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J. Differential Equations 260 (2016) 7982–7999

Journal of Differential Equations

www.elsevier.com/locate/jde

Existence of traveling waves of auto-catalytic systems with decay

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Received 30 October 2015; revised 14 January 2016

Available online 17 March 2016

Abstract

This article establishes the existence of traveling waves of a class of reaction–diffusion systems which model the pre-mixed isothermal autocatalytic chemical reaction of order m ($m > 1$) between two chemical species, a reactant and an auto-catalyst, and a linear decay. Moreover, our result shows that the set of speed is contained in a bounded interval for any fixed initial value at $x = -\infty$. This is in strong contrast to either the reaction–diffusion systems of autocatalytic chemical reaction of the order m without decay, or to the systems which have the same order of decay, which were shown by various authors (e.g. [8,17,13,26]) that the set of traveling wave speeds contains $[c_*, \infty)$ for some $c_* > 0$.

The same systems also appear in a mathematical model of microbial growth and competition in a flow reactor; see [2,24].

MSC: 34C20; 34C25; 92E20

Keywords: Autocatalytic chemical systems; Microbial growth and competition; Traveling wave

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http://dx.doi.org/10.1016/j.jde.2016.02.009
0022-0396/

1. Introduction

In this paper we consider a reaction–diffusion system

$$u_t = d_1 u_{xx} - k_1 u v^m, \qquad v_t = d_2 v_{xx} + k_1 u v^m - k_2 v, \tag{1.1}$$

where $d_1 > 0$, $d_2 > 0$, $m > 1$ and $k_1 > 0$, $k_2 > 0$ are constants. It models an auto-catalytic chemical reaction with a decay step:

$$A + mB \longrightarrow (m+1)B \quad \text{with rate } k_1[A][B]^m, \qquad B \longrightarrow C \quad \text{with rate } k_2[B],$$

where C is an inert product. It also appears in a mathematical model of microbial growth and competition in a flow reactor; see [2] and [24].

By scaling, we can without loss of generality assume that $d_1 = 1$, $d_2 = d$, $k_1 = 1$, and $k_2 = 1$.

It is easy to verify that all equilibrium points of (1.1) are in the form $(a, 0)$ with $a \in R$. Hence, any traveling wave $u(x,t) = u(x - ct)$, $v(x,t) = v(x - ct)$, with $c > 0$ as the speed, must link one equilibrium point $(u_0, 0)$ at $x = -\infty$ to another one $(u_1, 0)$ at $x = \infty$ with $u_1 > u_0 > 0$, and the equilibrium points are saddle in nature. Thus, we consider traveling wave problem

$$\begin{cases} u'' + cu' = uv^m, & u' > 0 \quad \text{in } \mathbb{R}, \\ dv'' + cv' = v - uv^m, & v > 0 \quad \text{in } \mathbb{R}, \\ u(-\infty) = u_0, \quad v(-\infty) = 0, & v(\infty) = 0, \quad u(\infty) < \infty. \end{cases} \tag{1.2}$$

The advantage of (1.1), from a modeling point of view, over the system

$$u_t = d_1 u_{xx} - k_1 u v^m, \qquad v_t = d_2 v_{xx} + k_1 u v^m,$$

which models auto-catalytic chemical reaction without a decay step, is that it was demonstrated in [5] by asymptotic analysis and numerical computation, and rigorously proved for $m = 1$ in [9] that any small amount of B introduced locally with uniform initial distribution of A can generate traveling wave. This feature, which should hold true for general m order reaction, seems to be contradictory to the fact that in the relevant experimental result of chemical reactions, the initiation of traveling wave calls for sufficient amount of B to be added [27]. To overcome this deficiency in modeling for lacking of threshold phenomenon, Gray [12] made the observation that the auto-catalyst B cannot be stable indefinitely and should be used to produce other chemicals. In particular, it was suggested in [19,20] that B decays to an inert product C at a rate of order n,

$$B \longrightarrow C \qquad \text{with rate } kv^n.$$

The resulting PDE system is

$$\begin{cases} u_t = u_{xx} - uv^m \\ v_t = dv_{xx} + uv^m - kv^n, \end{cases}$$

where $m, n \geq 1$ and $k > 0$ is a rate constant, after a simple scaling.

If $m = n$, the above system takes the form:

$$(II)\begin{cases} u_t = u_{xx} - uv^m \\ v_t = dv_{xx} + uv^m - kv^m, \end{cases} \tag{1.3}$$

which was extensively studied in the literature.

It was proved for $m = 1$ in [17,25] that for any $0 < u_0 < k$, there exists a traveling wave solution for (II) if $c \geq c^* = c^*(d, k, u_0)$. See also [14–16] for some related results. In case $m = n > 1$, results similar to the situation of $m = 1$ were derived in [13,26], again demonstrating the same phenomenon as mono-stable nonlinearity for a single equation. A more recent work [11] studies the case of $m = 1$, $n > 1$ shows the same kind of result for existence of traveling waves.

In this paper we prove the following:

Theorem 1. *Let $d > 0$, $m > 1$, and $u_0 > 0$ be given constants. There exists a positive constant c such that (1.2) admits a solution. In addition, the set of speeds for existence lies in a bounded interval for a given value of $u_0 > 0$. Furthermore, the speed c must satisfy*

$$c^2 < 2d\left[\max(1,d)\left(\frac{m+1}{2u_0}\right)^{m/(m-1)}\frac{m+1}{m-1} + m - 1\right].$$

Remark. It is interesting to note that the same kind of result as above can be derived if the linear decay term v is replaced by v^n with $1 < n < m$. This will be proved in a forthcoming work [30].

Our theorem is pathbreaking in the sense that it provides a phenomenon that is not seen in the literature. Indeed, as far as we know, for scalar equations or for systems of equations, almost all results for traveling waves can be put into two categories (cf. [1,3,4,10,21,22]): (i) Mono-stable type in which there exists c_* such that for each $c \geqslant c_*$, there exists a traveling wave of speed c, (ii) Bistable type in which there exists a unique traveling wave speed. So far, all results on traveling waves for autocatalytic systems are mono-stable type: there exists $c_* > 0$ such that for every $c \geqslant c_*$, there exists a traveling wave of speed c; see, for example, [6,8,9,18,25,17,14–16,13,26] and the references therein. The existence of a finite maximum traveling wave speed shows that (1.2) is not in the mono-stable category. It is also not in the bi-stable category since in our forthcoming paper [7] we shall show that the number of wave speeds increases to infinity as $u_0 \to \infty$.

Our proof of Theorem 1 is original. Most of the past work on traveling wave problems for autocatalytic system is by means of three classical methods (i) invariant region, (ii) phase plane, say using u as an independent variable and expressing $v = P(u)$ and $u' = Q(u)$ and (iii) classical shooting argument using the boundary condition at $-\infty$ (or at ∞) as the changing parameter. Here we use a novel shooting argument that is new in the literature for the type of problem at hand.

