or* $(\alpha, \beta, w)$ *to* (3.3) *has the following properties*:

1. $\alpha_z > 0 > \beta_z$ *on* $\mathbb{R}$;
2. $\alpha + \beta < 1$ *on* $\mathbb{R}$ *if* $D < 1$, $\alpha + \beta \equiv 1$ *if* $D = 1$, *and* $\alpha + \beta > 1$ *if* $D > 1$;
3. $v = \int_{-\infty}^{\infty} \alpha(z)\beta^m(z) + k\alpha(z)\beta^n(z)\,dz > 0$;
4. *The equilibrium point* $(0,1,0)$ *of* (2.3) *is a saddle with a two-dimensional stable manifold and a one-dimensional unstable manifold. The eigenvalues and associated eigenvectors are*:

$$\lambda_1 = -vD^{-1}, \qquad \mathbf{e}_{\lambda_1} = (0, -1, -\lambda_1)^T,$$
$$\lambda_2 = -\frac{1}{2}\big(\sqrt{v^2 + 4(1+k)} + v\big), \qquad \mathbf{e}_{\lambda_2} = \big(\lambda_2(D\lambda_2 + v), -(1+k), -(1+k)\lambda_2\big)^T,$$
$$\lambda_3 = \frac{1}{2}\big(\sqrt{v^2 + 4(1+k)} - v\big), \qquad \mathbf{e}_{\lambda_3} = \big(\lambda_3(D\lambda_3 + v), -(1+k), -(1+k)\lambda_3\big)^T;$$

5. *When* $m = 1$, *the equilibrium point* $(1,0,0)$ *is a sink. The eigenvalues and associated eigenvectors are*:

$$\mu_1 = -v, \qquad \mathbf{e}_{\mu_1} = (1,0,0)^T,$$
$$\mu_2 = \frac{1}{2D}\big(-v + \sqrt{v^2 - 4D}\big), \qquad \mathbf{e}_{\mu_2} = \big(-(v + D\mu_2)/(v + \mu_2), 1, \mu_2\big)^T,$$
$$\mu_3 = -\frac{1}{2D}\big(v + \sqrt{v^2 - 4D}\big), \qquad \mathbf{e}_{\mu_3} = \big(-(v + D\mu_3)/(v + \mu_3), 1, \mu_3\big)^T;$$

6. *When* $m > 1$, *the equilibrium point* $(1,0,0)$ *is degenerate*; *it has a two-dimensional stable manifold and a one-dimensional center manifold. The eigenvalues and associated eigenvectors are*:

$$\mu_1 = -v, \qquad \mathbf{e}_{\mu_1} = (1,0,0)^T,$$
$$\mu_2 = -vD^{-1}, \qquad \mathbf{e}_{\nu_2} = \big(0, 1, -vD^{-1}\big)^T,$$
$$\mu_3 = 0, \qquad \mathbf{e}_{\mu_3} = (1,-1,0)^T.$$

All items except (3) were proven in either [6] or [12]. The equation in (3) is obtained by integrating the equation involving $\alpha_{zz}$ in (3.2) with the boundary conditions $\alpha(\infty) = 1$ and $\alpha(-\infty) = 0$.

The third property in the proposition demonstrates that $v > 0$. The fourth property clearly tells us that the travelling wave we are looking for is indeed the one-dimensional unstable manifold associated with the equilibrium $(0, 1, 0)$. Hence, given $v > 0$, a travelling wave of speed $v$, if it exists, is unique up to a translation.

### 3.2. *New setting—A non-autonomous 2 × 2 system*

Different from earlier work in [12], here we shall use a transformation turning the third order autonomous system (3.3) into a second order non-autonomous system, using $u := 1 - \beta$ as the independent variable. This is allowed since for the solution of interest, $\beta_z < 0$, so $z \to 1 - \beta(z)$ has an inverse. To make the resulting system as simple as possible, we also scale the variables. Hence, we introduce

$$u = 1 - \beta, \qquad A = \frac{D\alpha}{v^2}, \qquad y = \frac{vz}{D}, \qquad \sigma := \frac{D}{v}.$$

