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Self-similar singular solutions of a p-Laplacian evolution equation with absorption ☆

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Abstract

We consider, for $p\in(1,2)$ and $q>1$, self-similar singular solutions of the equation $v_t = \mathrm{div}(|\nabla v|^{p-2}\nabla v) - v^q$ in $R^n\times(0,\infty)$; here by *self-similar* we mean that v takes the form $v(x,t)=t^{-\alpha}w(|x|t^{-\alpha\beta})$ for $\alpha=1/(q-1)$ and $\beta=(q+1-p)/p$, whereas *singular* means that v is non-negative, non-trivial, and $\lim_{t\searrow 0} v(x,t)=0$ for all $x\neq 0$. That is, we consider the ODE problem

$$\begin{cases} (|w'|^{p-2}w')' + (n-1)|w'|^{p-2}w'/r + \alpha(\beta r w' + w) - w^q = 0 & \forall r>0,\\ w'(0)=0, \quad w(r)\geqslant 0 \quad \text{in } [0,\infty), \quad \lim_{r\to\infty} r^{1/\beta}w(r)=0. \end{cases} \tag{0.1}$$

We show that this ODE problem has a non-trivial solution if and only if $q<p-1+p/n$ (i.e., if and only if $n\beta<1$), and in case of existence, the solution is unique and the corresponding self-similar singular solution of the PDE satisfies

$$\int_{R^n} v^{n\beta}(x,\cdot)\,dx = \text{constant}, \quad \lim_{t\searrow 0}\int_{|x|<\varepsilon} v(x,t)\,dx = \infty \quad \forall \varepsilon>0.$$

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1. Introduction and main results

In this paper, we consider, for $p \in (1,2)$ and $q > 1$, self-similar singular solutions of the p-Laplacian evolution equation with absorption

$$v_t = \mathrm{div}(|\nabla v|^{p-2}\nabla v) - v^q \quad \text{in } R^n \times (0,\infty). \tag{1.1}$$

Here by a *singular* solution we mean a non-negative and non-trivial solution which is continuous in $R^n \times [0,+\infty)\backslash\{(0,0)\}$ and satisfies

$$\lim_{t\searrow 0} \sup_{|x|>\varepsilon} v(x,t) = 0 \quad \forall \varepsilon > 0. \tag{1.2}$$

That is, we restrict our attention to solutions having an isolated singularity at $(0,0)$. Also, by *self-similar* we mean that v has the form

$$v(x,t) = \left(\frac{\alpha}{t}\right)^{\alpha} u\left(|x|\left(\frac{\alpha}{t}\right)^{\alpha\beta}\right), \quad \alpha := \frac{1}{q-1}, \quad \beta := \frac{q+1-p}{p}, \tag{1.3}$$

where u, defined on $[0,\infty)$, solves

$$(|u'|^{p-2}u')' + \frac{n-1}{r}|u'|^{p-2}u' + \beta r u' + u - |u|^{q-1}u = 0 \quad \forall r > 0. \tag{1.4}$$

Note that condition (1.2) is equivalent to, if v is given by (1.3),

$$\lim_{r\to\infty} r^{1/\beta}u(r) = 0. \tag{1.5}$$

Singular solutions were first studied for the semilinear heat equation

$$v_t = \Delta v - v^q. \tag{1.6}$$

Brezis and Friedman [1] in 1983 proved that when $q \geqslant 1 + 2/n$, (1.6) has no singular solution, whereas when $q \in (1, 1+2/n)$, it has, for every $c \in (0,\infty)$, a unique singular solution that satisfies $v(\cdot,0) = c\delta(\cdot)$, i.e.,

$$\lim_{t\searrow 0} \int_{|x|<\varepsilon} v(x,t)\,\mathrm{d}x = c \quad \forall \varepsilon > 0. \tag{1.7}$$

Such a singular solution is referred as a *fundamental solution* (FS for short) with initial mass c. Shortly after the work of Brezis and Friedman [1], Brezis et al. [2] discovered that when $q \in (1, 1+2/n)$, (1.6) admits a unique singular solution which is more singular than any FS. Such a singular solution is termed as a *very singular solution* (VSS for short) and it satisfies (1.7) with $c = \infty$. See also [6,7] and references therein. All singular solutions of (1.6) was later classified by Oswald [14].

These pioneer works on (1.6) were soon extended to the porous medium equation

$$v_t = \Delta v^m - v^q. \tag{1.8}$$

For $m>1$, i.e., slow diffusion case, Kamin, Peletier and Vazquez [9], after earlier works of Kamin and Peletier [8], Kamin and Veron [12] and Peletier and Terman [15], classified in 1989 all singular solutions of (1.8) as follows: (1) any singular solution is either an FS or a VSS; (2) when $q \geqslant m+2/n$, (1.8) has no singular solution at all; (3) when $q \in (m, m+2/n)$, there exists a unique VSS and in addition, for every $c>0$, a unique FS with initial mass c; and (4) when $q \in (1, m]$, there does not exist any VSS but there exists, for every $c>0$, a unique FS with initial mass c.

For fast diffusion case, i.e., $m \in (0,1)$, there were some partial results of Peletier and Zhao [17] in 1990 who proved that when $m \in (\max\{n-2, 0\}/n,\ 1)$, (1.8) has both FS and VSS if $q \in (1, m+2/n)$ and no singular solution when $q \geqslant m+2/n$, and of Leoni [13] in 1996 who proved that when $m \in (0,1)$ and $q>1$, (1.8) has a *self-similar* VSS if and only if $m > \frac{1}{n}\max\{0, n-2\}$ and $q \in (1,\ m+2/n)$.

In a forthcoming paper [3], we shall completely classify all singular solutions of (1.8) in a manner similar to that of [9] for the case $m \in (0,1),\ \ q>1$.