Since the phenomenon we discovered is new in the literature and most classical tools do not apply, our result may appear preliminary. But, since the shape of v is not monotone, and can have a large number of oscillations before approaching zero, see [7], it is quite challenge to study the existence of traveling wave analytically. There are quite a number of unanswered questions, for example:

1. The number of traveling waves. Denoting by $n(u_0)$ the number of traveling wave speeds. We know from [7], where we treat the case of $u_0 \gg 1$, that $\lim_{u_0\to\infty} n(u_0) = \infty$. We speculate that $n(u_0) = 1$ when $0 < u_0 \ll 1$.

2. The stability of traveling waves. We speculate that each traveling wave is stable in certain function space.
3. The dynamical behavior of initial value problem. Suppose $(u, v)|_{t=0} = (u_0, 0)$ for $x < 0$ and $(u, v)|_{t=0} = (u_1, 0)$ for $x > 1$, where $0 < u_0 < u_1$. What would be the asymptotic behavior of the solution of (1.1) as $t \to \infty$? If (c, U, V) is a traveling wave of (1.2) and $u_1 = U(\infty)$, we would expect that the solution of (1.1) approaches a translation of the traveling wave. However, it is not clear what happens if $u_1 \neq U(\infty)$ for any traveling wave (c, U, V).

We hope that this paper opens a new avenue for the study of not only this important model but also related traveling wave problems.

Our problem (1.2) is very different from (1.3) and the approaches adopted in the works [13–17,25,26] cannot be applied to the present case. In particular, an important fact in (1.3) is that v must be bell-shaped for a traveling wave. But, the same cannot be said of (1.2) and some preliminary computation indicates that v may oscillate before decaying to zero at ∞ for different choices of u_0. In addition, a key insight of [17] and [13,26] is that the behavior of solution with $u_0 \sim k$ is very different from that with u_0 small. We do not have the same observation because the nonlinear term behaves very differently. The interplay of uv^m and kv generate complex dynamics for v. Furthermore, the solution structure of (1.2) is more complex than that of (1.3).

Finally, the stability of traveling wave, which is always the central question in the study of traveling wave, is a very challenge question. Different from (1.3), we speculate that (1.1) admits stacked traveling waves of different speeds; see Fife and McLeod [10] for description of stacked traveling waves. We note some related works on multi-dimensional steady-state for (1.2) and (1.1) in [23,28,29].

Our approach is based on a shooting argument for the speed set $c \in [0, \infty)$. Because of the special nature of our argument, we briefly explain how our proof works. The basic idea is that for each $c \geqslant 0$ we consider the ode system in (1.2) with "initial" value prescribed at $-\infty$. According to the behavior of the solution at ∞, every $c \in [0, \infty)$ must belong to one of the three non-intersecting subsets, $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ which have different characteristic of the underlying solutions (detailed definition will be given in Section 2). We show $\mathcal{A}$ is open and contains 0 and $\mathcal{B}$ is open and contains all large c. The set $\mathcal{C}$ which corresponds to traveling wave must be non-empty.

The organization of this paper is as follows. In section 2 we show some preliminary results to classify solutions of "initial value problem" into three different types, which justifies the definition of the sets $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$. We prove that the set $\mathcal{A}$ is open and contains 0 in section 3. In section 4, we show $\mathcal{B}$ is open and contains a neighborhood of ∞, and therefore, $\mathcal{C}$, which corresponds to the traveling wave, is bounded and non-empty. The existence of traveling wave follows immediately.

2. Preliminary

For each constant $c \geqslant 0$, we consider the initial value problem, for $(u, v) = (u(x, c), v(x, c))$,

$$\begin{cases} u'' + cu' = uv_+^m & \text{in } \mathbb{R}, \\ dv'' + cv' - v = -uv_+^m & \text{in } \mathbb{R}, \\ [u, v] = [u_0, e^{\lambda x}] + O(1)e^{m\lambda x} & \text{as } x \to -\infty, \end{cases} \tag{2.1}$$

where λ is the positive root of $d\lambda^2 + c\lambda = 1$ and $v_+ := \max\{v, 0\}$.

Lemma 2.1. *For each $c \geqslant 0$, problem (2.1), with $\lambda = 2/(\sqrt{c^2+4d}+c)$ and $v_+ := \max\{v, 0\}$, admits a unique solution. The solution depends on c continuously and satisfies $u' > 0$ in $\mathbb{R}$. In addition, define*

$$x_1(c) := \sup\{z \in \mathbb{R} \mid v' > 0 \text{ in } (-\infty, z)\}, \qquad x_2(c) := \sup\{z \in \mathbb{R} \mid v > 0 \text{ in } (-\infty, z)\}; \tag{2.2}$$

then $x_1(c) < \infty$, $v'(x_1(c), c) = 0 > v''(x_1(c), c)$, and one and only one of the following holds:

(1) *$x_2(c) < \infty$ and $v(x_2(c), c) = 0$;*
(2) *$x_2(c) = \infty$ and $\lim_{x\to\infty} u(x, c) = \infty$;*
(3) *$x_2(c) = \infty$ and $\lim_{x\to\infty} u(x, c) < \infty$. In this case $\lim_{x\to\infty} v(x, c) = 0$, so (c, u, v) solves (1.2).*

We denote

$$\begin{aligned}
\mathcal{A} &:= \{c \geqslant 0 \mid x_2(c) < \infty\}, \\
\mathcal{B} &:= \{c \geqslant 0 \mid x_2(c) = \infty, \lim_{x\to\infty} u(x, c) = \infty\}, \\
\mathcal{C} &:= \{c \geqslant 0 \mid x_2(c) = \infty, \lim_{x\to\infty} u(x, c) < \infty\}.
\end{aligned} \tag{2.3}$$

We shall show that $\mathcal{A}$ and $\mathcal{B}$ are open, $0 \in \mathcal{A}$, and $[C, \infty) \subset \mathcal{B}$ for some $C \gg 1$. Thus, $\mathcal{C}$ is non-empty and problem (1.2) admits a solution for some $c > 0$. In addition, if c is a wave speed, i.e., (1.2) admits a solution, then after a translation, the solution of (1.2) is also a solution of (2.1), so $c \in \mathcal{C}$ and $0 < c < C$; namely, the set of speeds is a bounded set. In a forthcoming paper [7], we shall investigate the set $\mathcal{C}$ when u_0 is sufficiently small; in particular we shall show that $\mathcal{C}$ contains more than one point (when $0 < u_0 \ll 1$).

Proof of Lemma 2.1. We divide the proof in several steps.

1. Consider the dynamical system

$$u' = p, \quad v' = q, \quad p' = -cp + uv_+^m, \quad dq' = -cq + v - uv_+^m. \tag{2.4}$$

On the (u, v, p, q) phase space, the equilibrium point $(u_0, 0, 0, 0)$ is a saddle, with a one-dimensional unstable manifold. Thus, there exists a unique solution satisfying $[u, v, p, q] = [u_0, e^{\lambda x}, 0, \lambda e^{\lambda x}] + O(e^{m\lambda x})$ as $x \to -\infty$. We denote the solution by $(u(x), v(x))$ or by $(u(x, c), v(x, c))$ when necessary. The solution can be extended to a maximal existence interval $(-\infty, X)$. In the existence interval, we can integrate the equation $(e^{cx}u')' = e^{cx}uv_+^m$ to derive that

$$u'(x) = \int_{-\infty}^{x} e^{c(y-x)} u(y) v_+^m(y)\, dy. \tag{2.5}$$

It then follows that $u > 0$ and $u' > 0$ in the maximal existence interval.

