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LONG TIME BEHAVIOR OF SOLUTIONS TO P-LAPLACIAN EQUATION WITH ABSORPTION*

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Abstract. In this paper, we study the long time behavior of solutions to the Cauchy problem of $u_t = \operatorname{div}(|\nabla u|^{p-2}\nabla u) - u^q$ in $R^n \times (0,\infty)$, with nonnegative initial value $u(x,0) = \phi(x)$ in R^n, where $(2n)/(n+1) < p < 2$ and $q > 1$. For initial data of various decay rates, especially the critical decay $\phi = O(|x|^{-\mu})$ with $\mu = p/(q+1-p)$, we show that the solution converges as $t \to \infty$ to a self-similar solution. This extends the recent result of Escobedo, Kavian, and Matano for the semilinear case of $p = 2$. Here an essential role is played by singular and very singular self-similar solutions established in our previous works [X. Chen, Y. Qi, and M. Wang, *J. Differential Equations*, 190 (2003), pp. 1–15; X. Chen, Y. Qi, and M. Wang, preprint, Department of Mathematics, HKUST, Hong Kong, 1998].

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1. Introduction.

In this paper, we study the long time behavior of solutions to the Cauchy problem

$$\text{(I)} \qquad \begin{cases} \mathcal{L}u := u_t - \operatorname{div}(|\nabla u|^{p-2}\nabla u) = -u^q & \text{in } R^n \times (0,\infty), \\ u(x,0) = \phi(x) & \text{in } R^n, \end{cases}$$

where $2n/(n+1) < p < 2$, $q > 1$, and ϕ is a nonnegative continuous function which decays to 0 as $|x| \to \infty$. The existence, uniqueness, and Hölder continuity of nonnegative weak solutions of (I) are well established by Chen and DiBenedetto [1, 2] and DiBenedetto and Herrero [7] under much weaker conditions, say, if $\phi \in L^1_{loc}(R^n)$. Therefore, our continuity assumption on ϕ is more than necessary and is used only for the convenience of stating our results.

Our major concern is the behavior of $u(\cdot,t)$ as $t \to \infty$ and how it is influenced by

(a) the decay rate of $\phi(x)$ as $|x| \to \infty$, and

(b) the competition between the diffusion $\operatorname{div}(|\nabla u|^{p-2}\nabla u)$ and the absorption u^q.

It turns out that a critical decay rate for ϕ is $O(|x|^{-\mu})$, and a critical exponent of absorption is q^*, where μ and q^* are defined by

$$(1.1) \qquad q^* := p - 1 + \frac{p}{n}, \qquad \mu := \frac{p}{q+1-p} = \frac{n}{1 + n(q-q^*)/p}.$$

The exponents μ and q^* arise naturally from finding radially symmetric and scaling invariant solutions, i.e., self-similar solutions of the form $t^{-\beta}w(|x|t^{-\gamma})$. For later

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references, we cite several relevant results on self-similar solutions [3, 5, 6, 8, 9, 14, 15, 17, 20, 21, 22].

PROPOSITION 1.1. *Let* $p > 2n/(n+1)$ *and* $\mathcal{L}u = u_t - (|\nabla u|^{p-2}\nabla u)$.

(1) *A self-similar solution to* $\mathcal{L}u = 0$ *has the form* $t^{\gamma\nu}w(xt^{\gamma})$ *with* $\gamma = -1/[p+(p-2)\nu]$ *and some* $\nu \in \mathbb{R}$. *Self-similar solutions to* $\mathcal{L}u = -u^q$ *have the form* $t^{\gamma\mu}w(xt^{\gamma})$ *with* $\gamma = -1/[p+(p-2)\mu]$.

(2) *For every* $\nu \in (0,n)$ *and* $B \geq 0$, *the solution* G_B^{ν} *to* $\mathcal{L}u = 0$ *with initial value* $B|x|^{-\nu}$ *is self-similar of the form* $t^{-\gamma\nu}w(|x|t^{-\gamma})$ *with* $\gamma = 1/[p+(p-2)\nu]$.

For each $c > 0$, *there is a unique fundamental solution* E_c *of* $\mathcal{L}u = 0$ *with initial mass* c. *It is self-similar of the form* $t^{-\gamma n}w(|x|t^{-\gamma})$ *with* $\gamma = 1/[p+(p-2)n]$.

(3) *For each* $A > 0$, *there is a unique radially symmetric self-similar solution* W_A *to* $\mathcal{L}u = -u^q$ *such that* $W(x, 0+) = A|x|^{-\mu}$ *for all* $x \neq 0$.

When $1 < q < q^*$ *(i.e.,* $\mu > n$*),* W_A *is monotonic in* A, *and* $W_0 := \lim_{A\searrow 0} W_A$ *is the unique very singular solution (VSS) of* $\mathcal{L}u = -u^q$ *in the sense that*

$$\lim_{t\searrow 0}\sup_{|x|>\varepsilon} W_0(x,t) = 0 \quad \textit{and} \quad \lim_{t\searrow 0}\int_{|x|\leq\varepsilon} W_0 dx = \infty \qquad \forall\, \varepsilon > 0.$$

Also, for every $c > 0$, $\mathcal{L}u = -u^q$ *admits a unique fundamental solution* u^c *with initial mass* c. *Fundamental solutions are not self-similar, but* $\lim_{c\to\infty} u^c = W_0$ *is the self-similar VSS.*

When $q = q^*$ *(i.e.,* $\mu = n$*),* W_A *is monotonic in* A *and* $\lim_{A\searrow 0} W_A = 0$ *in* $\mathbb{R}^n \times (0,\infty)$. *There are neither fundamental solutions nor VSSs to* $\mathcal{L}u = -u^{q^*}$.

When $q > q^*$ *(i.e.,* $\mu < n$*),* W_A *is the solution with the initial data* $A|x|^{-\mu} \in L^1_{loc}(\mathbb{R}^n)$. *There are neither fundamental solutions nor VSSs to* $\mathcal{L}u = -u^q$.

In what follows, $W_0 \equiv 0$ when $q \geq q^*$, and W_0 is the VSS when $1 < q < q^*$.