The system of differential equations (2.2) becomes

$$\begin{cases} u_{yy} + u_y = A\big((1-u)^m + k(1-u)^n\big) & \text{on } \mathbb{R}, \\ A_y = \sigma^2(u + u_y) - DA & \text{on } \mathbb{R}. \end{cases}$$

Since $u_y > 0$ for the solution of interest, we can use $u$ as the independent variable. Introducing $P(u) = u_y$, we have an equivalent system of second order non-autonomous (singular) ODEs

$$\begin{cases} PP' = A\big[(1-u)^m + k(1-u)^n\big] - P & \forall u \in [0,1], \\ PA' = \sigma^2[P + u] - DA & \forall u \in [0,1], \\ P(u) > 0, \quad A(u) > 0 & \forall u \in (0,1), \\ P(0) = 0, \quad A(0) = 0. \end{cases} \tag{3.4}$$

**Lemma 3.1.** *For every $D > 0$ and $\sigma > 0$, (3.4) admits a unique solution. In addition,*

$$P(u) = \lambda u + O\big(u^2\big), \qquad A(u) = \frac{\lambda(1+\lambda)}{1+k} u + O\big(u^2\big) \quad \text{as } u \searrow 0, \tag{3.5}$$

*where*

$$\lambda := \frac{1}{2}\big(\sqrt{4\sigma^2(1+k) + D^2} - D\big) \quad \big(\text{the only positive root to } \lambda(\lambda + D) = \sigma^2(1+k)\big).$$

*Furthermore, $A'(u) > 0$ for all $u \in [0, 1)$ and there are only two possible cases:*

(a) $P(1) > 0$; *there does not exist any travelling wave solution to* (2.2).
(b) $P(1) = 0$; *there exists a travelling wave solution to* (2.2), *unique up to translation.*

**Proof.** The proof is now a routine exercise, and we omit it. □

### 3.3. A scalar equation

Next we review the existence of travelling wave of unit speed to the equation

$$u_{zz} + u_z = lu\big[(1-u)^m + k(1-u)^n\big], \quad 0 \leqslant u \leqslant 1 \text{ on } \mathbb{R}, \qquad u(-\infty) = 0, \qquad u(\infty) = 1. \tag{3.6}$$

Here $n > m \geqslant 1$ are parameters, and $l$ and $k$ are positive constants. We seek upper bounds on $l$ for the existence of a solution, for given $m$, $n$ and $k$. Since a solution, if it exists, satisfies $u_z > 0$ on $\mathbb{R}$, we can write $u' = Q(u)$ and work on the $(u, Q)$ phase plane. The resulting equation on the phase plane is

$$\begin{cases} QQ' + Q = lu\big[(1-u)^m + k(1-u)^n\big] & \forall u \in [0,1], \\ Q(0) = 0, \quad Q > 0 \text{ on } (0,1). \end{cases} \tag{3.7}$$

There is a one-to-one correspondence between solutions to (3.6) and solutions to (3.7) satisfying the additional requirement $Q(1) = 0$.

**Lemma 3.2.** *For each $n > m \geqslant 1$ and $l, k > 0$, there exists a unique solution $Q = Q(m,n,k,l;\cdot)$ to* (3.7)*. In addition, there exists a positive constant $L(m,n,k)$ such that $Q(m,n,k,l;1) = 0$ if $l \in (0, L(m,n,k)]$ and $Q(m,n,k,l;1) > 0$ if $l \in (L(m,n,k), \infty)$. Consequently,* (3.7) *admits a solution if and only if $l \in (0, L(m,n,k)]$.*

*In addition, $L(m,n,k)$ is a strictly increasing function of $m$ and $n$. Moreover, if $m = 1$ and $n = 2$,*

$$L(1,2,k) = \begin{cases} \frac{1}{4} & \text{if } k \leqslant 2, \\ \frac{2k}{(2+k)^2} & \text{if } k > 2. \end{cases} \tag{3.8}$$

**Proof.** The existence of $Q$ and $L$ follows by the comparison principle. The exact value of $L(1,2,k)$ is calculated in [4,30]. We omit details, because it is a standard argument. □

### 3.4. The case of $D > 1$

In this subsection we demonstrate how to deal with the case of $D > 1$.