2. Next we show that $X = \infty$. Let $\mu = (\sqrt{c^2+4d}+c)/(2d) = c/d + \lambda = 1/(d\lambda)$ be the positive root of $d\mu^2 - c\mu = 1$. Then

$$d[e^{\mu x}(v'-\lambda v)]' = e^{\mu x}[dv'' + d(\mu-\lambda)v' - d\mu\lambda v] = e^{\mu x}[dv'' + cv' - v] = -e^{\mu x}uv_+^m.$$

After integration, we obtain

$$\lambda v - v' = \frac{1}{d}\int_{-\infty}^{x} e^{\mu(y-x)}u(y)v_+^m(y)dy > 0. \tag{2.6}$$

Hence, $(e^{-\lambda x}v(x))' < 0$, so $v(x) < e^{\lambda x}$ and $v_+^m(x) < e^{m\lambda x}$ for every $x \in (-\infty, X)$. Consequently, we obtain from (2.5) that $0 < u'(x) \leqslant u(x)e^{m\lambda x}/(c+m\lambda)$. This in turn, implies that $u_0 \leqslant u(x) \leqslant u_0 \exp(e^{m\lambda x}/(m\lambda[c+m\lambda]))$. Hence, uv_+^m is locally bounded. Thus $X = \infty$.

Using an integral formulation, one can show that the solution depends on c continuously.

3. Now define $x_1(c)$ as in (2.2). We claim that $x_1(c) < \infty$. Suppose not. Then $v' > 0$ in $\mathbb{R}$. By (2.5), we find that $u'(x) \geqslant u(0)v^m(0)[1-e^{-cx}]/c$ for every $x > 0$, so $\lim_{x\to\infty} u(x) = \infty$. Now let $y > 0$ be a point such that $u(y)v^{m-1}(y) = 2$. Then in $[y, \infty)$, $dv'' + cv' = v[1 - uv^{m-1}] \leqslant -v(y)$ and $dv'(x) \leqslant dv'(y) + c[v(y) - v(x)] - v(y)[x-y] = dv'(y) + [y - x + c]v(y) - cv(x) + yv(y) \to -\infty$ as $x \to \infty$, contradicting $v' > 0$ in $\mathbb{R}$. Hence, we must have $x_1(c) < \infty$. By continuity, $v'(x_1(c)) = 0 \geqslant v''(x_1(c))$. Since $v''(x_1(c)) = 0$ would imply that $dv'''(x_1(c)) = -u'v^m < 0$ contradicting $v' > 0$ in $(-\infty, x_1(c))$, we must have $v''(x_1(c)) < 0$.

4. Define $x_2(c)$ as in (2.2). Then, by definition, $x_2(c) > x_1(c)$ and $v > 0$ in $(-\infty, x_2(c))$. If $x_2(c) < \infty$, then by continuity, $v(x_2(c)) = 0$ so $c \in \mathcal{A}$. If $x_2(c) = \infty$, then since $u' > 0$ in $\mathbb{R}$, we have either $\lim_{x\to\infty} u(x) = \infty$ or $\lim_{x\to\infty} u(x) < \infty$, so $c \in \mathcal{B}\cup\mathcal{C}$. Hence, $[0,\infty) = \mathcal{A}\cup\mathcal{B}\cup\mathcal{C}$.

5. Finally, we consider the case $c \in \mathcal{C}$, i.e., $x_2(c) = \infty$ and $u(\infty) := \lim_{x\to\infty} u(x) < \infty$. Then along a sequence $x \to \infty$, we have $u'(x) \to 0$. Consequently, integrating $u'' + cu' = uv^m$ over $(-\infty, x)$ and sending $x \to \infty$ along the sequence $u'(x) \to 0$ we obtain

$$c[u(\infty) - u_0] = \int_{\mathbb{R}} u(y)v^m(y)\,dy < \infty.$$

This implies that $\liminf_{x\to\infty} v(x) = 0$. Next integrating $dv'' + cv' = v - uv^m$ over $(-\infty, x)$ and sending $x \to \infty$ along a sequence on which $dv'(x) + cv(x) \to 0$ we obtain

$$\int_{\mathbb{R}} v(y)dy = \int_{\mathbb{R}} u(y)v^m(y)dy.$$

Hence,

$$dv' + cv = h(x) := \int_{-\infty}^{x} [v - uv^m]dy = \int_{x}^{\infty} [uv^m - v]dy \to 0 \text{ as } x \to \infty.$$

Using integrating factor $e^{cx/d}$, we obtain, for every $x > z$, $v(x) \leqslant v(z) + \frac{1}{c}\sup_{[z,\infty)}|h|$. Sending $z \to \infty$ along a sequence on which $v(z) \to 0$ we then obtain $\lim_{x\to\infty} v(x) = 0$. Thus, (u, v) is a solution of (1.2). This completes the proof of the lemma. □

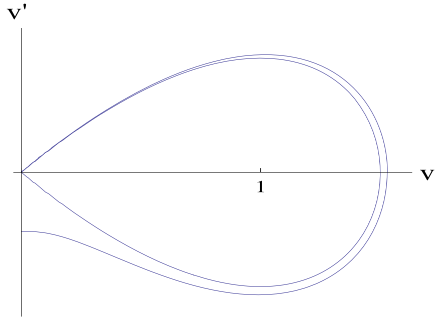

Fig. 1. The level set $E := \frac{dv'^2}{2} - \frac{v^2}{2} + \frac{\bar{u}v^{m+1}}{m+1} = 0$ and $E = \delta$, an increasing function.

3. The set $\mathcal{A}$

Lemma 3.1. *The set $\mathcal{A}$ defined in (2.3) is open. In addition,* $0 \in \mathcal{A}$.

Proof. First we show that $\mathcal{A}$ is open. Suppose $c \in \mathcal{A}$. Then by continuity, $v(x_2(c), c) = 0$. Since $v'(x_2(c), c) = 0$ would imply $v \equiv 0$ which is impossible, we must have $v_2'(x_2(c), c) < 0$. Consequently, by the uniqueness of solutions of initial value problem of ode system, we have $v_+(x, c) = 0$ and

$$v(x,c) = \frac{v'(x_2(c),c)}{\mu + c}[e^{\lambda[x - x_2(c)]} - e^{\mu[x_2(c) - x]}] < 0,$$
$$u(x,c) = u(x_2(c),c) + \frac{u'(x_2(c),c)}{c}[1 - e^{c[x - x_2(c)]}]$$

for every $x > x_2(c)$. Thus, $v(x_2(c) + 1, c) < 0$. By the continuity of solution with respect to the parameter c, we can find a $\delta > 0$ such that $v(x_2(c)+1, \tilde{c}) < 0$ for every $\tilde{c} \in [0, \infty) \cap (c-\delta, c+\delta)$. This implies that $[0, \infty) \cap (c - \delta, c + \delta) \subset \mathcal{A}$. Hence, $\mathcal{A}$ is open.