To study the long time behavior of solutions of (I), it is convenient to divide the decay rates of ϕ ($\phi \not\equiv 0$) as follows:

(A1) $\lim_{|x|\to\infty} |x|^{\mu}\phi(x) = \infty$;
(A2) $\lim_{|x|\to\infty} |x|^{\mu}\phi(x) = A \in [0,\infty)$;
(B1) $q > q^*$ and $\lim_{|x|\to\infty} |x|^{\nu}\phi(x) = B \in [0,\infty)$ for some $\nu \in (\mu, n)$;
(B2) $q > q^*$ and $\phi \in L^1(\mathbb{R}^n)$.
(B3) $q = q^*$ and $\lim_{x\to\infty} |x|^{\alpha}\phi(x) = A > 0$ for some $\alpha > n$.

Note that the analogues of (B1), (B2) are not needed for $1 < q < q^*$ since $\mu > n$ then.

For $p = 2$, questions (a) and (b) have been discussed by Gmira and Veron [12], Kamin and Peletier [14, 15], Escobedo and Kavian [9], and recently by Escobedo, Kavian, and Matano [10] and Herraiz [13].

PROPOSITION 1.2. *Let* $p = 2, q > 1$, *and* u *solve* (I). *Set* $q^* = 1 + 2/n$, $\mu = 2/(q-1)$, *and* $\gamma = 1/2$.

(1) (See [12].) *If* (A1) *holds, then*

$$\lim_{t\to\infty}\sup_{|x|\leq at^{\gamma}} |t^{1/(q-1)}u(x,t) - (q-1)^{-1/(1-q)}| = 0 \quad \forall\, a > 0. \tag{1.2}$$

(2) (See [10].) *If* $1 < q < q^*$ *and* (A2) *holds, then*

$$\lim_{t\to\infty}\sup_{|x|<at^{\gamma}} t^{1/(q-1)}|u(x,t) - W_A(x,t)| = 0 \quad \forall\, a > 0. \tag{1.3}$$

(3) (See [15].) *If* (B1) *holds, then*

$$\lim_{t\to\infty}\sup_{|x|\leq at^{\gamma}} t^{\gamma\nu}|u(x,t) - G_B^{\nu}(x,t)| = 0 \quad \forall\, a > 0. \tag{1.4}$$

(4) (See [12].) *If* (B2) *holds, then for some* $c \in [0, \infty)$

$$\lim_{t\to\infty} \sup_{|x|<at^\gamma} t^{\gamma n}|u(x,t) - E_c(x,t)| = 0 \quad \forall\, a > 0. \tag{1.5}$$

(5) (See [13].) *If* $q > q^*$ *and* $\lim_{|x|\to\infty} |x|^n \phi(x) = B > 0$, *then*

$$\lim_{t\to\infty} \sup_{|x|<at^\gamma} t^{\gamma n}|u/\log t - E_c(x,t)| = 0 \quad \forall\, a > 0, \tag{1.6}$$

where $c = \omega_n B/2$ *with* ω_n *the area of unit sphere* S^{n-1}.

(6) (See [13].) *If* (B3) *holds, then*

$$\lim_{t\to\infty} \sup_{|x|<at^\gamma} t^{n/2}|u(x,t)(\log t)^{n/2} - E_{C_n}(x,t)| = 0 \quad \forall\, a > 0 \tag{1.7}$$

for a unique constant $C_n > 0$.

Remark. It is of great interest to know whether the results in Propositions 1.1 and 1.2 are still valid if the precise pointwise assumptions on initial values are replaced by more general integral estimates. However, we do not have a clear idea of how to approach a situation like that. One possible difficulty is the loss of scaling laws.

For $p \neq 2$, similar but not as complete results were obtained by Zhao [23].

PROPOSITION 1.3. *Let* $p > 2n/(n+1)$, $q > 1$, *and* u *solve* (I).

(1) *If* $\lim_{|x|\to\infty} |x|^{\mu-\varepsilon}\phi(x) = \infty$ *for some* $\varepsilon > 0$, *then* (1.2) *holds with* $\gamma = 1/[p + (p-2)\mu]$.

(2) *If* $1 < q < q^*$ *and* $\lim_{|x|\to\infty} |x|^{\mu+\varepsilon}\phi(x) = 0$ *for some* $\varepsilon > 0$, *then* (1.3) *holds with* $A = 0$ *and* $\gamma = 1/[p + (p-2)\mu]$. *(The result for when* $p > 2$ *and* ϕ *has compact support was obtained by Kamin and Vasquez* [17]*.)*

(3) *If* (B1) *holds, so does* (1.4) *with* $\gamma = 1/[p + (p-2)\nu]$.

(4) *If* $\lim_{|x|\to\infty} |x|^{n+\varepsilon}\phi(x) = 0$ *for some* $\varepsilon > 0$, *then* (1.5) *holds with* $\gamma = 1/[p + (p-2)n]$ *and some* $c \in [0, \infty)$.

There are also analogous results for the porous medium equation

$$\text{(II)} \qquad \begin{cases} u_t = \triangle u^m - u^q & \text{in } R^n \times (0,\infty), \\ u(x,0) = \phi(x) & \text{in } R^n. \end{cases}$$

In this case, the critical exponents should be defined as $\mu = 2/(q-m)$ and $q^* = m + 2/n$. For radially symmetric self-similar solutions and for $m > (1-2/n)_+$ and $q > \max\{1, m\}$, Proposition 1.1 holds with $\mathcal{L}$ defined as $\mathcal{L}u = u_t - \Delta u^m$; see [4, 12, 16, 17, 19, 21]. The results on noncritical decay rates were obtained by Kamin and Peletier [16] for $m > 1$ and Peletier and Zhao [21] for $(1-2/n)_+ < m < 1$, and the result on the critical decay rate was obtained by Kwak [18].

PROPOSITION 1.4. *Let* $m > (1-2/n)_+$, $q > \max\{1, m\}$, *and* u *be a solution to* (II).

(1) *If* (A1) *holds, so does* (1.2) *with* $\gamma = 1/[2 + (m-1)\mu]$.

(2) *Suppose* $1 < q < q^*$. *If* (A2) *holds with* $A > 0$ *or* $A = 0$ *and in addition* $\lim_{|x|\to\infty} |x|^{\mu+\varepsilon}\phi(x) = 0$ *for some* $\varepsilon > 0$, *then* (1.3) *holds with* $\gamma = 1/[2 + (m-1)\mu]$.

(3) *The condition* (B1) *implies* (1.4) *with* $\gamma = 1/[2 + (m-1)\nu]$.

(4) *The condition* (B2) *implies* (1.5) *with* $\gamma = 1/[2 + (m-1)n]$.