Investigation on singular solutions was at the same time also carried out for the p-Laplacian evolution equation (1.1). For $p>2$, i.e., the degenerate case, after works of Kamin and Vazquez [10] on FSs and [5,16] on existence and uniqueness of self-similar VSSs, Kamin and Vazquez [11] provided a complete classification for all singular solutions of (1.1), which is the same as that for the porous medium equation (1.8) except the borderlines $q=m+2/n$ and $q=m$ are replaced by $q=p-1+p/n$ and $q=p-1$, respectively. Actually, they established the above classification for $u_t = \mathrm{div}(|\nabla u|^{p-2}\nabla u) - \phi(u)$ where $\phi(\cdot)$ is in certain class of non-negative functions including u^q.

In another forthcoming paper [4], we shall, based on the result of the current paper, give a complete classification as that in [11] for all singular solutions of (1.1) for $p \in (1,2),\ \ q>1$.

Here, in this paper, we consider only existence and uniqueness of self-similar singular solutions; namely we consider non-negative and non-trivial solutions of (1.4) that satisfies (1.5). Throughout this paper, we always assume that

$$1<p<2, \quad q>1. \tag{1.9}$$

Our main result is the following:

Theorem 1.1. *Assume* (1.9). *Then* (1.4) *has a non-negative and non-trivial solution that satisfies* $u'(0)=0$ *and* (1.5) *if and only if* $q<p-1+p/n$. *Also, in case of existence* (*i.e.*, $q<p-1+p/n$), *the solution is unique and the corresponding function* v *given by* (1.3) *is a self-similar VSS of* (1.1) *satisfying*

$$\int_{R^n} v^{n\beta}(x,\cdot)\,dx = constant, \quad \lim_{t\searrow 0}\int_{|x|<\varepsilon} v(x,t)\,dx = \infty \quad \forall \varepsilon>0.$$

Note that $q<p-1+p/n$ is equivalent to $n\beta<1$. Our theorem implies that if (1.1) has a self-similar singular solution v, then $n\beta<1$ and v is a VSS. This conclusion can be readily seen from the following arguments (where discussion of the case $n\beta=1$ is formal):

Let $v(x,t)$ be non-negative, non-trivial, and self-similar of the form (1.3) with u satisfying (1.5). Then for every $t>0$ and $\varepsilon>0$,

$$\int_{|x|<\varepsilon} v(x,t)\,dx = \left(\frac{t}{\alpha}\right)^{\alpha(n\beta-1)} \int_{|y|<\varepsilon(\alpha/t)^{\alpha\beta}} u(y)\,dy. \tag{1.10}$$

If $n\beta>1$, then (1.5) yields $\int_{|y|<R} u(y) = o(R^{n-1/\beta})$ as $R\to\infty$, so (1.10) gives that $\int_{|x|<\varepsilon} v(x,t)\,dx = o(1)$ where $o(1)\to 0$ as $t\to 0$. As a singular solution is either an FS or VSS (cf. [4] or [9]), we see that v cannot be a singular solution of (1.1). Thus (1.1) cannot have any self-similar singular solution.

If $n\beta<1$, then (1.10) implies that $\int_{|x|<\varepsilon} v(x,t)\,dx\to\infty$ as $t\searrow 0$, so v must be a VSS if it is a self-similar singular solution.

Finally, we consider the case $n\beta=1$. Assume further that (1.5) is strengthened to $r^{1/\beta}u = O(r^{-\delta})$ for some $\delta>0$. Then letting $\varepsilon\to\infty$ in (1.10) one sees that the $L^1(R^n)$ norm of $v(\cdot,t)$ is independent of t. As (1.1) contains an absorption term v^q, v cannot be a solution of (1.1), so (1.1) does not have any self-similar singular solution.

To prove Theorem 1.1, it suffices to consider the solution of (1.4) with initial value

$$u(0)=a, \quad u'(0)=0. \tag{1.11}$$

We need only consider the case $a\in(0,1)$ since $a\geqslant 1$ implies that the solution is non-decreasing. Theorem 1.1 follows from the following more detailed result on solutions of the initial value problem (1.4) and (1.11).

Theorem 1.2. *Assume* (1.9). *For each* $a\in(0,1)$ *let* $u(r;a)$ *be the solution of* (1.4), (1.11). *Then the following hold*:

(I) *If* $n\beta\geqslant 1$, *then* $u>0$ *and* $u'<0$ *in* $(0,\infty)$ *and* $\liminf_{r\to\infty} r^{1/\beta}u(r;a)>0$.

(II) *If* $n\beta<1$, *then there exists* $a^*\in(0,1)$ *such that the following hold*:

(a) *If* $a\in(0,a^*)$, *then there exists* $R(a)<\infty$ *such that* $u'<0$ *in* $(0,R(a)]$ *and* $u(R(a);a)=0$.

(b) *If* $a\in(a^*,1)$, *then* $u'<0$, $u>0$, $u_a := \frac{d}{da}u>0$, *and* $(r^{p/(2-p)}u)'>0$ *in* $(0,\infty)$. *In addition*, $\lim_{r\to\infty} r^{1/\beta}u(r;a)$ *has a finite limit* $k(a)$ *which, as a function of* a *defined on* $(a^*,1)$, *is positive, continuous and non-decreasing, and satisfies* $\lim_{a\searrow a^*} k(a)=0$ *and* $\lim_{a\nearrow 1} k(a)=\infty$.

(c) *If* $a=a^*$, *then* $\lim_{r\to\infty} r^{\mu}u(r;a^*)=J^*$, *where*

$$\mu=\frac{p}{2-p}, \quad J^* = \left\{\frac{\mu^{p-1}(\mu-n)}{\beta\mu-1}\right\}^{1/(2-p)}. \tag{1.12}$$

Observe that $\beta\mu = (q+1-p)/(2-p) > 1$. The assertion II(b) of the above theorem implies the following:

Theorem 1.3. *Assume that* $p \in (1,2)$ *and* $1 < q < p - 1 + p/n$. *Then for every* $k > 0$, *there exists a self-similar solution* v_k *to* (1.1) *such that*

$$v(x,0) = k|x|^{-1/\beta}, \quad i.e., \quad \lim_{t \searrow 0} v_k(x,t) = k|x|^{-1/\beta}, \quad \forall x \neq 0.$$

In addition, v_k *is strictly increasing in* k *and* $\lim_{k\searrow 0} v_k$ *is the unique self-similar* (*very*) *singular solution of* (1.1).