**Theorem 3.1.** *Suppose $D \geqslant 1$ and $n > m \geqslant 1$. There exists a positive constant $v_{\min}$ such that* (3.2) *admits a travelling wave if and only if $v \geqslant v_{\min}$. In addition, $v_{\min}$ is bounded by*

$$\sqrt{\frac{D}{L(m,n,k)}} \leqslant v_{\min} \leqslant \sqrt{\frac{D}{L(m,n,k)}} \frac{1}{\sqrt{1 - (1 - \frac{1}{D})\frac{\sqrt{\Delta+1}-1}{\sqrt{\Delta+1}+1}}},$$

*where $\Delta = 4L(m,n,k)(1+k)$, with $L(m,n,k)$ being the same constant as in Lemma* 3.2*. In particular, when $m = 1, n = 2$,*

$$2\sqrt{D} \leqslant v_{\min} \leqslant \frac{2\sqrt{D}}{\sqrt{1 - (1 - \frac{1}{D})\frac{\sqrt{2+k}-1}{\sqrt{2+k}+1}}} \quad \text{if } k \leqslant 2,$$

$$\sqrt{D}\left(\sqrt{\frac{k}{2}} + \sqrt{\frac{2}{k}}\right) \leqslant v_{\min} \leqslant \sqrt{D}\left(\sqrt{\frac{k}{2}} + \sqrt{\frac{2}{k}}\right) \frac{1}{\sqrt{1 - (1 - \frac{1}{D})\frac{k}{2(k+1)}}} \quad \text{if } k > 2.$$



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**Remark.** It is easy to show that when $m=1$, $n=2$, and $k \geqslant 2$, if

$$k > \frac{D-1+\sqrt{D^2+6D+1}}{2},$$

the upper bound of $v_{\min}$ given in Theorem 3.1 is better than that derived in [12].

**Lemma 3.3.** *Suppose $D \geqslant 1$. Then $DA(u) \geqslant \sigma^2 u$ for all $u \in [0,1]$. Consequently, there is no travelling wave solution to* (3.2) *when $\sigma^2 > DL(m,n,k)$, i.e., when $v < \sqrt{D/L(m,n,k)}$.*

**Proof.** The proof is exactly as in Lemma 2.3, and we omit the details. □

The following lemma is very much similar to their counterpart of Lemmas 2.4 in Section 2 and we therefore only state the results.

**Lemma 3.4.** *Suppose $D > 1$. Then,*

$$A(u) < \frac{\lambda(1+\lambda)}{1+k}u, \qquad P(u) < \lambda u \quad \forall u \in (0,1).$$

*Consequently, there exists a travelling wave solution to* (3.2) *when $\lambda(\lambda+1) \leqslant (1+k)L(m,n,k)$, i.e. when*

$$v \geqslant \sqrt{\frac{D}{L(m,n,k)}} \frac{1}{\sqrt{1-(1-\frac{1}{D})\frac{\sqrt{\Delta+1}-1}{\sqrt{\Delta+1}+1}}},$$

*where $\Delta = 4L(m,n,k)(1+k)$.*

**Proof of Theorem 3.1.** The upper and lower bounds of $v_{\min}$ follow directly from Lemmas 3.3 and 3.4. When $m=1$, $n=2$, use the explicit expression of $L(1,2,k)$. As to the existence of minimum speed $v_{\min}$, the proof follows exactly from the same argument as in Theorem 2.1, and we omit it. □

### 3.5. The case of $D < 1$

Like the situation of single auto-catalysis reaction, the case of $D < 1$ needs more elaborate approach. But, luckily most of the arguments in Section 2 can be carried over with modification, which greatly simplifies the proof here.