(5) *If* $q > q^*$ *and* $\lim_{|x|\to\infty} |x|^n\phi(x) = A > 0$, *then*

$$\lim_{t\to\infty} \sup_{|x|<at^\gamma} \tau^{\gamma n}|u(x,t)/\log\tau - E_c(x,\tau)| = 0 \quad \forall\, a > 0, \tag{1.8}$$

where $c = c(m, n, a) > 0$, $t = c^{1-m}\tau(\log \tau)^{1-m}$, *and* $\gamma = 1/[2 + (m-1)n]$.

One may notice that the recent results of Escobedo, Kavian, and Matano [10] listed in Proposition 1.2 (2) and that of Kwak [18] in Proposition 1.4 (2) address the critical decay of ϕ with sharp results for semilinear and porous media cases, respectively. We would like to mention that though some ideas of Escobedo, Kavian, and Matano [10] and Kwak [18] can be adopted to study (I), their approach cannot be extended easily to the present case.

In this paper, we shall extend the results of Propositions 1.2, 1.3, and 1.4 to the case in which $p \in (2n/(n+1), 2)$. Our main results are the following.

Assume that $2n/(n+1) < p < 2$ *and* $q > 1$. *Let* μ *and* q^* *be defined as in* (1.1), *and let* u *be a solution to* (I). *Let* E_c, G_B^ν, *and* W_A *be defined as in Proposition* 1.1. *Denote* $L^\infty = L^\infty(\mathbb{R}^n)$.

THEOREM 1.5. *Assume* (A1). *Then* (1.2) *holds with* $\gamma = 1/[p + (p-2)\mu]$.

THEOREM 1.6. *Assume* (A2). *Then* (1.3) *holds with* $\gamma = 1/[p + (2-p)\mu]$; *more precisely,*

$$\lim_{t\to\infty} t^{\gamma\mu}\|u(\cdot,t) - W_A(\cdot,t)\|_{L^\infty} = \lim_{t\to\infty} \|t^{\gamma\mu}u(yt^\gamma,t) - W_A(y,1)\|_{L^\infty} = 0. \tag{1.9}$$

In particular, when $A = 0$ *and* $q = q^*$,

$$\lim_{t\to\infty} t^{1/(q^*-1)}\|u(\cdot,t)\|_{L^\infty} = 0 \quad \text{and} \quad \lim_{t\to\infty} t^{1/(q^*-1-\varepsilon)}u(0,t) = \infty \quad \forall\varepsilon > 0. \tag{1.10}$$

THEOREM 1.7. *Assume* (B1). *Then* (1.4) *holds with* $\gamma = 1/[p + (p-2)\nu]$; *more precisely,*

$$\lim_{t\to\infty} t^{\gamma\nu}\|u(\cdot,t) - G_B^\nu(\cdot,t)\|_{L^\infty} = \lim_{t\to\infty} \|t^{\gamma\nu}u(yt^\gamma,t) - G_B^\nu(y,1)\|_{L^\infty} = 0. \tag{1.11}$$

THEOREM 1.8. *Assume* (B2). *Then for* $\gamma = 1/[p + (p-2)n]$ *and some constant* $c > 0$,

$$\lim_{t\to\infty} t^{\gamma n}\|u(\cdot,t) - E_c(\cdot,t)\|_{L^\infty} = \lim_{t\to\infty} \|t^{\gamma n}u(yt^\gamma,t) - E_c(y,1)\|_{L^\infty} = 0. \tag{1.12}$$

Remark. We make some interesting observations.

1. Theorem 1.5 says that when ϕ decays too slowly, absorption dominates diffusion.

2. When $1 < q < q^*$, $W_0 = VSS > 0$, so Theorem 1.6 indicates that the absorption is still strong, but diffusion plays a role to balance it.

When $q \geq q^*$, $W_0 \equiv 0$, so the conclusion of Theorem 1.6 for $A = 0$ is insufficient. Theorems 1.7 and 1.8 exactly compensate for this insufficiency for the case in which $q > q^*$ but not for $q = q^*$.

3. Here we show explicitly that the constant c in Theorem 1.8 is positive, which is not mentioned in [12, 16, 21]. With this improvement, we can conclude that when ϕ decays quickly enough (say, of compact support), then in large time, the solution approaches the VSS when $q \in (1, q^*)$ or the fundamental solution E_c of the pure diffusion equation when $q > q^*$.

4. For the case in which $q = q^*$ and ϕ decays quickly (say, of compact support or L^1), except the two limits in (1.10), we do not have a more precise description. Here, different scaling is needed. For the case in which $p = 2$, we refer interested readers to the work of Herraiz [13] and some insightful observation of Galaktionov, Kurdyumov, and Samarskii [11].

5. The most striking fact, of course, is that in each and every case being discussed, the limiting behavior of $u(x,t)$ as $t \to \infty$ is characterized by a self-similar solution of the same equation or the corresponding equation without absorption.

6. We intend to answer questions (a) and (b) as follows: The absorption dynamics ($u_t = -u^{-q}$) decreases the solution (in time) with a $O(t^{-1/(q-1)})$ rate. The diffusion process ($u_t = \text{div}(|\nabla u|^{p-2}\nabla u)$) makes mass diffuse, thereby decreasing u at a maximum rate of $O(t^{-n/(p+[p-2]n)}) = O(t^{-1/(q^*-1)})$, as can be seen from the fundamental solutions. Hence, when $1 < q < q^*$, the diffusion-related decay is slower than that of the absorption, so absorption is strong and will dominate, or at least not subordinate to, the diffusion. When $q > q^*$, as diffusion can diminish u at a rate ranging from $O(1)$ (for constant initial data) to $O(t^{-1/(q^*-1)})$ (for L^1 initial data), the relative strengths of absorption and diffusion then heavily depend on the decay rate of the initial data. For fast decay, diffusion dominates; for slow decay, absorption prevails; the critical decay rate is $O(|x|^{-\mu})$.

We shall first prove Theorem 1.8 in the next section since, with the results we established in [3, 5], the proof becomes very easy. Although Theorem 1.7 had already been proven by Zhao [23] (cf. Proposition 1.3 (3)), we provide a different proof in section 2 for completeness. In section 3, we first prove Theorem 1.6 for the case in which $A \in (0,\infty)$. Then the cases in which $A = \infty$ and $A = 0$ can be handled by taking appropriate limits.