Our paper is organized as follows. In Section 2, we shall first establish the well posedness of the initial value problem (1.4) and (1.11), and exclude the trivial cases $a = 0$ and $\geqslant 1$. Then we show Theorem 1.2(I), which implies the non-existence part of Theorem 1.1. Also we provide certain properties of the solution $u(r;a)$; in particular, we show that u is monotonic in a in the interval where $J := r^{\mu}u$ is non-decreasing. Next in Sections 3 and 4 we prove (IIa) and (IIb) of Theorem 1.2, respectively. Finally, in Section 5, we prove (IIc).

2. Preliminary

2.1. The initial value problem (1.4) and (1.11)

Lemma 2.1. *For every* $a \in R$, *there exists a unique solution* $u(\cdot;a)$ *to* (1.4), (1.11) *in some right-half neighborhood of the origin. In addition, the following hold*:

(1) *If* $a = 0$, *then* $u \equiv 0$.
(2) *If* $a = 1$, *then* $u \equiv 1$.
(3) *If* $a > 1$, *then* $u' > 0$ *in its existence interval.*
(4) *Assume* $a \in (0,1)$ *and let* $(0, R(a))$ *be the maximal existence interval in which* $u \in (0,1)$. *Then* $u' < 0$ *in* $(0, R(a))$ *and either* (i) $R(a) = \infty$ *and* $\lim_{r\to\infty} u(r;a) = 0$, *or* (ii) $R(a) < \infty$ *and* $u(R(a);a) = 0$. *In addition, as* $r \searrow 0$,

$$u(r;a) = a - \frac{p-1}{p}(a - a^q)^{1/(p-1)} n^{-1/(p-1)} r^{p/(p-1)}[1 + o(r)]. \tag{2.1}$$

Proof. First we derive an integral equation equivalent to the initial value problem (1.4), (1.11). Assume that u is a solution of (1.4) and (1.11) near the origin. Integrating over $(0,r)$, Eq. (1.4) multiplied by r^{n-1}, we obtain

$$-|u'|^{p-2}u' = \beta r u + \frac{1}{r^{n-1}} \int_0^r \rho^{n-1}[1 - n\beta - |u|^{q-1}]u \, d\rho =: \mathscr{G}[u](r). \tag{2.2}$$

Integrating the $\frac{1}{p-1}$th power of both sides then gives

$$u(r;a) = a - \int_0^r |\mathscr{G}[u](\rho)|^{(2-p)/(p-1)}\mathscr{G}[u](\rho)\,d\rho. \tag{2.3}$$

On the other hand, it is easy to show that if a continuous function u satisfies the above integral equation, then it solves (1.4) and (1.11). Thus, the initial value problem (1.4) and (1.11) is equivalent to (2.3).

Since $(2-p)/(p-1) > 0$ and $q > 1$, the existence, uniqueness, as well as the differentiability with respect to a, of solutions to (2.3) then follows from standard Picard's iteration and Gronwall's inequality technique. We omit the details.

Assertions (1) and (2) follow from the uniqueness of the solution. When $a > 1$, $\mathscr{G}[u](r) < 0$ for all sufficiently small positive r so $u' > 0$ for all small positive r. As (1.4) forbids u' from attaining a first zero (since at which $u' = 0$, $(|u'|^{p-2}u')' \leqslant 0$ and $u > 1$), we conclude that $u' > 0$ in its existence interval. Similarly, we can show that if $a \in (0,1)$, then $u' < 0$ as long as $u > 0$, so that either (i) $R(a) = \infty$ and $u(r;a) \searrow 0$ as $r \nearrow \infty$ or (ii) $R(a) < \infty$ and $u(R(a);a) = 0$. Finally, (2.1) follows by substituting the right-hand side of (2.3) by $u = a + o(r)$. $\square$

The first three assertions of the lemma indicate that we need only consider the solution of (1.4), (1.11) for $a \in (0,1)$. Hence, in the sequel, we always assume that $a \in (0,1)$.

2.2. *Non-existence of self-similar solution when* $q \geqslant p - 1 + p/n$

Now we are ready to prove Theorem 1.2I, which, together with Lemma 2.1(1)–(3), implies the non-existence part of Theorem 1.1.

Proof of Theorem 1.2(I). Multiplying (1.4) by $r^{1/\beta - 1}$ we have, for $r \in (0, R(a))$,

$$(r^{1/\beta-1}|u'|^{p-2}u' + \beta r^{1/\beta}u)' = (n - 1/\beta)r^{1/\beta-2}|u'|^{p-1} + r^{1/\beta-1}u^q > 0 \tag{2.4}$$

since $n\beta \geqslant 1$. Thus, the function $g(r) := r^{1/\beta-1}|u'|^{p-2}u' + \beta r^{1/\beta}u$ is strictly increasing in $(0, R(a))$. Observe that $\lim_{r\searrow 0} g(r) = 0$ since (2.2) implies that $|u'|^{p-2}u' = O(r)$. Therefore, $g > 0$ in $(0, R(a))$. Since $u' < 0$ in $(0, R(a))$, we then conclude that $R(a) = \infty$ and $u \searrow 0$ as $r \nearrow \infty$.