Next we show that $0 \in \mathcal{A}$. Assume that $c = 0$. Recall that $v'(x_1(0), 0) = 0 > v''(x_1(0), 0)$. We can define

$$\hat{x}_2 = \sup\{z > x_1(0) \mid v'(\cdot, 0) < 0 < v(\cdot, 0) \text{ in } (x_1(0), z)\}.$$

Set $\bar{u} = u(x_1(0), 0)$. Since $u' > 0$ in $\mathbb{R}$, we find that $(\bar{u} - u)v' \geqslant 0$ in $(-\infty, \hat{x}_2)$. Hence, integrating $v'[dv'' - v + \bar{u}v^m] = v'[\bar{u} - u]v^m$ we obtain, for each $x < \hat{x}_2$,

$$\frac{dv'^2}{2} - \frac{v^2}{2} + \frac{\bar{u}v^{m+1}}{m+1} = \delta(x) := \int_{-\infty}^{x} [\bar{u} - u(y)]v^m(y)v'(y)\,dy > 0.$$

Note that $\delta(x)$ is a strictly increasing function on $(-\infty, \hat{x}_2)$. By a v-v' phase plane analysis (cf. Fig. 1), we conclude that $\hat{x}_2 < \infty$ and $v(\hat{x}_2, 0) = 0$. Hence, $0 \in \mathcal{A}$. This completes the proof of the lemma. □

4. The set $\mathcal{B}$

The main purpose of this section is to show that $\mathcal{B}$ is open and non-empty. The proof is very technical and involves a number of estimates. In the sequel, $c > 0$ is a fixed constant. For notational simplicity, we write the solution $(u(x,c), v(x,c))$ of (2.1) simply as $(u(x), v(x))$. We denote

$$x_1 := \max\{z \in \mathbb{R} \mid v' > 0 \text{ in } (-\infty, z)\}, \quad x_2 = \sup\{z \in \mathbb{R} \mid v > 0 \text{ in } (-\infty, z)\}.$$

4.1. Basic estimate of v

First we investigate all critical points of v.

Lemma 4.1. *Suppose $v'(x) = 0$. Then $v > 0$ in $(-\infty, x]$ and exactly one of the following holds:*

(1) *$v''(x) > 0$, so x is a point of local positive minimum;*
(2) *$v''(x) < 0$, so x is a point of local positive maximum;*
(3) *$v''(x) = 0$ and $v'''(x) < 0$, so v is strictly decreasing near x.*

As a consequence, all critical points of v are isolated and can only accumulate at ∞.

Proof. Note that if $v'(x) = 0$, we must have $x < x_2$. In addition, if $v''(x) = 0$, then $dv'''(x) = -u'(x)v^m(x) < 0$. The assertion of the lemma thus follows. □

Lemma 4.2. *If $v'(z) > 0$, there exists $b > z$ such that $v' > 0$ in (z, b), $v'(b) = 0$, and $v''(b) < 0$.*

Proof. Let $b = \sup\{x > z \mid v' > 0 \text{ in } (z, x)\}$. Using the same argument as Step 3 of the proof of Lemma 2.1, we can show that $b < \infty$. Thus, $v'(b) = 0$ and by Lemma 4.1, $v''(b) < 0$. □

Lemma 4.3. *Suppose b is a point of local maximum of v. Then $v'(b) = 0 > v''(b)$, $\sigma(x) := u(x)v^{m-1}(x)$ satisfies $\sigma(b) > 1$, and*

$$v(x) < v(b) \quad \forall x \in (b, \infty).$$

Proof. By Lemma 4.1, $v'(b) = 0 > v''(b)$. Also, $\sigma(b) = 1 - dv''(b)/v(b) > 1$. Now let $[b, a)$ be the maximum interval on which $v' \leqslant 0$.

(i) If $a = \infty$, then $v' \leqslant 0$ in $[b, \infty)$ and the assertion of the lemma holds.

(ii) If $a < \infty$, then we must have $v'(a) = 0$ and by Lemma 4.1 and definition of a, $v''(a) > 0$. Let $(a, \tilde{b})$ be the maximum interval on which $v' > 0$. Then $v' \leqslant 0$ in (b, a) and $v' > 0$ in $(a, \tilde{b})$. Hence, b and $\tilde{b}$ are neighboring points of local maximum of v.

Note that $[u(a) - u]v' \leqslant 0$ in $(b, \tilde{b})$. Hence, for any $x \in (b, \tilde{b}]$, integrating $v'[dv'' + u(a)v^m - v] = [u(a) - u]v^m v' - cv'^2 \leqslant 0$ over $[b, x]$ we obtain

$$\frac{dv'(x)^2}{2} + \int_{v(b)}^{v(x)} [u(a)s^{m-1} - 1]s\,ds < 0.$$

Note that $u(a)v(b)^{m-1} \geqslant u(b)v^{m-1}(b) = \sigma(b) > 1$, we then obtain $v(x) < v(b)$ for every $x \in (b, \tilde{b}]$.

If $\tilde{b} = \infty$, we obtain the assertion of the lemma. If $\tilde{b} < \infty$, then $\tilde{b}$ is a point of local maximum of v and $v(\tilde{b}) < v(b)$. Since v has only finitely many critical points in any finite interval, one can use a mathematical induction to derive the assertion of the lemma. This completes the proof. □

Lemma 4.4. *Suppose a is a point of local minimum of v. Then*

$$v(x) \leqslant \Big(\frac{m+1}{2u(a)}\Big)^{\frac{1}{m-1}} \quad \forall x \geqslant a.$$

In particular, taking $a = -\infty$ we have

$$v(x) \leqslant \Big(\frac{m+1}{2u_0}\Big)^{\frac{1}{m-1}} \quad \forall x \in \mathbb{R}. \tag{4.1}$$

Proof. Let (a, b) be the maximal interval on which v is strictly increasing. Then $v'(a) = 0$ and $v' > 0$ on (a, b). Since $u' > 0$ on $\mathbb{R}$, we can integrate $v'v = v'[dv'' + cv' + uv^m]$ over (a, x) with $x \in (a, b)$ to obtain

$$\frac{v^2(x) - v^2(a)}{2} = \frac{dv'^2(x)}{2} + c\int_a^x v'^2 + \int_a^x v^m v' u dx$$

$$> u(a)\int_a^x v^m v' dx = \frac{u(a)}{m+1}(v^{m+1}(x) - v^{m+1}(a)).$$

Thus

$$\frac{m+1}{2u(a)} > \frac{v^{m+1}(x) - v^{m+1}(a)}{v^2(x) - v^2(a)} = v^{m-1}(x)\frac{1-t^{m+1}}{1-t^2}\Big|_{t=\frac{v(a)}{v(x)}} \geqslant v^{m-1}(x).$$

In view of Lemma 4.3, we then obtain the assertion of the Lemma. □

Lemma 4.5. *Let $\lambda = \frac{2}{c+\sqrt{c^2+4d}}$ and $\alpha = \frac{1}{m-1}$. Then for every $z \in \mathbb{R}$,*

$$v(x) \leqslant \frac{1}{c\lambda}\max\Big\{v(z), \frac{1}{u^\alpha(z)}\Big\} \qquad \forall x \geqslant z.$$

Proof. Fix $z \in \mathbb{R}$.