2. Proof of Theorems 1.7 and 1.8. For each $\nu \in [\mu, p/(2-p))$ and $\lambda \geq 1$, we define

$$\gamma = 1/[p + (p-2)\nu], \quad \sigma = \gamma(q+1-p)[\nu - \mu], \quad u_\lambda = \lambda^{\gamma\nu} u(\lambda^\gamma x, \lambda t). \tag{2.1}$$

Then

$$\begin{cases} \mathcal{L}u_\lambda = -\lambda^{-\sigma} u^q & \text{in } \mathbb{R}^n \times (0,\infty), \\ u_\lambda(x,0) = \phi_\lambda(x) := \lambda^{\gamma\nu}\phi(\lambda^\gamma x) & \text{on } \mathbb{R}^n \times \{0\}. \end{cases} \tag{2.2}$$

Proof of Theorem 1.8. Set $\nu = n$. Let ψ_λ be the solution to $\mathcal{L}\psi_\lambda = 0$ with initial value ϕ_λ. Define $c_0 = \int_{\mathbb{R}^n} \phi$. Then $\int_{R^n} \phi_\lambda(x)\, dx = c_0$ for every $\lambda > 1$, and $\lim_{\lambda\to\infty} \int_{|x|>\varepsilon} \phi_\lambda(x)\, dx = 0$ for all $\varepsilon > 0$. Hence $\{\frac{1}{c_0}\phi_\lambda\}$ is a δ-sequence (i.e., a sequence approaching the δ function). It then follows from Theorem 3.1 of [5] that

$$\lim_{\lambda\to\infty} \psi_\lambda = E_{c_0} \qquad \text{in} \quad L^\infty((\varepsilon,\varepsilon^{-1});\ L^1(R^n) \cap C(\mathbb{R}^n)) \qquad \forall\, \varepsilon > 0.$$

Now we consider $\{u_\lambda\}_{\lambda>1}$. First by applying an L^∞ estimate (cf. [6, p. 127] or [7]) for ψ_λ and then by comparison, we obtain $u_\lambda \leq \psi_\lambda \leq M(p,n,c_0)t^{-\gamma n}$ for all $\lambda > 0$ and $t > 0$. By the regularity results in [1, 2, 7], $\{u_\lambda\}_{\lambda>1}$ is an equicontinuous family in any compact subset of $\mathbb{R}^n \times (0,\infty)$. Hence there exist a function U and a sequence $\{\lambda_j\}$ with $\lim_{j\to\infty} \lambda_j = \infty$ such that $\lim_{j\to\infty} u_{\lambda_j} = U(x,t)$ uniformly in any compact subset of $R^n \times (0,\infty)$. As σ is positive, (2.2) implies $\mathcal{L}U = 0$ in $\mathbb{R}^n \times (0,\infty)$. In addition, from $u_\lambda \leq \psi_\lambda$, there holds $U \leq E_{c_0}$ in $R^n \times (0,\infty)$ so that $\lim_{t\searrow 0} \sup_{|x|>\varepsilon} U(x,t) = 0$ for all $\varepsilon > 0$. Furthermore, for every fixed $\tau > 0$,

$$\begin{aligned} \limsup_{j\to\infty} \int_{\mathbb{R}^n} |U - u_{\lambda_j}|(x,\tau)dx &= \lim_{M\to\infty} \limsup_{j\to\infty} \int_{|x|>M} |U - u_{\lambda_j}|(x,\tau)|dx \\ &\leq \lim_{M\to\infty} \limsup_{j\to\infty} \int_{|x|\geq M} (E_c + \psi_{\lambda_j})(x,\tau)dx = 0. \end{aligned} \tag{2.3}$$

Note that $\int_{\mathbb{R}^n} u(x,t)dx$ is a nonincreasing function of t, so $c := \lim_{t\to\infty}\int_{\mathbb{R}^n} u(x,t)dx$ exists. Consequently, for every $\tau > 0$,

$$\int_{\mathbb{R}^n} U(x,\tau)dx = \lim_{j\to\infty}\int_{\mathbb{R}^n} u_{\lambda_j}(x,\tau)dx = \lim_{j\to\infty}\int_{\mathbb{R}^n} u(y,\lambda_j\tau)dy$$
$$= \lim_{t\to\infty}\int_{\mathbb{R}^n} u(x,t)dx = c.$$

Thus U is a fundamental solution with mass c; i.e., $U = E_c$. As c is independent of $\{\lambda_j\}$, the whole sequence $\{u_\lambda\}$ must converge to E_c in any compact subset of $\mathbb{R}^n \times (0,\infty)$. A similar argument to that in (2.3) yields $u_\lambda(\cdot,\tau) \to E_c(\cdot,\tau)$ in $L^1(\mathbb{R}^n)$ for every $\tau > 0$. Local regularity estimates then imply that $u_\lambda(\cdot,1) \to E_c(\cdot,1)$ in $L^\infty(\mathbb{R}^n)$. The assertion (1.12) thus follows from the definition of u_λ and the scaling invariance of E_c.

It remains to show $c > 0$. As $u \le \psi_1 \le Mt^{-\gamma n}$ for all $t > 0$,

$$-\frac{d}{dt}\int_{R^n} u(x,t)\ dx = \int_{R^n} u^q(x,t)\ dx \le Mt^{-(q-1)\gamma n}\int_{R^n} u(x,t)\ dx.$$

An integration over $[1,t]$ then gives

$$\log\int_{R^n} u(x,t)\ dx \ge \log\int_{R^n} u(\cdot,1)\ dx + \frac{M(1-t^{1-(q-1)\gamma n})}{1-(q-1)\gamma n} \qquad \forall\, t > 1.$$

Simple calculation shows $1-(q-1)\gamma n = \gamma n[q^* - q] < 0$. Thus $\lim_{t\to\infty}\log\int_{R^n} u(\cdot,t) > -\infty$; i.e., $c > 0$. This completes the proof. ☐

Proof of Theorem 1.7. Let $\nu \in (\mu, n)$ be as in the statement of the theorem, and let $\gamma, \sigma, u_\lambda$, and ϕ_λ be as in (2.1). Let G_B^ν be the solution to $\mathcal{L}G = 0$ with initial data $B|x|^{-\nu}$. Fix an arbitrary $\delta \in (0, (n/\nu - 1)/q)$ (so that $\nu(1+q\delta) < n$). To prove the theorem, it suffices to show

$$\lim_{\lambda\to\infty}\sup_{0<t<2, x_0\in\mathbb{R}^n}\int_{|x-x_0|<1} |u_\lambda(x,t) - G_B^\nu(x,t)|^{1+\delta}dx = 0. \tag{2.4}$$

Indeed, by local regularities of solutions, the above limit implies that $\|u_\lambda(\cdot,1) - G_B^\nu(\cdot,1)\|_{L^\infty(\mathbb{R}^n)} \to 0$ as $\lambda \to \infty$, and therefore (1.11) follows from the definition of u_λ and the scaling invariance of G_B^ν. We now prove (2.4).