As $g(\cdot)$ is increasing, $\lim_{r\to\infty} (r^{1/\beta-1}|u'|^{p-2}u' + \beta r^{1/\beta}u) = \lim_{r\to\infty} g(r) = g_\infty$ exists, where g_∞ is either a positive constant or ∞. As $u' < 0$ in $(0, \infty)$, we then conclude that $\liminf_{r\to\infty} r^{1/\beta}u \geqslant g_\infty/\beta > 0$. This completes the proof. $\square$

2.3. A monotonicity lemma

From now on, we shall always assume that $p\in(1,2)$ and $1<q<p-1+p/n$. Note the condition $1<q<p-1+p/n$ implies that $1<p-1+p/n$, i.e., $p>2n/(n+1)$. It then follows that

$$\beta := \frac{q+1-p}{p} < \frac{1}{n}, \quad \mu := \frac{p}{2-p} > n. \tag{2.5}$$

To study the behavior of the solution $u(r;a)$, we introduce a function J defined by

$$J(r;a) := r^{\mu} u(r;a). \tag{2.6}$$

Since $rJ' - \mu J = r^{\mu+1}u' < 0$ in $(0, R(a))$, a substitution of $u = r^{-\mu}J$ into (1.4) gives

$$\begin{aligned}
&(p-1)r^2 J'' + [n-1-2\mu(p-1)]rJ' + \mu(\mu-n)J \\
&\quad + (\mu J - rJ')^{2-p}\{\beta r J' + (1-\beta\mu)J - r^{\mu(1-q)}J^q\} = 0.
\end{aligned} \tag{2.7}$$

In addition, a differentiation in a gives, for $J_a := \frac{\partial J}{\partial a}$,

$$\begin{aligned}
\mathscr{L}(J_a) :=\ & (p-1)r^2 J_a'' + [n-1-2\mu(p-1)]rJ_a' + \mu(\mu-n)J_a \\
& + (2-p)(\mu J - rJ')^{1-p}(\mu J_a - rJ_a') \\
& \times \{\beta r J' + (1-\beta\mu)J - r^{\mu(1-q)}J^q\} \\
& + (\mu J - rJ')^{2-p}\{\beta r J_a' + (1-\beta\mu) \\
& \times J_a - q r^{\mu(1-q)}J^{q-1}J_a\} = 0.
\end{aligned} \tag{2.8}$$

Lemma 2.2. *If $J' > 0$ in a finite interval $(0, r_1)$, then $J_a = r^{\mu}u_a > 0$ on $(0, r_1]$.*

Proof. Applying the differential operator $r\frac{d}{dr}$ to (2.7) and using the identity $r[r^2J'']' = r^2[rJ']''$, one obtains

$$\mathscr{L}(rJ') = \mu(1-q)r^{\mu(1-q)}(\mu J - rJ')^{2-p}J^q < 0 \quad \text{in } (0, R(a)).$$

We first show that $\mu a J_a > rJ'$ in $(0, r_1)$. Since $J_a = r^{\mu}u_a$ and $rJ' = r^{\mu}(ru' + \mu u)$, we need only show that $\mu a u_a > ru' + \mu u$ in $(0, r_1)$. Using (2.1) one sees that $\mu a u_a - (ru' + \mu u) > 0$ for all r sufficiently small. Hence, if the assertion that $\mu a u_a - (ru' + \mu u) > 0$ in $(0, r_1)$ is not true, then there exists $r_2 \in (0, r_1)$ such that $\mu a u_a - (ru' + \mu u)$ vanishes at $r = r_2$ and is positive in $(0, r_2)$.

Since $ru' + \mu u$ is uniformly positive on $[0, r_2]$, there exists a positive number ℓ such that $\max_{r\in[0,r_2]}\{\ell u_a - (ru' + \mu u)\} = 0$. Let $r_3 \in [0, r_2]$ be the point such that $\{\ell u_a - (ru' + \mu u)\}|_{r=r_3} = 0$. Since $\mu a u_a - (ru' + \mu u)$ is positive in $(0, r_2)$ and vanishes at $r = 0$ and r_2, $\ell < \mu a$ and $r_3 \in (0, r_2)$. As $\ell J_a - rJ' = r^{\mu}[\ell u_a - (ru' + \mu u)]$, one sees that r_3 is an interior zero maximum of $\ell J_a - rJ'$. Hence, at $r = r_3$, $\ell J_a - rJ' = 0$, $(\ell J_a - rJ')' = 0$, and $(\ell J_a - rJ')'' \leqslant 0$, which lead to $\ell \mathscr{L}(J_a) \leq \mathscr{L}(rJ')$. But this is impossible since $\mathscr{L}(J_a) = 0$ and $\mathscr{L}(rJ') < 0$ at $r = r_3$. Thus, $\mu a u_a - (ru' + \mu u) > 0$ and in turn $\mu a J_a > rJ' > 0$ in $(0, r_1)$.

It remains to show that $J_a > 0$ at r_1. For a later application, here we provide an elaborated proof.

Let $r_0 = \min\{1, r_1/2\}$, $\ell_0 = (rJ'/J_a)|_{r=r_0}$ and $k_0 = (\ell_0 J_a - rJ')'|_{r=r_0}$. We claim that $k_0 > 0$. In fact, if $k_0 < 0$, then $\ell_0 J_a > rJ'$ near the left-hand side of r_0. As $\ell_0 < \mu a$, by (2.1), $\ell_0 J_a - rJ'$ is negative near $r = 0$. Thus, there exists $r_4 \in (0, r_0)$ such that $\ell_0 J_a - rJ'$ is positive in (r_4, r_0) and vanishes at $r = r_4$ and $r = r_0$. It then follows from the same argument as above that for some $\ell \in (0, \ell_0)$ the function $\ell J_a - rJ'$ obtains a zero maximum at some point in (r_4, r_0) which is impossible. Similarly, if $k_0 = 0$, then a comparison of the second-order derivatives of $\ell_0 J_a$ and rJ' at r_0 still shows that $\ell_0 J_a > rJ'$ near the left-hand side of r_0, and still we can derive a contradiction. Thus $k_0 > 0$.

Now let ψ be the solution to $\mathscr{L}(\psi) = 0$ in $(0, R(a))$ with the initial values $\psi(r_0) = 0$ and $\psi'(r_0) = 1$. Then $\psi > 0$ in $(r_0, r_1]$ since between any two zeros of ψ there is a zero of J_a.