**Theorem 3.2.** *Suppose $D < 1$. Then, there exists no travelling wave to* (III) *if*

$$v < \sqrt{\frac{D}{L(m,n,k)}} \frac{1}{\sqrt{1-(1-\frac{1}{D})\frac{\sqrt{\Delta+1}-1}{\sqrt{\Delta+1}+1}}}.$$

*But, there exists a travelling wave to* (III) *if*

$$v \geqslant \begin{cases} \frac{4D(1+k)}{\sqrt{1+4D(1+k)^{1/2}}} & \text{if } m \geqslant 2, \\ \frac{2D}{(-D^2+v^2)^{1/2}} & \text{if } 1 < m < 2, \end{cases}$$

*where*

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$$\nu=\begin{cases}\frac{(m-1)(1-D)+\sqrt{(m-1-4D)^2+4(5-m)D(1+k)}}{5-m} & \text{if } n\leqslant 2m,\\ \frac{m-1+\sqrt{(m-1-4D)^2+16D(1+k)}}{4} & \text{if } n>2m.\end{cases}$$

**Lemma 3.5.** *Suppose $D<1$ and $m\geqslant 1$, then*

$$(1+k)^{1/2}(1-u)^{m/2}A-\lambda(P+u)\leqslant 0 \quad \forall u\in[0,1] \tag{3.9}$$

*and*

$$(1+k)A\geqslant \lambda(P+u) \quad \forall u\in[0,1]. \tag{3.10}$$

**Proof.** Calculation shows that

$$\begin{aligned}&P\big((1+k)^{1/2}(1-u)^{m/2}A-\lambda(P+u)\big)'\\&\quad=(1+k)^{1/2}(1-u)^{m/2}\big[\sigma^2(P+u)-DA\big]\\&\qquad-\frac{(1+k)^{1/2}}{2}mPA(1-u)^{m/2-1}-\lambda A\big[(1-u)^m+k(1-u)^n\big]\\&\quad<-\left(D+\frac{\lambda}{(1+k)^{1/2}}(1-u)^{m/2}+\frac{k\lambda}{(1+k)^{1/2}}(1-u)^{n-m/2}\right)\big((1+k)^{1/2}(1-u)^{m/2}A-\lambda(P+u)\big)\\&\qquad+(1+k)^{1/2}(P+u)\left(\left(\sigma^2-\frac{\lambda^2}{(1+k)}\right)(1-u)^{m/2}-\frac{k\lambda^2}{k+1}(1-u)^{n-m/2}-\frac{\lambda D}{(1+k)^{1/2}}\right)\\&\quad<-\left(D+\frac{\lambda}{(1+k)^{1/2}}(1-u)^{m/2}+\frac{k\lambda}{(1+k)^{1/2}}(1-u)^{n-m/2}\right)\big((1+k)(1-u)^{m/2}A-\lambda(P+u)\big)\end{aligned}$$

since $\sigma^2(1+k)=\lambda^2+\lambda D$. Hence, (3.9) follows from a standard argument.

Similarly, to show (3.10), we compute

$$\begin{aligned}P\big[(1+k)A-\lambda(P+u)\big]'&=(1+k)\big[\sigma^2(P+u)-DA\big]-\lambda A\big[(1-u)^m+k(1-u)^n\big]\\&<-\left(D+\frac{\lambda}{(1+k)}(1-u)^m+\frac{k\lambda}{(1+k)}(1-u)^n\right)\big((1+k)A-\lambda(P+u)\big)\\&\quad+(1+k)(P+u)\left(\sigma^2-\frac{\lambda^2}{(1+k)^2}(1-u)^m-\frac{k\lambda^2}{(k+1)^2}(1-u)^n-\frac{\lambda D}{1+k}\right)\\&\geqslant-\left(D+\frac{\lambda}{(1+k)}(1-u)^m+\frac{k\lambda}{(1+k)}(1-u)^n\right)\big((1+k)A-\lambda(P+u)\big)\end{aligned}$$

since $\sigma^2(1+k)=\lambda^2+\lambda D$. The rest follows from the Gronwall's inequality. Thus, (3.10) holds. □

**Lemma 3.6.** *Suppose $D<1$. Then $A\geqslant\lambda(1+\lambda)u/(1+k)$ and $\mu_0u(1-u)^m\leqslant P\leqslant\lambda u$ in $[0,1]$, where*

$$\mu_0=\begin{cases}\frac{\lambda(1+\lambda)}{1+k} & \text{if } k>\lambda \text{ and } n\leqslant 2m,\\ \left(\frac{\lambda(1+\lambda)k}{1+k}\right)^{1/2} & \text{if } k\leqslant\lambda \text{ and } n\leqslant 2m,\\ (-1+\sqrt{1+4\lambda(1+\lambda)/(1+k)})/2 & \text{if } n>2m.\end{cases}$$