Let $x_0 \in \mathbb{R}^n$ and $R > 2$ be arbitrarily given. We can find a smooth cut-off function $\zeta(x)$ satisfying $0 \le \zeta \le 1$ in $\mathbb{R}^n$, $\zeta = 1$ when $|x - x_0| < 1$, $\zeta = 0$ when $|x - x_0| > R$, and $|\nabla\zeta| \le 3/R$ in $\mathbb{R}^n$. Let $s = (1+q\delta)p/(2-p)$. In what follows, all positive constants depending only on p, n, q, δ, and ν will be denoted by the same letter C.

Integrating the identity $0 = (1+q\delta)u_\lambda^{q\delta}\zeta^s(\mathcal{L}u_\lambda + \lambda^{-\sigma}u_\lambda^q)$ over $\mathbb{R}^n$ and using integration by parts, the assumption $|\nabla\zeta| \le 3/R$, and the Cauchy inequality

$$su^{q\delta}\zeta^{s-1}|\nabla u_\lambda|^{p-2}\nabla u_\lambda\cdot\nabla\zeta \le mu_\lambda^{q\delta-1}|\nabla u_\lambda|^p\zeta^s + \frac{1}{q\delta+1}u_\lambda^{q\delta+1}\zeta^s$$
$$+ C|\nabla\zeta|^{(q\delta+1)p/(2-p)},$$

we obtain

$$\frac{d}{dt}\int_{R^n} u_\lambda^{1+q\delta}\zeta^s dx + \int_{R^n}\{(1+q\delta)\lambda^{-\sigma}u_\lambda^{q+q\delta}\zeta^s - u_\lambda^{1+q\delta}\zeta^s\}dx$$
$$\le CR^{n-(q\delta+1)p/(2-p)}. \tag{2.5}$$

Before proceeding further, we first establish an elementary algebraic inequality. Since $p \in (1,2)$, there are positive constants $C(p,n)$ and $c(p,n)$ such that for every $\mathbf{a}, \mathbf{b} \in \mathbb{R}^n$,

$$| \, |\mathbf{a}|^{p-2}\mathbf{a} - |\mathbf{b}|^{p-2}\mathbf{b} \, | \le C(p,n)\{|\mathbf{a}-\mathbf{b}|^2(|\mathbf{a}|+|\mathbf{b}|)^{p-2}\}^{1-1/p},$$
$$(|\mathbf{a}|^{p-2}\mathbf{a} - |\mathbf{b}|^{p-2}\mathbf{b}) \cdot (\mathbf{a}-\mathbf{b}) \ge c(p,n)|\mathbf{a}-\mathbf{b}|^2(|\mathbf{a}|+|\mathbf{b}|)^{p-2}.$$

(They can be proven first for $1 = |\mathbf{a}| \ge |\mathbf{b}|$, and then for the general case by scaling.) It then follows that for any $\mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathbb{R}^n$,

$$(|\mathbf{a}|^{p-2}\mathbf{a} - |\mathbf{b}|^{p-2}\mathbf{b}) \cdot ((\mathbf{a}-\mathbf{b}) + \mathbf{c}) \ge -\hat{C}(p,n)|\mathbf{c}|^p. \tag{2.6}$$

Denote $v = u_\lambda - G_B^\nu$. Multiplying the difference of the equations for u_λ and G_B^ν by $|v|^{\delta-1}v\zeta^s$, integrating the resulting equation over $\mathbb{R}^n$, and using integration by parts and the inequality (2.6) with $\mathbf{a} = \nabla u_\lambda, \mathbf{b} = \nabla G_B^\nu$, and $\mathbf{c} = sv\delta^{-1}\zeta^{-1}\nabla\zeta$, we then obtain

$$\frac{d}{dt}\int_{R^n} |v|^{1+\delta}\zeta^s dx \le \int_{R^n} \{(1+\delta)\lambda^{-\sigma}|v|^\delta u_\lambda^q \zeta^s dx + C|v|^{\delta-1+p}\zeta^{s-p}|\nabla\zeta|^p\}.$$

Cauchy's inequality and the assumptions $|\nabla\zeta| < 3/R$ and $s = (1+q\delta)p/(2-p)$ then yield

$$\frac{d}{dt}\int_{R^n} |v|^{1+\delta}\zeta^s dx - \int_{R^n} \{|v|^{1+\delta}\zeta^s + C\lambda^{-\sigma-\sigma\delta}u_\lambda^{q+q\delta}\zeta^s\}dx \le CR^{n-(1+\delta)p/(2-p)}.$$

Now adding to it a $C\lambda^{-\sigma\delta}$ multiple of (2.6), we obtain

$$\begin{aligned}\frac{d}{dt}\int_{\mathbb{R}^n} \{|v|^{1+\delta} + C\lambda^{-\sigma\delta}u_\lambda^{1+q\delta}\}\zeta^s dx - \int_{\mathbb{R}^n}\{|v|^{1+\delta} + C\lambda^{-\sigma\delta}u_\lambda^{1+q\delta}\}\zeta^s dx \\ \le CR^{n-(1+\delta)p/(2-p)}\end{aligned}$$

for all $\lambda > 1$. Gronwall's inequality then yields

$$\sup_{0<t<2} e^{-t}\int_{\mathbb{R}^n} |v|^{1+\delta}\zeta^s dx \le CR^{n-(1+\delta)p/(2-p)} + \int_{\mathbb{R}^n} \zeta^s\{|v(\cdot,0)|^{1+\delta} + C\lambda^{-\sigma\delta}\phi_\lambda^{1+q\delta}\}dx.$$

Denote by $B_r(x_0)$ the ball of radius r and center x_0. That ζ is a cut-off function then gives, after replacing v by $u_\lambda - G_B^\nu$,

$$\begin{aligned}&\sup_{x_0\in\mathbb{R}^n}\sup_{0<t<2}\int_{B_1(x_0)} |u_\lambda - G_B^\nu|^{1+\delta}dx \le CR^{n-(1+\delta)p/(2-p)} \\ &\quad + C\sup_{x_0\in\mathbb{R}^n}\int_{B_R(x_0)} |\phi_\lambda - B|x|^{-\nu}|^{1+\delta}dx + C\sup_{x_0\in\mathbb{R}^n}\lambda^{-\sigma\delta}\int_{B_R(x_0)}\phi_\lambda^{1+q\delta}dx.\end{aligned} \tag{2.7}$$

Set $M = \sup_{x\in\mathbb{R}^n}|x|^\nu\phi(x)$. Then the definition of ϕ_λ implies $\phi_\lambda(x) \le M|x|^{-\nu}$. Consequently, the last term in (2.7) converges to zero as $\lambda \to \infty$.