1. First we consider the case $v'(z) > 0$. Let $(a, b) \ni z$ be the maximum interval on which $v' > 0$. Then $b < \infty$, $v'(a) = 0 = v'(b)$, and $\sigma(a) < 1 < \sigma(b)$. In addition, since $u' > 0$ and $v' > 0$ in (a, b), we have $\sigma' > 0$ in (a, b), so there exists $\bar{z} \in (a, b)$ such that $\sigma(\bar{z}) = 1$. Consider two cases.

(i) Suppose $z \geqslant \bar{z}$. Then for $x \in [z, b]$, we have $dv'' + cv' = v[1 - \sigma] \leqslant 0$ in $[z, x]$. After integration, we obtain,

$$v(x) \leqslant \frac{-dv'(x)+dv'(z)+cv(z)}{c} \leqslant \frac{\lambda dv(z)+cv(z)}{c} = \frac{v(z)}{c\lambda},$$

since $v'(x) \geqslant 0$, $v'(z) < \lambda v(z)$ (cf. (2.6)), and $\lambda(d\lambda + c) = 1$.

(ii) Suppose $z < \bar{z}$. Then $v(x) < v(\bar{z})$ for $x \in [z, \bar{z})$ and $v(x) \leqslant v(\bar{z})/(c\lambda)$ when $x \in [\bar{z}, b)$. Hence, for $x \in [z, b]$,

$$v(x) \leqslant \frac{v(\bar{z})}{c\lambda} = \frac{1}{c\lambda u^{\alpha}(\bar{z})} \leqslant \frac{1}{c\lambda u^{\alpha}(z)}.$$

Combing the two cases, we obtain $v(x) \leqslant (c\lambda)^{-1}\max\{v(z), u^{-\alpha}(z)\}$ for every $x \in [z, b]$. After applying Lemma 4.3, we obtain the assertion of the lemma.

2. Next, suppose $v'(z) \leqslant 0$. Let $[z, a)$ be the maximum interval on which $v' \leqslant 0$. In the case $a = \infty$, we have $v(x) \leqslant v(z)$ for all $x \in [z, a) = [z, \infty)$, so the assertion of the lemma is true.

If $a < \infty$, then $v(x) \leqslant v(z)$ for every $x \in [z, a]$. Using the first step, we have, for every $x > a$, $v(x) \leqslant (c\lambda)^{-1}\max\{v(a), u^{-\alpha}(a)\} \leqslant (c\lambda)^{-1}\max\{v(z), u^{-\alpha}(z)\}$. This completes the proof of the lemma. □

Lemma 4.6. $\lim_{x\to\infty} v_+(x) = 0$.

Proof. If $x_2 < \infty$, then $v_+(x) = 0$ for all $x \geqslant x_2$, so the assertion is automatically true. We now consider the case $x_2 = \infty$. Then $v > 0$ in $\mathbb{R}$.

First we show that $\liminf_{x\to\infty} v(x) = 0$. Suppose not. Then there exists $A > 0$ such that $v(x) > A$ for all $x > 0$. Consequently, $u'' + cu' = uv^m > u(0)A^m$ in $(0, \infty)$. This implies that $\lim_{x\to\infty} u(x) = \infty$. Hence, there exists $z > 0$ such that $uv^{m-1} > 2$ in $[z, \infty)$. Consequently, in $[z, \infty)$, $dv'' + cv' = v(1 - uv^{m-1}) < -v < -A$. This implies that $v'(x) < v'(z)e^{c(z-x)/d} - A[1 - e^{c(z-x)/d}]/d$ for all $x > z$ so $\limsup_{x\to\infty} v'(x) < 0$, contradicting the assumption that $x_2 = \infty$. Thus, we must have $\liminf_{x\to\infty} v(x) = 0$.

Next, we consider two cases. (i) $\lim_{x\to\infty} u(x) = \infty$ and (ii) $\lim_{x\to\infty} u(x) < \infty$.

(i) Suppose $\lim_{x\to\infty} u(x) = \infty$. Then by Lemma 4.5, we have $\lim_{x\to\infty} v(x) = 0$.

(ii) Suppose $\lim_{x\to\infty} u(x) < \infty$. Then by Step **4** of the proof of Lemma 2.1, we obtain $\lim_{x\to\infty} v(x) = 0$.

The assertion of the Lemma thus follows. □

4.2. The function $\rho := u'/u$

Lemma 4.7. *Let $\rho = u'/u$. Then for every $z \in \mathbb{R}$ and $h \geqslant 0$,*

$$\rho(z+h) \leqslant \frac{e^{-ch}}{c}\Big(\frac{m+1}{2u_0}\Big)^{\frac{m}{m-1}} + \frac{1-e^{-ch}}{c} \sup_{[z,\infty)} v_+^m, \tag{4.2}$$

$$\rho(z+h) \leqslant \max\Big\{\rho(z), \quad \frac{1}{c}\sup_{[z,\infty)} v_+^m\Big\}. \tag{4.3}$$

Consequently, $\lim_{x\to\infty} \rho(x) = 0$ *and*

$$0 < \rho(x) \leqslant \frac{1}{c}\Big(\frac{m+1}{2u_0}\Big)^{\frac{m}{m-1}} \quad \forall x \in \mathbb{R}. \tag{4.4}$$

Proof. Using $u' = \rho u$ and $u'' = u[\rho' + \rho^2]$ we obtain from the equation $u'' + cu' = uv_+^m$ that

$$\rho' + \rho^2 + c\rho - v_+^m = 0 \text{ in } \mathbb{R}, \qquad \rho(-\infty) = 0.$$

Set $M(z) = \sup_{x \geqslant z} v_+^m(x)$. Using integrating factor e^{cx} we obtain, for $z \in \mathbb{R}$ and $h > 0$,

$$\begin{aligned}\rho(z+h) &= e^{-c(z+h)} \int_{-\infty}^{z+h} e^{cy}\Big(v_+^m(y) - \rho^2(y)\Big)dy \\ &\leqslant e^{-c(z+h)}\Big(\int_{-\infty}^{z} e^{cy} M(-\infty)dy + \int_{z}^{z+h} e^{cy} M(z)dy\Big) \\ &= \frac{M(-\infty)}{c} e^{-ch} + \frac{M(z)}{c}[1 - e^{-ch}].\end{aligned}$$

After using Lemma 4.4 for $M(-\infty) = \sup_{\mathbb{R}} v_+^m$, we obtain (4.2). Setting $h = 0$ we obtain (4.4). The assertion (4.3) follows by comparing $\rho(\cdot)$ with the constant $\max\{\rho(z), c^{-1} \sup_{[z,\infty)} v_+^m\}$ on interval $[z, \infty)$.