We consider the function $\varphi = \ell_0 J_a - k_0 \psi$. It is obvious that $\mathscr{L}(\varphi) = 0$ in $(0, R(a))$ and $0 = \varphi - rJ' = \{\varphi - rJ'\}'$ at $r = r_0$. Since $\mathscr{L}(rJ') < 0$, we must have $\{\varphi - rJ'\}''|_{r=r_0} > 0$, i.e., $\varphi - rJ'$ is positive near the right-hand side of r_0. Hence, following the same argument as that in proving $\mu a J_a > rJ'$ in $(0, r_1)$, we have $\varphi > rJ'$ in (r_0, r_1); in particular, by continuity, $\varphi \geqslant rJ' \geqslant 0$ at $r = r_1$. Therefore, $\ell_0 J_a = \varphi + k_0 \psi \geqslant k_0 \psi > 0$ at $r = r_1$. This completes the proof of the present lemma. $\square$

For convenience, we denote

$$\begin{aligned} \mathscr{A} = \{a \in (0,1) \mid &\text{ there exists } R_1(a) \in (0, R(a)) \text{ such that} \\ & J'(R_1(a); a) = 0\}, \\ \mathscr{B} = \Big\{a \in (0,1) \mid &\, J'(\cdot; a) > 0 \text{ in } (0, \infty), \ \lim_{r\to\infty} J(r; a) < \infty\Big\}, \\ \mathscr{C} = \Big\{a \in (0,1) \mid &\, J'(\cdot; a) > 0 \text{ in } (0, \infty), \ \lim_{r\to\infty} J(r; a) = \infty\Big\}. \end{aligned}$$

Since $J' > 0$ near the origin, if $a \in (0,1)$ is not in $\mathscr{A}$, then $J' > 0$ in $(0, R(a))$, which implies $R(a) = \infty$, so that $a \in \mathscr{B} \bigcup \mathscr{C}$. Thus, $\mathscr{A}$, $\mathscr{B}$ and $\mathscr{C}$ are disjoint and $\mathscr{A} \bigcup \mathscr{B} \bigcup \mathscr{C} = (0, 1)$.

3. Characterization of the set $\mathscr{A}$

Lemma 3.1. *Let $a \in (0,1)$. The following statements are equivalent*:

(i) $a \in \mathscr{A}$;
(ii) *there exists* $R_1 \in (0, R(a))$ *such that* $J' > 0$ *in* $(0, R_1(a))$, $J''(R_1(a); a) < 0$, *and* $J' < 0$ *in* $(R_1(a), R(a))$;
(iii) $\sup_{r \in (0,R(a))} J < J^*$ *where* J^* *is as in* (1.12);
(iv) *there exists* $r_1 \in (0, R(a))$ *such that* $\int_0^{r_1} \rho^{n-1}(1 - n\beta - u^{q-1}) u d\rho > 0$;
(v) $R(a) < \infty$ *and* $u'(R(a); a) < 0$;
(vi) $R(a) < \infty$.

Proof. (i) $\Rightarrow$ (ii). Let $(0, R_1(a))$ be the maximal interval where $J' > 0$. Since $a \in \mathscr{A}$, $R_1(a) < R(a)$ and $J'(R_1(a); a) = 0$. We claim that $J''(R_1(a); a) < 0$. In fact, if $J''(R_1(a); a) = 0$, then differentiating (2.7) with respect to r and evaluating the resulting equation at $r = R_1(a)$, we obtain $J'''(R_1(a); a) < 0$. This contradicts the fact that $J' > 0$ in $(0, R_1)$. Thus, $J''(R_1(a); a) < 0$.

Next we show that $J' < 0$ in $(R_1(a), R(a))$. In fact, if this is not true, then there exists $R_2(a) \in (R_1(a), R(a))$ such that $J'(R_2(a); a) = 0$ and $J' < 0$ in $(R_1(a), R_2(a))$. Evaluating (2.7) at $r = R_1(a)$ with $J'(R_1; a) = 0$ and $J''(R_1; a) < 0$, and at $r = R_2$ with $J'(R_2; a) = 0$ and $J''(R_2; a) \geqslant 0$, and using the fact that $r^{\mu(1-q)} J^{q-1} = u^{q-1}$ we obtain

$$\begin{aligned} &(\beta\mu + u^{q-1} - 1)(\mu J)^{2-p}|_{r=R_1(a)} \\ &\quad < \mu(\mu - n) \leqslant (\beta\mu + u^{q-1} - 1)(\mu J)^{2-p}|_{r=R_2(a)}. \end{aligned} \tag{3.1}$$

But this is impossible since $\beta\mu > 1$, $J(R_1; a) > J(R_2; a)$ and $u(R_1; a) > u(R_2; a)$. Hence $J' < 0$ in $(R_1(a), R(a))$.

(ii) $\Rightarrow$ (iii). Note that the maximum of J is obtained at $r = R_1(a)$, so the assertion follows from the first inequality of (3.1).

(iii) $\Rightarrow$ (iv). Assume for the contrary that $\int_0^r \rho^{n-1}(1 - n\beta - u^{q-1}) u \, d\rho \leqslant 0$ for all $r \in (0, R(a))$. Then from (2.2), for all $r \in (0, R(a))$, $-|u'|^{p-2}u' \leqslant \beta r u$, i.e. $-u^{-1/(p-1)}u' \leqslant (\beta r)^{1/(p-1)}$. Upon integrating this inequality over $(0, r)$ we have

$$u(r; a) \geqslant \left(a^{(p-2)/(p-1)} + \frac{2-p}{p} \beta^{1/(p-1)} r^{p/(p-1)} \right)^{(p-1)/(p-2)} \quad \forall r \in (0, R(a)).$$

It then follows that $R(a) = \infty$ and, upon recalling (2.6), that $\hat{J} := \liminf_{r\to\infty} J > 0$.

Note that either $J' > 0$ in $(0, \infty)$, or if J' changes sign, then $a \in \mathscr{A}$, so that $J' < 0$ in $(R_1(a), \infty)$. Hence, $\lim_{r\to\infty} J = \hat{J}$ and $\liminf_{r\to\infty} |rJ'| = 0$.