To estimate the first integral in (2.7), one notices that $\phi_\lambda(x) \to B|x|^{-\nu}$ uniformly in $\mathbb{R}^n \setminus B_\varepsilon(0)$ for any $\varepsilon > 0$. As $\int_{B_\varepsilon(0)}|x|^{-\nu(1+\delta)}dx = O(\varepsilon^{n-(1+\delta)\nu})$, we see also that, as $\lambda \to \infty$, the first term in (2.7) converges to zero. Hence, sending $\lambda \to \infty$, we obtain from (2.7) that

$$\limsup_{\lambda\to\infty}\sup_{x_0\in\mathbb{R}^n, 0<t<2}\int_{B_1(x_0)}|u_\lambda - G_B^\nu|^{1+\delta}dx \le CR^{n-(1+\delta)p/(2-p)}.$$

Sending $R \to \infty$, we then obtain (2.4), thereby completing the proof. ∎

3. Proof of Theorems 1.5 and 1.6. Set $\gamma = 1/[p+(p-2)\mu]$. Then $(q-1)\gamma\mu = 1$. For some $\delta > 0$ to be chosen later, we make a self-similar transformation:

$$u(x,t) = [(t+\delta)/\gamma\mu]^{-\gamma\mu} w(y,\tau), \quad y = [(t+\delta)/\gamma\mu]^{-\gamma} x, \qquad \tau = \gamma\mu \, \log[(t+\delta)/\delta]. \tag{3.1}$$

Then w satisfies

$$\begin{cases} w_\tau = \operatorname{div}(|\nabla w|^{p-2}\nabla w) + \mu^{-1} y \cdot \nabla w + w - w^q, & y \in R^n, \quad \tau > 0, \\ w(y,0) = [\delta/\gamma\mu]^{\gamma\mu} \phi([\delta/\gamma\mu]^{\gamma} y), & y \in \mathbb{R}^n. \end{cases} \tag{3.2}$$

Now we fix $\delta = \gamma\mu \|\phi\|_{L^\infty(\mathbb{R}^n)}^{-1/\gamma\mu}$. Then

(1) $0 \le w(y,0) \le 1$ in R^n;

(2) $\lim_{|y|\to\infty} |y|^\mu w(y,0) = \lim_{|x|\to\infty} |x|^\mu \phi(x) = A$.

To prove Theorem 1.6 for $A \in (0,\infty)$, we need only to show that

$$\lim_{\tau\to\infty} w(y,\tau) = W_A(y,1) = w_A(|y|) \quad \text{uniformly in } y \in \mathbb{R}^n; \tag{3.3}$$

here $w_A(r)$, $r \in [0,\infty)$, is the unique solution to the following boundary value problem:

$$\begin{cases} \mathcal{L}_r w := |w'|^{p-2}[(p-1)w'' + (n-1)r^{-1}w'] + w + \mu^{-1} r w' - w^q = 0, \quad r > 0, \\ w_A'(0) = 0, \quad w(r) > 0 \quad \text{on } [0,\infty), \quad \text{and} \quad \lim_{r\to\infty} r^\mu w(r) = A. \end{cases} \tag{3.4}$$

We prove (3.3) by using sub- and supersolutions. To ease the computation, it is convenient to introduce new dependent and independent variables

$$w(r) = r^{-\mu} J(\theta), \qquad r = Re^\theta, \tag{3.5}$$

where R is a parameter of our choice. Then $\mathcal{L}_r w = r^{-\mu q}\mathcal{L}_\theta J$, where, denoting $\dot{} = d/d\theta$,

$$\mathcal{L}_\theta J := |\dot J - \mu J|^{p-2}\{a(\ddot J - \mu \dot J) + b(\dot J - \mu J)\} + \mu^{-1} R^l e^{l\theta} \dot J - J^q,$$
$$a = (p-1), \quad b = (n-1) - (\mu+1)a, \quad l = \mu(q-1).$$

LEMMA 3.1. *For every $A \in (0,\infty)$, there exists $R_0(A) > 0$ such that for every $R > R_0$, there are two functions $w^+_{A,R}(r)$ and $w^-_{A,R}(r)$ with the following properties:*

(a) *$w^+_{A,R}(r) = 1$ in $[0,R]$, $Ar^{-\mu} \le w^+_{A,R}(r) \le 1$ in $[R,\infty)$;*

(b) *$w^-_{A,R}(r) = 0$ in $[0,R]$, $0 \le w^-_{A,R}(r) \le Ar^{-\mu}$ in $[R,\infty)$;*

(c) *$\mathcal{L}_r w^+_{A,R} \le 0$ and $\mathcal{L}_r w^-_{A,R} \ge 0$ in $(0,\infty)$ in the distributional sense;*

(d) *$\lim_{r\to\infty} r^\mu w^+_{A,R}(r) = \lim_{r\to\infty} r^\mu w^-_{A,R}(r) = A$.*

Proof. (I) Construction of $w^+_{A,R}(r)$. We define $w^+_{A,R} = 1$ in $[0,R]$ and $w^+_{A,R} = r^{-\mu}[A + (R^\mu - A)(R/r)^l]$ in $[R,\infty)$. Then $w^+_{A,R}$ satisfies (a) and (d). In addition, $(w^+_{A,R})'' \le 0$ at $r = R$ in the distributional sense. It remains to check $\mathcal{L}_r w^+_{A,R} \le 0$ in (R,∞) or, equivalently, $\mathcal{L}_\theta J \le 0$ in $(0,\infty)$, where

$$J = A + (R^\mu - A)e^{-l\theta}.$$

Note that $\mu J - \dot J = \mu A + (R^\mu - A)(l+\mu)e^{-l\theta} > \mu A$ in $(0,\infty)$. Also, assuming $R > 2A$, then $R^\mu - A > R^\mu/2$. Hence, using $p - 2 < 0$, we can calculate, for all $\theta \in (0,\infty)$,

$$\mathcal{L}_\theta J = |rJ' - \mu J|^{p-2}\{(l+\mu)(al-b)(R^\mu - A)e^{-l\theta} - \mu b A\}$$
$$-l\mu^{-1}(R^\mu - A)R^l - J^q \le (\mu A)^{p-2}(l+\mu)(al+|b|)R^\mu - lR^\mu R^l/(2\mu) \le 0$$

if R is sufficiently large (depending on A). The construction of $w^+_{A,R}(r)$ is now complete.