Next, no matter $x_2 < \infty$ or $x_2 = \infty$, we always have $\lim_{x\to\infty} v_+(x) = 0$, so by (4.2), $\lim_{x\to\infty} \rho(x) = 0$. This completes the proof of the lemma. □

4.3. *The function* $w = u^{\frac{1}{m-1}} v$

Let $\alpha = \frac{1}{m-1}$ and consider the function $w = u^\alpha v$. We have

$$\begin{aligned} dw'' + cw' &= u^\alpha[dv'' + cv'] + 2d(u^\alpha)'v' + \alpha u^{\alpha-1}[du'' + d(\alpha-1)u^{-1}u'^2 + cu']v \\ &= w - w_+^m + \frac{2d(u^\alpha)'}{u^\alpha}\Big([u^\alpha v]' - [u^\alpha]'v\Big) \\ &\quad + \frac{\alpha w}{u}\Big\{duv_+^m + (1-d)cu' + d(\alpha-1)\frac{u'^2}{u}\Big\}. \end{aligned}$$

Hence, introduce $\rho = u'/u$ we obtain

$$dw'' + cw' - w + w_+^m = \eta_1 w' + \eta_2 w, \tag{4.5}$$

where

$$\eta_1 = 2\alpha d\rho, \qquad \eta_2 = \alpha d v_+^m + \alpha[1-d]c\rho - d\alpha[\alpha+1]\rho^2. \tag{4.6}$$

Define $\eta(x) = \max_{z>x}[|\eta_1(z)| + |\eta_2(z)|]$. Then by Lemma 4.6 and Lemma 4.7 we have

$$\lim_{x\to\infty} \eta(x) = 0. \tag{4.7}$$

Lemma 4.8. *Let $w = u^\alpha v$ with $\alpha = 1/(m-1)$. Suppose that $x_2 = \infty$. Then one of the following holds:*

(1) $\lim_{x\to\infty} w(x) = 1$ *and* $\lim_{x\to\infty} u(x) = \infty$*, so* $c \in \mathcal{B}$*;*
(2) $\lim_{x\to\infty} w(x) = 0$ *and* $\lim_{x\to\infty} u(x) < \infty$*, so* $c \in \mathcal{C}$*. In addition,*

$$\lim_{x\to\infty} \frac{v'(x)}{v(x)} = -\mu = -\frac{c+\sqrt{c^2+4d}}{2d}. \tag{4.8}$$

Proof. One and only one of the following holds:

(1) $\liminf_{x\to\infty} w^{m-1}(x) > 1 + \frac{c^2}{4d}$;
(2) $\liminf_{x\to\infty} w^{m-1}(x) \leqslant 1 + \frac{c^2}{4d}$ and $\limsup_{x\to\infty} w(x) > \liminf_{x\to\infty} w(x)$;
(3) $\liminf_{x\to\infty} w^{m-1}(x) = \limsup_{x\to\infty} w^{m-1}(x) \leqslant 1 + \frac{c^2}{4d}$.

We shall show that only case (3) happens.

(1) Suppose $\liminf_{x\to\infty} w^{m-1}(x) > 1 + \frac{c^2}{4d}$. Consider the function $\hat{v}(x) = v(x)e^{cx/2d}$. Then $v = e^{-cx/2d}\hat{v}$ so

$$d\hat{v}'' = \hat{v}Q, \qquad Q := 1 + \frac{c^2}{4d} - w^{m-1}.$$

Note that $\liminf_{x\to\infty} Q(x) < 0$. It then follows from the Sturm–Liouville nodal comparison theorem that $\hat{v}$ oscillates, i.e., $\hat{v}$ changes sign infinitely many times as $x \to \infty$, contradicting the assumption that $x_2 = \infty$. Hence, case (1) does not happen.

(2) Suppose $l := \liminf_{x\to\infty} w^{m-1}(x) \leqslant 1 + \frac{c^2}{4d}$ and $L := \limsup_{x\to\infty} w^{m-1}(x) > l$. Then w oscillates infinitely many times as $x \to \infty$.

Let $\{b_j\}_{j=1}^\infty$ be a sequence of point of local maximum of w such that $b_j \to \infty$ and $w^{m-1}(b_j) \to L$ as $j \to \infty$. By (4.7), we can assume that $\eta(b_1) < \min(1, c)$. For each $j \geqslant 2$, let (a_j, b_j) be the largest interval such that $w' > 0$ in (a_j, b_j). Then $w'(a_j) = w'(b_j) = 0$. Integrating $dw'w'' + w^m w' = [-c + \eta_1]w'^2 + [1 + \eta_2]ww' \leqslant [1 + \eta(a_j)]ww'$ over $[a_j, b_j]$ we obtain

$$\frac{w^{m+1}(b_j) - w^{m+1}(a_j)}{m+1} \leqslant \frac{[1+\eta(a_j)][w(b_j)^2 - w^2(a_j)]}{2}.$$

This implies that

$$w^{m-1}(b_j) \leqslant \frac{(m+1)[1+\eta(a_j)]}{2} \sup_{0<t<1} \frac{1-t^2}{1-t^{m+1}} \leqslant \frac{(m+1)[1+\eta(a_j)]}{2}.$$

Sending $j \to \infty$ we find that $L \leqslant \frac{m+1}{2}$. Now set $w_j(x) = w(b_j + x)$. Then

$$\lim_{j\to\infty} \left(w_j(0),\ w_j'(0) \right) = (L^\alpha,\ 0).$$

Hence, from (4.5) and (4.7), we see that $w_* := \lim_{j\to\infty} w_j$ exists and satisfies

$$dw_*'' + cw_*' + w_*^m - w_* = 0 \ \text{ in } \mathbb{R}, \qquad w_*'(0) = 0, \qquad w_*(0) = L^\alpha = \sup_{x\in\mathbb{R}} w_*(x) > 0. \quad (4.9)$$

Since $c > 0$ and $w_* \geqslant 0$, the only solution is $w_* \equiv 1$. Hence, $L = 1$.

Similarly, let $\{\hat{a}_j\}_{j=1}^\infty$ be a sequence of local minimum of w such that as $j \to \infty$, $\hat{a}_j \to \infty$ and $w(\hat{a}_j) \to l$. For each $j \geqslant 2$, let $(\hat{b}_j, \hat{a}_j)$ be the maximum interval on which $w' < 0$. First of all, we derive from $w'(\hat{b}_j) = 0 \geqslant w''(\hat{b}_j)$ that $0 \geqslant dw''(\hat{b}_j) = w(\hat{b}_j)[1 - w^{m-1}(\hat{b}_j) + \eta_2(\hat{b}_j)]$, so $w(\hat{b}_j) \geqslant [1 + \eta_2(\hat{b}_j)]^{\frac{1}{m-1}}$. As $\limsup_{x\to\infty} w(x) = L = 1$, we find that $\lim_{j\to\infty} w(\hat{b}_j) = 1$. Next, integrating $w'[dw'' + w^m] = [-c + \eta_1]w'^2 + [1 + \eta_2]ww' \leqslant [1 - \eta(\hat{b}_j)]ww'$ over $[\hat{b}_j, \hat{a}_j]$ we derive that

$$\frac{w^{m+1}(\hat{a}_j) - w^{m+1}(\hat{b}_j)}{m+1} \leqslant \frac{[1 - \eta(\hat{b}_j)][w(\hat{a}_j)^2 - w^2(\hat{b}_j)]}{2}.$$

Sending $j \to \infty$ we derive that $\frac{l^{(m+1)\alpha}}{m+1} - \frac{\ell^{2\alpha}}{2} \leqslant \frac{1}{m+1} - \frac{1}{2}$. This implies that $l = 1$. Hence, $\lim_{x\to\infty} w(x) = 1$. But this contradicts $\ell < L$. Hence case (2) does not happen.