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**Proof.** $A \geqslant \lambda(1+\lambda)u/(1+k)$ and $P \leqslant \lambda u$ follow from the same line of argument as in Lemmas 2.6 and 2.7. To prove $\mu_0 u(1-u)^m \leqslant P$, simple computation shows

$$\begin{aligned} &P\big[P-\mu u(1-u)^m\big]' \\ &\quad = -\big[P-\mu u(1-u)^m\big]\big[1+\mu(1-u)^m-\mu m u(1-u)^{m-1}\big] \\ &\qquad + A(1-u)^m + Ak(1-u)^n - \mu u(1-u)^m - \mu^2 u(1-u)^{2m} + \mu^2 m u^2(1-u)^{2m-1}. \end{aligned}$$

The crucial thing is to prove that the first two positive terms with $A$ in them can bound the next two negative terms. Notice that if $n \leqslant 2m$, $k > \lambda$ and $\mu < \lambda(1+\lambda)/(1+k)$ imply $\mu^2 < \lambda(1+\lambda)k/(1+k)$. Hence,

$$P\big[P-\mu u(1-u)^m\big]' > -\big[P-\mu u(1-u)^m\big]\big[1+\mu(1-u)^m-\mu m u(1-u)^{m-1}\big],$$

and when $u$ is very close to zero, $P > \mu_0 u(1-u)^m$, it follows that $P \geqslant \mu_0 u(1-u)^m$ in $[0,1]$ when $n \leqslant 2m$, $k > \lambda$. Similarly, the inequality holds when $n \leqslant 2m$, $k \leqslant \lambda$. The final case of $n > 2m$ is easy, because it just uses the first term $A(1-u)^m$ to bound the two negative terms. This completes the proof of lemma. □

The result of Lemma 3.5, which is good for proving Theorem 3.2 when $m \geqslant 2$, is not so useful when $1 < m < 2$. The better estimate is given in the following lemma.

**Lemma 3.7.** *Suppose $D < 1$ and $1 < m < 2$. Then,*

$$(1-u)^{m-1}A - \eta(P+u) \leqslant 0 \quad \forall u \in [0,1] \tag{3.11}$$

*provided*

$$\eta \geqslant \Theta \equiv \frac{\sigma^2}{(m-1)\mu_0 + D} > \frac{\lambda}{1+k}. \tag{3.12}$$

**Proof.**

$$\begin{aligned} &P\big[(1-u)^{m-1}A - \eta(P+u)\big]' \\ &\quad = (1-u)^{m-1}\big[\sigma^2(P+u) - DA\big] - (m-1)(1-u)^{m-2}PA - \eta A(1-u)^m - \eta Ak(1-u)^n \\ &\quad \leqslant -\big[(1-u)^{m-1}A - \eta(P+u)\big]\big[D + \eta(1-u) + \eta k(1-u)^{n+1-m} + (m-1)\mu u(1-u)^{m-1}\big] \\ &\qquad + (P+u)\big[(1-u)^{m-1}\sigma^2 - \eta^2(1-u) - \eta^2 k(1-u)^{n+1-m} - (m-1)\mu_0\eta u(1-u)^{m-1} - \eta D\big], \end{aligned}$$

where we used the result in Lemma 3.6 that $P \geqslant \mu_0 u(1-u)^m$. It is clear that if

$$(1-u)^{m-1}\sigma^2 - \eta^2(1-u) - \eta^2 k(1-u)^{n+1-m} - (m-1)\mu_0\eta u(1-u)^{m-1} - \eta D \leqslant 0,$$

then the desired result is a straight forward application of Gronwall's inequality. The inequality holds at $u = 0$ if $\eta \geqslant \lambda/(1+k)$. Let

$$H(u) = \sigma^2 - \eta^2(1-u)^{2-m} - \eta^2 k(1-u)^{n+2-2m} - (m-1)\mu_0\eta u - \eta D(1-u)^{1-m}.$$

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Simple calculation shows

$$H(u) \leqslant \sigma^2 - \frac{(m-1)\eta\mu_0}{1+k} - \eta D(1-u)^{1-m} \leqslant 0$$

when (3.12) holds. This completes the proof of the lemma. □

**Proof of Theorem 3.2.** The non-existence follows directly from Lemma 3.6 since $A > \lambda(1+\lambda)u/(1+k)$ in $(0,1)$. Now, we prove the existence by considering different combination of $m$ and $n$.