(II) Construction of $w^-_{A,R}(r)$. We need only construct $J(\theta) := r^\mu w^-_{A,R}(r)|_{r=Re^\theta}$. Let $m \geq 1$ be the smallest integer such that $(p-1)(2m+1) \geq 1$. Define

$$\Theta(J) = \begin{cases} \mu[J - (J-\varepsilon)^{2m+1}], & 0 < J < 2\varepsilon, \\ (A-J)D, & 2\varepsilon \leq J < A, \end{cases}$$
$$D = \mu[2\varepsilon - \varepsilon^{2m+1}]/[A - 2\varepsilon],$$
$$\varepsilon = \min\{1/(2m+1),\ 1/(4A),\ l/(4\mu A)\}.$$

Clearly Θ is Lipschitz continuous and positive in $[0, A)$. In addition, $0 < D \leq l$.

Let $J(\theta), \theta \in [0,\infty)$, be the solution to

$$\dot{J} = \Theta(J) \quad \text{in} \ \ (0,\infty), \qquad J(0) = 0.$$

Since $\Theta > 0$ in $[0, A)$ and $\Theta(A) = 0$, J is strictly increasing, and $\lim_{\theta\to\infty} J(\theta) = A$.

Now we define $w^-_{A,R} = 0$ for $r \leq R$ and $w^-_{A,R} = r^{-\mu} J(\theta)|_{\theta=\log(r/R)}$ for $r > R$. Then $w^-_{A,R}$ satisfies conditions (b) and (d) in the lemma. In addition, $(w^-_{A,R})'' \geq 0$ at $r = R$ in the distributional sense. It remains to check that $\mathcal{L}_r w^-_{A,R} \geq 0$ in (R,∞), which is equivalent to checking that $\mathcal{L}_\theta J(\theta) \geq 0$ for all $\theta > 0$.

Let $\theta_1 := \int_0^{2\varepsilon} dJ/\Theta(J)$ be the unique point such that $J = 2\varepsilon$. We consider two cases: (i) $\theta \in (\theta_1, \infty)$ and (ii) $\theta \in (0, \theta_1]$.

Case (i). $\theta \in (\theta_1,\infty)$. Then $J = A-(A-2\varepsilon)e^{D(\theta_1-\theta)}$ and $\dot{J} = D(A-2\varepsilon)e^{D(\theta_1-\theta)}$. Also, $\mu J - \dot{J} = (\mu + D)J - DA \geq 2\varepsilon(\mu + D) - AD = \mu\varepsilon^{2m+1}$. Thus, as $p < 2$ and $D \leq l$,

$$\mathcal{L}_\theta J \geq -(\mu\varepsilon^{2m+1})^{p-2} C(A) + \mu^{-1} R^l D(A-2\varepsilon)e^{D\theta_1 + (l-D)\theta} - A^q \geq 0$$

provided that R is large enough.

Case (ii). $\theta \in (0, \theta_1]$. Then $J \in (0, 2\varepsilon]$ and $\dot{J} - \mu J = -\mu(J-\varepsilon)^{2m+1}$ so that

$$\begin{aligned} &|\dot{J} - \mu J|^{p-2}\{a(\ddot{J} - \mu\dot{J}) + b(\dot{J} - \mu J)\} \\ &= -\mu^{p-1}|J-\varepsilon|^{(p-2)(2m+1)+2m}\{(2m+1)a\dot{J} + b(J-\varepsilon)\} \geq -C(A) \end{aligned}$$

since $(p-2)(2m+1) + 2m \geq 0$. Noting that $\dot{J} = \mu[J - (J-\varepsilon)^{2m+1}] \geq \mu\varepsilon^{2m+1}$, we have

$$\mathcal{L}_\theta J \geq -C(A) + R^l e^{l\theta}\varepsilon^{2m+1} - (2\varepsilon)^q \geq 0$$

if R is large enough. This completes the proof of the lemma. ∎

Lemma 3.2. *For every $A \in (0,\infty)$, $R > R_0(A)$, let $W^\pm_{A,R}(y,\tau)$ be the solution to the PDE in* (3.2) *with initial value $w^\pm_{A,R}(|y|)$. Then*

$$\lim_{\tau\to\infty} \|W^\pm_{A,R}(\cdot,\tau) - w_A(|\cdot|)\|_{L^\infty(R^n)} = 0.$$

Proof. Since $w^+_{A,R}(|y|)$, as a function of (y,τ), is a supersolution to the PDE in (3.2), by comparison, $W^+_{A,R}(\cdot,\tau) \leq w^+_{A,R}(|\cdot|)$ for all $\tau > 0$. In turn, this implies that $W^+_{A,R}(\cdot,\tau+\tau_1) \leq W^+_{A,R}(\cdot,\tau_1)$ for all $\tau > 0$ and $\tau_1 \geq 0$. It then follows that $W^+_{A,R}(\cdot,\tau)$

is monotone decreasing in τ. Similarly, $W^-_{A,R}(\cdot,\tau)$ is monotone increasing in τ. Hence there exist $w^{\infty,\pm}_{A,R}$ such that

$$W^{\pm}_{A,R}(\cdot,\tau) \to w^{\infty,\pm}_{A,R}(\cdot) \qquad \text{as } \tau\to\infty \text{ pointwise in } R^n. \tag{3.6}$$

It is clear that $w^{\infty,\pm}_{A,R}$ solve $\mathcal{L}_r w^{\infty,\pm}_{A,R} = 0$ and are radially symmetric and that

$$w^-_{A,R}(|y|) \le w^{\infty,-}_{A,R}(y) \le w^{\infty,+}_{A,R}(y) \le w^+_{A,R}(|y|).$$

It follows that $\lim_{|y|\to\infty} |y|^\mu w^{\infty,\pm}_{A,R} = A$. By the uniqueness of solution to (3.4) (cf. [3]), $w^{\infty,\pm}_{A,R}(y) = w_A(|y|)$. Hence $\lim_{\tau\to\infty} W^{\pm}_{A,R}(\cdot,\tau) = w_A(|\cdot|)$. Since $W^{\pm}_{A,R}(y,\tau) \le w^+_{A,R}(|y|)$, by local regularity, the convergence is uniform in any compact subset of $\mathbb{R}^n$. Further, as $w^+_{A,R}(|y|) = O(|y|^{-\mu})$, the convergence is also in L^∞. This completes the proof. □

Proof of Theorems 1.5 *and* 1.6.