(3) Suppose $L := \lim_{x\to\infty} w^{m-1}(x)$ exists and is finite. Then we derive from (4.5) and (4.7) that either $L = 0$ or $L = 1$.

If $L = 1$, we derive from $u = (wv^{-1})^{1/\alpha}$ that $\lim_{x\to\infty} u(x) = \infty$, so $c \in \mathcal{B}$.

If $L = 0$, then we derive from $dv'' + cv' = v[1 - w^{m-1}] = [1 + o(1)]v$ that v decays to zero exponentially fast as $x \to \infty$. Consequently, both $\int_{\mathbb{R}} v dx$ and $\int_{\mathbb{R}} uv^m dx = \int_{\mathbb{R}} w^{m-1} v$ are finite. Using integrating factor e^{cx} we then obtain, for every $x \in \mathbb{R}$,

$$u(x) = u_0 + \int_{-\infty}^{x} e^{c(y-x)} \int_{-\infty}^{y} u(z)v^m(z)dzdy \leqslant u_0 + \frac{1}{c}\int_{\mathbb{R}} u(z)v^m(z)dz.$$

Hence, $\lim_{m\to\infty} u(x) < \infty$ so $c \in \mathcal{C}$. In addition, writing the equation for v as $dv'' + cv' - v = O(v^m)$ we obtain (4.8). This completes the proof of the Lemma. □

4.4. An invariant region

Let

$$g(s) = s^m - s, \quad G(s) = \int_1^s g(t)dt, \quad \hat{g}(s) = \begin{cases} g(s) & \text{if } s \in [0, 2], \\ g(2) + g'(2)(s-2) & \text{if } s > 2, \end{cases}$$

$$k_1 = \inf_{0<s\neq 1} \frac{dG(s)}{2\hat{g}^2(s)}, \quad k_2 := \inf_{\frac{1}{2}<s<2, s\neq 1} \frac{g^2(s)}{G(s)}, \qquad \theta = \min\left\{\sqrt{k_1}, \frac{c}{4g'(2)}, \frac{d}{4c}\right\}.$$

Since g is convex and $g(1) = 0 < g'(1)$, k_1 and k_2 are positive and in $[0, \infty)$,

$$g \geqslant \hat{g}, \quad G \geqslant \frac{2\theta^2}{d}\hat{g}^2, \quad \theta\hat{g}' \leqslant \frac{c}{4}, \quad \hat{g}\Big[g - \frac{2c\theta}{d}\hat{g}\Big] \geqslant \frac{1}{2}\hat{g}g \geqslant \frac{1}{2}\hat{g}^2.$$

Consider the "energy":

$$E := \frac{d}{2}w'^2 + G(w) + \theta\hat{g}(w)w'. \tag{4.10}$$

By the definition of θ, we have $|\theta\hat{g}(w)w'| \leqslant \frac{d}{4}w'^2 + \frac{\theta^2}{d}\hat{g}^2 \leqslant \frac{d}{4}w'^2 + \frac{1}{2}G$. Hence,

$$\frac{3}{4}dw'^2 + \frac{3}{2}G(w) \geqslant E \geqslant \frac{d}{4}w'^2 + \frac{1}{2}G(w). \tag{4.11}$$

We can calculate, when $x < x_2$,

$$\begin{aligned}
E' &= w'[dw'' + g(w)] + \theta\hat{g}(w)w'' + \theta\hat{g}'(w)w'^2 \\
&= w'[(\eta_1 - c)w' + \eta_2 w] + \frac{\theta}{d}\hat{g}[-g + (\eta_1 - c)w' + \eta_2 w] + \theta\hat{g}'w'^2 \\
&\leqslant [\eta_1 - c + \theta\hat{g}']w'^2 - \frac{\theta}{d}\hat{g}g + \Big(\frac{cw'^2}{4} + \frac{\eta_2^2 w^2}{c}\Big) + \\
&\quad \Big(\Big[\frac{c}{4} + \frac{\eta_1^2}{c}\Big]w'^2 + \frac{1}{d^2}\Big[c + \frac{c}{4}\Big]\theta^2\hat{g}^2\Big) + \Big(\frac{3c\theta^2\hat{g}^2}{4d^2} + \frac{\eta_2^2 w^2}{3c}\Big) \\
&= -\Big(\frac{c}{2} - \eta_1 - \frac{\eta_1^2}{c} - \theta\hat{g}'\Big)w'^2 - \frac{\theta}{d}\hat{g}\Big[g - \frac{2c\theta}{d}\hat{g}\Big] + \frac{4\eta_2^2}{3c}w^2 \\
&\leqslant -\Big(\frac{c}{4} - \eta_1 - \frac{\eta_1^2}{c}\Big)w'^2 - \frac{\theta}{2d}\hat{g}^2 + \frac{4\eta_2^2}{3c}w^2.
\end{aligned}$$

In particular, if $0 < \eta_1 < \frac{c}{9}$ and $|w - 1| \leqslant \frac{1}{2}$ we have

$$\begin{aligned}
E' &\leqslant -\frac{c}{8}w'^2 - \frac{\theta k_2}{2d}G(w) + \frac{3\eta_2^2}{c} \\
&\leqslant -\min\Big\{\frac{c}{6d}, \frac{\theta k_2}{3d}\Big\}E + \frac{3\eta_2^2}{c}.
\end{aligned} \tag{4.12}$$

Lemma 4.9. *Suppose $c > 0$ and there exists $z < x_2$ such that, with $\delta = \delta(c)$ defined in (4.14) below,*

$$\rho(z) \leqslant \frac{\delta}{c}, \quad v(z) \leqslant (c\lambda)\delta^{\frac{1}{m}}, \quad \frac{1}{u^\alpha(z)} \leqslant (c\lambda)\delta^{\frac{1}{m}}, \quad E(z) \leqslant \frac{1}{2}G\Big(\frac{1}{2}\Big), \tag{4.13}$$

then $x_2 = \infty$, $\lim_{x\to\infty} w(x) = 1$ and $\lim_{x\to\infty} u(x) = \infty$.