Let $\{r_j\}_{j=1}^{\infty}$ be a sequence such that $\lim_{j\to\infty} r_j = \infty$ and $\lim_{j\to\infty} (rJ')|_{r=r_j} = 0$. We claim that $\{r_j\}$ can be selected such that in addition $\lim_{j\to\infty} (r^2 J'')|_{r=r_j} = 0$.

In fact, if $|rJ'|$, which is positive for all large r, oscillates infinitely many times, then one can select $\{r_j\}$ to be local minimum points of $|rJ'|$ so that $0 = (rJ')' = rJ'' + J'$ on $\{r_j\}$. That is, $\lim_{j\to\infty} (r^2J'')|_{r=r_j} = -\lim_{j\to\infty} (rJ')|_{r=r_j} = 0$.

If $|rJ'|$ does not oscillate infinitely many times, then $|rJ'|$ eventually monotonically decreases to zero. So, one can select $\{r_j\}$ along which $r(|rJ'|)'$ approaches zero, namely, $r^2J'' = r(rJ')' - rJ'$ approaches zero along the sequence $\{r_j\}$.

Now evaluating (2.7) at r_j and sending $j\to\infty$ we obtain $\hat{J} = J^*$, contradicting to the assumption $\sup_{r\in(0,R(a))} J < J^*$.

(iv) $\Rightarrow$ (v). Since the function $z = 1 - n\beta - u^{q-1}$ strictly increases in $(0, R(a))$, $\int_0^{r_1} \rho^{n-1} zu > 0$ implies that $z > 0$ for all $r\in[r_1, R(a))$. It then follows that for some $\delta > 0$, $\int_0^r \rho^{n-1}(1 - n\beta - u^{q-1})u\, d\rho \geqslant \delta$ in $[r_1, R(a))$. Consequently, from (2.2)

$$-|u'|^{p-2}u' \geqslant \beta ru + \delta r^{1-n} \quad \forall r\in[r_1, R(a)). \tag{3.2}$$

If $n = 1$, then $|u'|^{p-1} > \delta$ in $(r_1, R(a))$ and assertion (V) is trivially true.

If $n\geqslant 2$, then $p/n\in(0, 1]$ so using the inequality $\beta ru + \delta r^{1-n} \geqslant (\beta ru)^{1-p/n}(r^{1-n}\delta)^{p/n} = \beta^{1-p/n}\delta^{p/n}u^{1-p/n}r^{1-p}$, we obtain from (3.2) that, for all $r\in(r_1, R(a))$,

$$-u' \geqslant r^{-1}u^{1-\nu}(\beta^{1-p/n}\delta^{p/n})^{1/(p-1)}, \tag{3.3}$$

where $\nu = [(n+1)p - 2n]/[n(p-1)] > 0$. Multiplying both sides of (3.3) by $u^{\nu-1}$ and integrating over $[r_1, r)$, $r < R(a)$, one immediately concludes that $R(a) < \infty$. In addition, it follows from (3.2) that $u'(R(a); a) < 0$.

(v) $\Rightarrow$ (vi) is trivially true. (vi) $\Rightarrow$ (i) is also trivially true since $u(R(a); a) = 0$ implies that $J = r^\mu u$ has an interior maximum in $(0, R(a))$. This completes the proof of the lemma. □

Theorem 3.1. *There exists* $a_* \in((1 - n\beta)^{1/(q-1)}, 1]$ *such that* $\mathscr{A} = (0, a_*)$.

Proof. When $a\in(0, (1 - n\beta)^{1/(q-1)}]$, $u(r; a) < a$ for all $r\in(0, R(a))$, so (iv) of Lemma 3.1 holds. It then follows that $a\in\mathscr{A}$. Thus, $(0, (1 - n\beta)^{1/(q-1)}]\subset\mathscr{A}$.

For any given $\tilde{a}\in\mathscr{A}$, since $J''(R_1(\tilde{a}); \tilde{a}) < 0$, an implicit function theorem then yields that the equation $J'(R_1; a) = 0$ has a local unique C^1 solution $R_1 = R_1(a)$ in a neighborhood of $\tilde{a}$. Thus, $\mathscr{A}$ is open and $R_1(a)$ is C^1 in $\mathscr{A}$. Furthermore, defining $m(a) = J(R_1(a); a)$ we have $\frac{d}{da}m(a) = J'\frac{d}{da}R_1 + J_a = J_a > 0$ for all $a\in\mathscr{A}$.

To finish the proof, we need only to show that if $(a_1, a_2)\subset\mathscr{A}$ and $a_1 > 0$, then $a_1\in\mathscr{A}$.

In fact, by continuous dependence of initial data,

$$\sup_{r \in (0, R(a_1))} J(r; a_1) \leqslant \limsup_{a \searrow a_1} m(a) < m((a_1 + a_2)/2) < J^*,$$

so that by Lemma 3.1(iii), $a_1 \in \mathscr{A}$. This completes the proof of the theorem. □

4. Characterization of the set C

Lemma 4.1. *Let* $a \in (0, 1)$. *Then* $a \in \mathscr{C}$ *if and only if* $\sup_{r \in (0, R(a))} J(r; a) > J^*$.