Case 1. $0 < A < \infty$. Let $w(y,\tau)$ be the solution to (3.2). We need to prove (3.3). Since we have the limit $\lim_{|y|\to\infty} |y|^\mu w_0(y) = A$, for every $\varepsilon > 0$, there exists $R_\varepsilon > 0$ such that

$$A - \varepsilon < |y|^\mu w(y,0) < A + \varepsilon \qquad \forall\ |y| \ge R_\varepsilon.$$

It follows from Lemma 3.1 (a), (b) and $0 \le w(y,0) \le 1$ that

$$w^-_{A-\varepsilon,R_\varepsilon}(|\cdot|) \le w(\cdot,0) \le w^+_{A+\varepsilon,R_\varepsilon}(|\cdot|).$$

Consequently, by the comparison principle,

$$W^-_{A-\varepsilon,R_\varepsilon}(y,\tau) \le w(y,\tau) \le W^+_{A+\varepsilon,R_\varepsilon}(y,\tau).$$

Thus, by Lemma 3.2,

$$\limsup_{\tau\to\infty} \|w(y,\tau) - w_A(|y|)\|_{L^\infty} \le \|w_{A+\varepsilon} - w_A\|_{L^\infty} + \|w_{A-\varepsilon} - w_A\|_{L^\infty}.$$

However, since $\varepsilon > 0$ is arbitrary, we obtain (3.3).

Case 2. $\lim_{|x|\to\infty} |x|^{-\nu}\phi(x) = A = \infty$. By comparison,

$$\liminf_{\tau\to\infty} w(y,\tau) \ge \lim_{A\to\infty} w_A(|y|) = 1$$

uniformly in any compact subset of $y \in \mathbb{R}^n$. Since $w \le 1$ for all y and $\tau \ge 0$, we conclude that as $\tau\to\infty$, $w(y,\tau)\to 1$ uniformly in any compact subset of $\mathbb{R}^n$. Using the definition of $w(y,\tau)$ in (3.1) and the fact that $\gamma\mu = 1/(q-1)$, the assertion of Theorem 1.5 thus follows.

Case 3. $A = 0$. If $1 < q < q^*$, then

$$\limsup_{\tau\to\infty} w(y,\tau) \le \lim_{A\to 0} w_A(|y|) = w_0(|y|) = W_0(y,1),$$

where w_0 is the unique positive solution of (3.4) with $A = 0$ and W_0 is the unique VSS. It remains to show that

$$\liminf_{\tau\to\infty} w(y,\tau) \ge w_0(|y|). \tag{3.7}$$

To do this, we go back to the PDE (I). Set $\nu = \mu$, and define γ and u_λ as in (2.1). Then (2.2) holds with $\sigma = 0$. In addition,

$$\lim_{\lambda\to\infty} \int_{|x|\le\varepsilon} \phi_\lambda(x)dx = \lim_{\lambda\to\infty} \int_{|x|\le\varepsilon\lambda^\gamma} \lambda^{\gamma(\mu-n)}\phi(x)dx = \infty \quad \forall\, \varepsilon > 0.$$

Hence, for each $c > 0$, there exists a sequence $\{\psi_\lambda^c\}_{\lambda\ge1}$ such that $0 \le \psi_\lambda \le \phi_\lambda$ for all $\lambda \ge 1$ and $\{\frac{1}{c}\psi_\lambda\}_{\lambda>1}$ is a δ-sequence. Consequently, the solution u_λ^c of $\mathcal{L}u_\lambda = -u_\lambda^q$ with initial value $u_\lambda^c(x,0) = \psi_\lambda^c$ tends to the fundamental solution u^c of $\mathcal{L}u_\lambda = -u_\lambda^q$ with initial mass c as $\lambda \to \infty$; see [5]. Since $u_\lambda \ge u_\lambda^c$, we have

$$\liminf_{\lambda\to\infty} u_\lambda \ge \lim_{\lambda\to\infty} u_\lambda^c = u^c.$$

It follows that

$$\liminf_{\lambda\to\infty} u_\lambda(x,t) \ge \lim_{c\to\infty} u^c(x,t) = W_0(x,t).$$

A direct translation of the above relation in terms of $w(y,\tau)$ is exactly (3.7). Thus

$$\lim_{\tau\to\infty} w(y,\tau) = w_0(|y|) = W_0(y,1)$$

uniformly in any compact subset of $\mathbb{R}^n$. Since for $A = 1$ and some large R, $w(y,\tau) \le w_{A,R}^+(|y|)$ for all $\tau > 0$ and $w_{1,R}^+$ decays to zero as $|y| \to \infty$, the above limit is uniform in $\mathbb{R}^n$. This completes the proof of Theorem 1.6 for the case in which $A = 0$ and $1 < q < q^*$.

If $q \ge q^*$, then

$$\limsup_{\tau\to\infty} \|w(y,\tau)\|_{L^\infty(\mathbb{R}^n)} \le \lim_{A\to0+} \|w_A(|y|)\|_{L^\infty(\mathbb{R}^n)} = 0.$$

Finally, we prove the second limit in (1.10), where $q = q^*$. By taking smaller ϕ if necessary, we can assume that $\phi \le 1$ and has compact support. Then $u \le 1$ in $\mathbb{R}^n \times (0,\infty)$. Consequently, $\mathcal{L}u = -u^{q^*} \ge -u^{q^*-\varepsilon}$. Now let u^ε be the solution to $\mathcal{L}u^\varepsilon = -u^{q^*-\varepsilon}$ and initial data $u^\varepsilon(x,0) = \phi$. Then, from what we just proved, we have $\lim_{t\to\infty} t^{1/(q-\varepsilon-1)}u^\varepsilon(0,t) = W_0^{q^*-\varepsilon}(0,1)$, where $W_0^{q^*-\varepsilon}$ is the VSS for $q = q^*-\varepsilon$. Thus $\liminf_{t\to\infty} t^{1/(q^*-\varepsilon-1)}u(0,t) > 0$. As ε is arbitrary, the second limit in (1.10) follows. This completes the proof of Theorem 1.6. $\square$

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