**Case I.** $m \geqslant 2$.

$$\begin{aligned}\big[P - \eta u(1-u)\big]' &= -1 - \eta(1-2u) + \frac{A}{P}\big[(1-u)^m + k(1-u)^n\big] \\ &\leqslant -1 - \eta(1-2u) + \frac{\lambda}{(1+k)^{1/2}}\big((1-u)^{m/2} + k(1-u)^{n-m/2}\big) \\ &\quad + \frac{\lambda u}{(1+k)^{1/2}P}\big((1-u)^{m/2} + k(1-u)^{n-m/2}\big),\end{aligned}$$

by Lemma 3.5. We proceed to show that if $\lambda(1+k)^{1/2} \leqslant 1/4$, there exists $\eta > \lambda$ such that $[P - \eta u(1-u)] \leqslant 0$ in $[0,1]$, and hence the resulting solution is a travelling wave to (III). If $P - \eta u(1-u) = 0$ at some $u > 0$, then at this point,

$$\begin{aligned}\big[P - \eta u(1-u)\big]' &\leqslant -1 - \eta(1-2u) + \frac{\lambda}{(1+k)^{1/2}}\big((1-u)^{m/2} + k(1-u)^{n-m/2}\big) \\ &\quad + \frac{\lambda}{(1+k)^{1/2}\eta}\big((1-u)^{m/2-1} + k(1-u)^{n-m/2-1}\big),\end{aligned}$$

but the last term is negative in $[0,1]$ with probable choice of $\eta > \lambda$, a contradiction. Hence, $[P - \eta u(1-u)] \leqslant 0$ in $[0,1]$.

**Case II.** $1 < m < 2$ and $n \leqslant 2m$.

Similar to the previous case,

$$\begin{aligned}\big[P - \eta u(1-u)\big]' &\leqslant -1 - \eta(1-2u) + \Delta(1-u) + \frac{\Theta u(1-u)}{P} \\ &\quad + k\Theta(1-u)^{n+1-m} + k\Theta\frac{(1-u)^{n+1-m}}{P}\end{aligned} \tag{3.13}$$

by using the result of (3.11) in Lemma 3.7, where

$$\Theta = \frac{\sigma^2}{(m-1)\mu_0 + D}.$$

It is then easy to verify that there exists $\eta > \lambda$ such that $[P - \eta u(1-u)] \leqslant 0$ in $[0,1]$ if $\Theta \leqslant 1/4$. Now, choose $\mu_0 = \lambda(1+\lambda)/(1+k)$, with $\lambda < k$ satisfied if $4\sigma^2(1+k) < k^2$. The two conditions

$$\Theta \leqslant 1/4 \quad \text{with } \mu_0 = \lambda(1+\lambda)/(1+k) \quad \text{and} \quad 4\sigma^2(1+k) < k^2$$

are equivalent to

Q1

$$\nu > \max\left(\frac{2D(1+k)^{1/2}}{k}, \frac{2D(1+k)^{1/2}}{(-D^2+\nu^2)^{1/2}}\right)$$

with

$$\nu = \frac{(m-1)(1-D) + \sqrt{(m-1-4D)^2 + 4(5-m)D(1+k)}}{5-m}.$$

**Case III.** $1 < m < 2$ and $n > 2m$.

In this case, what we need to do is to use the $\Theta$ with $\mu_0 = (-1 + \sqrt{1 + 4\lambda(1+\lambda)/(1+k)})/2$ in (3.13). To simplify the computation, we can replace $\mu_0$ in the definition of $\Theta$ by $\lambda/(1+k)$, since $\mu_0 > \lambda/(1+k)$. The condition $\Theta \leqslant 1/4$ then yields

$$\nu \geqslant \frac{2D(1+k)^{1/2}}{(-D^2+\nu^2)^{1/2}} \quad \text{with } \nu = \frac{m-1+\sqrt{(m-1-4D)^2 + 16D(1+k)}}{4}.$$

This proves the theorem. □

## Uncited references

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