Proof. The only if part follows from the definition of $\mathscr{C}$. Now if $\sup_{r \in (0, R(a))} J(r; a) > J^*$, then by Lemma 3.1(iii), $a \notin \mathscr{A}$, so $a \in \mathscr{B} \bigcup \mathscr{C}$. However, if $\hat{J} := \lim_{r \to \infty} J$ is finite, then one can find a sequence $\{r_j\}$ along which rJ' and $r^2 J''$ approach zero. This implies, from Eq. (2.7), that $\hat{J} = J^*$. But this contradicts the assumption that $\sup_{r \in (0, R(a))} J > J^*$. □

Theorem 4.1. *There exists* $a^* \in (0, 1)$ *such that* $\mathscr{C} = (a^*, 1)$. *In addition, for every* $a \in \mathscr{C}$, *there exists* $k(a) > 0$ *such that*

$$\lim_{r \to \infty} r^{1/\beta} u(r; a) = k(a).$$

Furthermore, $k(a)$, *as a function of* $a \in (a^*, 1)$, *is positive, continuous, strictly increasing, and*

$$\lim_{a \searrow a^*} k(a) = 0, \quad \lim_{a \nearrow 1} k(a) = \infty.$$

Proof. *Step* 1. We first show that $\mathscr{C}$ is open and non-empty. Since $a \in \mathscr{C}$ if and only if $\sup_{r \in (0, R(a))} J(r; a) > J^*$, by the continuous dependence of initial data, $\mathscr{C}$ is open. Also, as $\lim_{a \nearrow 1} u(r; a) = u(r; 1) \equiv 1$ uniformly in any compact subset of $[0, \infty)$, $\lim_{a \nearrow 1} J((2J^*)^{1/\mu}; a) = 2J^*$, so that $(1 - \varepsilon, 1) \subset \mathscr{C}$ for some sufficiently small positive ε.

As $\mathscr{A} = (0, a_*)$, $[a_*, 1) \subset \mathscr{B} \cup \mathscr{C}$, so that $J' > 0$ for all $r \in (0, \infty)$ and all $a \in [a_*, 1)$. Consequently, by Lemma 2.2, $J_a > 0$ for all $r \in (0, \infty)$ and all $a \in [a_*, 1)$. This implies that $\mathscr{C} = (a^*, 1)$ where $a^* = \inf\{a \geqslant a_* \mid \lim_{r \to \infty} J(r; a) > J^*\}$.

As a by product, $\mathscr{B} = [a_*, a^*] = \{a \mid R(a) = \infty, \text{ and } J(r; a) \nearrow J^* \text{ as } r \to \infty\}$.

Step 2. We now study the behavior of the solution $u(\cdot; a)$ for $a \in \mathscr{C}$. For simplicity, we write $u(r; a)$ and $J(r; a)$ as $u(r)$ and $J(r)$, respectively.

It is convenient to use the variable $\tau = \ln r$. Since u is positive, we can write $u(e^\tau) = u(1) \times \exp(-\int_0^\tau \Lambda(s)\, ds)$. Since $u' < 0$ and $J' = r^{\mu - 1}(ru' + \mu u) > 0$ for all $r > 0$, we have $0 < \Lambda(\tau) < \mu$ for all $\tau \in (-\infty, \infty)$.

Substituting this transformation into (1.4) and using the relations $r\frac{d}{dr} = \frac{d}{d\tau}, r^2\frac{d^2}{dr^2} = \frac{d^2}{d\tau^2} - \frac{d}{d\tau}$ and $r^p u^{2-p} = J^{2-p}$, we obtain, writing $\dot{\Lambda} = d\Lambda/d\tau$,

$$(p-1)\dot{\Lambda} = F(\Lambda,\tau) := (p-1)\Lambda^2 + (p-n)\Lambda + \Lambda^{2-p}[1-\beta\Lambda - u^{q-1}]J^{2-p}. \tag{4.1}$$

Here we consider Λ as an unknown function whereas $u = u(e^\tau)$ and $J = J(e^\tau)$ as known functions of τ.

Since $a \in \mathscr{C}$, as $\tau \nearrow \infty, u \searrow 0$ and $J \nearrow \infty$. It then follows that for any $\varepsilon > 0$, there exists $\tau_\varepsilon > 0$ such that $F(\Lambda,\tau) > 0$ for all $\Lambda \in (0,(1-\varepsilon)/\beta]$, $\tau > \tau_\varepsilon$ and $F(\Lambda,\tau) < 0$ for all $\Lambda \in ((1+\varepsilon)/\beta, \mu]$, $\tau > \tau_\varepsilon$. It then follows from an invariant region that

$$\lim_{\tau\to\infty} \Lambda(\tau) = 1/\beta.$$

Step 3. Next we show that, as $\tau \to \infty$, Λ approaches $1/\beta$ exponentially fast, with an exponent at least $v = \frac{1}{2}\min\{\frac{q-1}{\beta}, (2-p)(\mu - 1/\beta)\}$.

Consider the function $\Lambda^-(\tau) = \frac{1}{\beta}[1 - \frac{1}{2}e^{v(T-\tau)}]$ defined on $[T,\infty)$. We want to show that Λ^- is a sub-solution to $(p-1)\dot{\Lambda} = F(\Lambda,\tau)$ in $[T,\infty)$ provided that T is sufficiently large.

First let T be large enough such that $u^{q-1}(e^T) < \frac{1}{4}$ and $\Lambda(\tau) > 1/(2\beta)$ for all $\tau > T$. Then $u^{q-1}(e^\tau) = u^{q-1}(e^T)\exp(-(q-1)\int_T^\tau \Lambda) < \frac{1}{4}e^{v(T-\tau)}$ for all $\tau \geqslant T$.

Next, taking a larger T if necessary, we assume that $\Lambda(\tau) \leqslant 1/\beta + \frac{1}{2}(\mu - 1/\beta)$ for all $\tau \geqslant T$. Then

$$J^{2-p}(e^\tau) = J^{2-p}(e^T)\exp\left((2-p)\int_T^\tau (\mu - \Lambda)\right) \geqslant J^{2-p}(e^T)e^{v(\tau-T)} \quad \forall \tau \geqslant T.$$

Thus, $\{1 - \beta\Lambda^- - u^{q-1}(e^\tau)\}J^{2-p}(e^\tau) \geqslant \frac{1}{4}J^{2-p}(e^T)$ for all $\tau \geqslant T$. Consequently, for all $\tau \geqslant T$,

$$(p-1)\frac{d}{d\tau}\Lambda^- - F(\Lambda^-,\tau) \leqslant C(p,n,,v,\beta) - \frac{1}{4}(2\beta)^{p-2}J^{2-p}(T) < 0 \quad \forall \tau > T$$

if we take T large enough, since $J(e^T) \to \infty$ as $T \to \infty$.

Comparing $\Lambda(\tau)$ to $\Lambda^-(\tau)$ in $[T,\infty)$ we obtain that $\Lambda(\tau) > \Lambda^-(\tau) = \frac{1}{\beta}(1 - \frac{1}{2}e^{v(T-\tau)})$ in $[T,\infty)$.

In a similar manner, we can show that $\Lambda(\tau) \leqslant \Lambda^+(\tau) = \frac{1}{\beta}[1 + \frac{1}{2}(\mu - 1/\beta)e^{v(T-\tau)}]$.

Therefore, $|\Lambda - 1/\beta| \leqslant \mu e^{\nu(T-\tau)}$. Consequently, as $r \to \infty$,

$$r^{1/\beta}u(r) = u(1)\exp\left\{-\int_0^{\ln r} (\Lambda(\tau) - 1/\beta)\,d\tau\right\}$$
$$\to u(1)\exp\left\{-\int_0^{\infty} (\Lambda(\tau) - 1/\beta)\,d\tau\right\} =: k(a).$$

Since $u_a = r^{-\mu}J_a > 0$ in $(0, \infty)$, we know that $k(\cdot)$ is positive, continuous, and non-decreasing in $(a^*, 1)$.

Step 4. To study the behavior of the function $k(a)$ for a near the both ends of $\mathscr{C} = (a^*, 1)$, we consider the function $K(r;a) \coloneqq r^{1/\beta}u(r;a)$.

If $K(r;a)$ obtains a local maximum, say, at $r = r_1$, which is the first one, then at $r = r_1$, $K' = 0$, and $K'' \leqslant 0$ (if $K'' = 0$ then one can get $K''' < 0$ which contradicts $K' > 0$ in $(0, r_1)$), i.e., $\beta r u' + u = 0$ and $r^2u'' \leqslant (1+\beta)\beta^{-2}u$. Substituting this information into (1.4) then yields

$$K(r_1;a) < K^* \coloneqq \{\beta^{-p}[p - 1 + \beta(p-n)]\}^{1/(q+1-p)}.$$

Note that $p - 1 + \beta(p-n) = [p(q-1) + n(p - 1 + p/n - q)]/(q+1-p) > 0$. Similarly, if K obtains a local minimum, say, at $r = r_2$, then $K(r_2;a) \geqslant K^*$.

From this, we immediately conclude that one of the following holds:

(i) There exists r_1 such that $K' > 0$ in $(0, r_1)$, $K'(r_1;a) = 0$, $K(r_1;a) < K^*$, $K''(r_1;a) < 0$, and $K' < 0$ in (r_1, ∞); consequently, $K(r;a) \searrow k(a) < K^*$ when $r \nearrow \infty$.

(ii) $K'(r;a) > 0$ in $(0, \infty)$ and $K(r;a) \nearrow k(a)$ as $r \nearrow \infty$.

Now, if $a \in (0,1)$ is close to 1, case (i) will not occur since $\sup_{r\in(0,\infty)} r^{1/\beta}u(r;a) > K^*$. Thus, case (ii) occur and $\lim_{a\nearrow 1} k(a) \geqslant \lim_{r\to\infty} \lim_{a\nearrow 1} r^{1/\beta}u(r;a) = \infty$.

Now if $\lim_{a\searrow a^*} k(a) > 0$, we can derive from (i) or (ii) that $k(a^*) = \lim_{a\searrow a^*} k(a) > 0$, which implies that $a^* \in \mathscr{C}$. Since $\mathscr{C}$ is open, this is impossible. Hence, $\lim_{a\searrow a^*} k(a) = 0$.

Finally, as in the proof of the next section, we can show that $k_a = \lim_{r\to\infty} r^{1/\beta}u_a$ exists and is positive. This completes the proof of the theorem. □

5. Characterization of the set B

Theorem 5.1. $\mathscr{B} = \{a_*\} = \{a^*\}$ *and* $J(r;a^*) \nearrow J^*$ *as* $r \nearrow \infty$.

Proof. From the previous discussion we know that $\mathscr{B} = [a_*, a^*]$, and that for all $a \in \mathscr{B}$, $J_a > 0$ for all $r > 0$ and $J(r;a) \nearrow J^*$ as $r \nearrow \infty$. It remains to show that $a_* = a^*$.

We claim that if $a \in \mathscr{B}$, then $\lim_{r\to\infty} J_a(r;a) = \infty$.

To show this, we use the independent variable $\tau = \ln r$. Note that $rJ' = \dot{J}$ vanishes as $\tau \to \infty$. The linear operator $\mathscr{L}$ in (2.8) takes the form, for τ sufficiently large,

$$\mathscr{L}(\phi) = (p-1)\ddot{\phi} + [b + o(1)]\dot{\phi} - [c + o(1)]\phi,$$

where $o(1) \to 0$ as $\tau \to \infty$, b is a certain constant, and $c = p(\mu - n)$. As $c > 0$, it is easy to see that the solution to $\mathscr{L}(\phi_1) = 0$ in (T, ∞) with initial value $\phi_1(T) = 0$, $\dot{\phi}_1(T) = 1$ with T large enough will have the property that $\phi_1 \to \infty$ exponentially fast as $\tau \to \infty$.

Note that the function ψ constructed at the end of the proof of Lemma 2.2 is positive in (r_0, ∞). As J_a and ψ are linearly independent, one of them will be unbounded. Since $\ell_0 J_a \geqslant k_0 \psi$, we then know that $J_a \to \infty$ as $r \to \infty$.

Finally, we show that $a^* = a_*$. In fact, if $a^* > a_*$, then by Fatou's lemma

$$0 = \lim_{r \to \infty} (J(r; a^*) - J(r; a_*)) = \lim_{r \to \infty} \int_{a_*}^{a^*} J_a(r; a)\, da$$
$$\geqslant \int_{a_*}^{a^*} \liminf_{r \to \infty} J_a(r; a)\, da = \infty,$$

which is impossible. This completes the proof of the theorem. $\square$

Finally, we remark that all our main results in Section 1 follow directly from our demonstrations in Sections 2–5.

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