Proof. First of all, by Lemma 4.5,

$$\sup_{[z,\infty)} v \leqslant \frac{1}{c\lambda}\max\Big\{v(z), \frac{1}{u^\alpha(z)}\Big\} \leqslant \delta^{\frac{1}{m}}.$$

Next, by (4.3),

$$\sup_{[z,\infty)} \rho \leqslant \max\left\{\rho(z), \frac{1}{c}\sup_{[z,\infty)} v_+^m\right\} \leqslant \frac{\delta}{c}.$$

Assume that $\delta \leqslant c^2$. Then $\rho^2 \leqslant \delta^2/c^2 \leqslant \delta$. It then follows from the definition of η_i in (4.6) that in $[z,\infty)$,

$$|\eta_1| \le \frac{2\alpha d\delta}{c}, \quad |\eta_2| < \alpha \max\{d,1\}[2+\alpha]\delta.$$

Now we define

$$\delta(c) = \min\left\{c^2, \quad \frac{c^2}{18\alpha d}, \quad \frac{1}{\alpha\max\{d,1\}[2+\alpha]}\left(\frac{G(\frac{1}{2})}{6}\min\left\{\frac{c^2}{6d}, \frac{c\theta k_2}{3d}\right\}\right)^{\frac{1}{2}}\right\}. \tag{4.14}$$

Then in $[z,\infty)$ we have

$$|\eta_1| \leqslant \frac{c}{9}, \qquad -\min\left\{\frac{c}{6d}, \frac{\theta k_2}{3d}\right\}\frac{1}{2}G\left(\frac{1}{2}\right) + \frac{3\eta_2^2}{c} < 0.$$

Now set

$$\hat{x}_2 := \sup\left\{x \in [z, x_2) \mid 2E(x) \leqslant G\left(\frac{1}{2}\right)\right\}.$$

We claim that $\hat{x}_2 = x_2$.

Suppose the claim is not true. Then $\hat{x}_2 < x_2$. Consequently, at $\hat{x}_2$, we have $2E(\hat{x}_2) = G(\frac{1}{2})$ and $E'(\hat{x}_2) \geqslant 0$. That $2E(\hat{x}_2) \leqslant G(1/2)$ implies that $G(w(\hat{x}_2)) \leqslant G(1/2)$ so $|w(\hat{x}_2) - 1| \leqslant 1/2$. Consequently, by (4.12), we have

$$E'(x_2) \leqslant -\min\left\{\frac{c}{6d}, \frac{\theta k_2}{3d}\right\}E(x_2) + \frac{3\eta_2^2}{c} \leqslant -\min\left\{\frac{c}{6d}, \frac{\theta k_2}{3d}\right\}\frac{1}{2}G\left(\frac{1}{2}\right) + \frac{3\eta_2^2}{c} < 0,$$

a contradiction. Thus, we must have $\hat{x}_2 = x_2$.

Now we claim that $x_2 = \infty$. Indeed, if $x_2 < \infty$, we must have $v(x_2) = 0$ and $v'(x_2) < 0$. This implies that $w(x_2) = 0$ and by (4.11), $\lim_{x\nearrow x_2} 2E(x_2) \geqslant G(0)$. But this contradicts $2E \leqslant G(1/2) < G(0)$ on $[z, x_2)$. Thus, we must have $x_2 = \infty$.

Finally, since $\lim_{x\to\infty} \eta_2 = 0$, we derive from (4.12) that $\lim_{x\to\infty} E(x) = 0$. This implies that $\lim_{x\to\infty} w(x) = 1$ and completes the proof of the Lemma. □

4.5. *The openness of $\mathcal{B}$*

Now we are ready to show the following:

Lemma 4.10. *The set $\mathcal{B}$ defined in (2.3) is open.*

Proof. Suppose $c_0 \in \mathcal{B}$. Then we have

$$\lim_{x\to\infty}\left(|\rho(x,c_0)| + |v(x,c_0)| + \frac{1}{|u(x,c_0)|} + |w'(x,c_0)| + |w(x,c_0) - 1|\right) = 0.$$

Hence, by continuous dependence of solution with respect to parameter, there exists $\epsilon > 0$ and $z \in \mathbb{R}$ such that (4.13) holds for every solution of (2.1) with $c \in (c_0 - \epsilon, c_0 + \epsilon)$. Then by Lemma 4.9, $c \in \mathcal{B}$ for every $c \in (c_0 - \epsilon, c_0 + \epsilon)$. Hence, $\mathcal{B}$ is open. □

4.6. The non-emptiness of $\mathcal{B}$

Lemma 4.11. *Let*

$$V_0 := \Big(\frac{m+1}{2u_0}\Big)^{\frac{1}{m-1}}, \quad M = \frac{m+1}{2}\Big[1 + \alpha \max\{d, 1\} V_0^m\Big], \quad C = \sqrt{4d(M-1)}. \tag{4.15}$$

Then $[C, \infty) \in \mathcal{B}$.

Proof. Suppose $c \geqslant C$. We show that $c \in \mathcal{B}$ by three steps.

1. By (4.1) we have $v \leqslant V_0$. By (4.4) we have $0 < \rho \leqslant V_0^m / c$. Hence, by the definition of η_1 and η_2,

$$0 < \eta_1(x) \leqslant \frac{2\alpha d V_0^m}{c} < c, \qquad \eta_2(x) \leqslant \alpha \max\{d, 1\} V_0^m.$$

2. Next we find an upper bound of $w := u^\alpha v$. Let $\hat{M} = (\sup_{\mathbb{R}} w)^{m-1}$. Then there exists a sequence $\{z_i\}_{i=1}^\infty$ in $(-\infty, x_2)$ such that $w'(z_i) > 0$ and $\lim_{i\to\infty} w^{m-1}(z_i) = \hat{M}$. For each $i \geqslant 1$, Let $(a, z_i]$ be the maximal interval in which $w' > 0$. Then integrating $w'w^m = -dw'w'' - (c - \eta_1)w'^2 + (1+\eta_2)ww' \leqslant -dw'w'' + [1 + \sup_{\mathbb{R}} \eta_2]ww'$ over (a, z_i) we obtain

$$\frac{w^{m+1}(z_i) - w^{m+1}(a)}{m+1} \leqslant \Big[1 + \sup_{\mathbb{R}} \eta_2\Big] \frac{w^2(z_i) - w^2(a)}{2}.$$

Thus,

$$w^{m-1}(z_i) \leqslant \frac{m+1}{2}\Big[1 + \sup_{\mathbb{R}} \eta_2\Big] \frac{1-t^2}{1-t^{m+1}}\Big|_{t=\frac{w(a)}{w(z_i)}} \leqslant \frac{m+1}{2}\Big[1 + \alpha \max\{d, 1\} V_0^m\Big] = M.$$

Sending $i \to \infty$ we obtain $(\sup_{\mathbb{R}} w)^{m-1} \leqslant M$.

3. We now show that $c \in \mathcal{B}$. Let $\hat{\mu} = \frac{c + \sqrt{c^2 + 4d(1-M)}}{2d}$. Then $d\hat{\mu}^2 - c\hat{\mu} + M - 1 = 0$. In $(-\infty, x_2)$,

$$\begin{aligned} d[v' + \hat{\mu}v]' + (c - d\hat{\mu})[v' + \hat{\mu}v] &= dv'' + cv' + (c\hat{\mu} - d\hat{\mu}^2)v = (c\hat{\mu} - d\hat{\mu}^2 + 1 - w^{m-1})v \\ &\geqslant (c\hat{\mu} - d\hat{\mu}^2 + 1 - M)v = 0. \end{aligned}$$

This implies that $(e^{[c/d - \hat{\mu}]x}[v' + \hat{\mu}v])' \geqslant 0$. Since $v' + \hat{\mu}v > 0$ for $x < x_1$, after integration, we derive that

$$v'(x) + \hat{\mu}v(x) > 0 \quad \forall x < x_2.$$

Since $x_2 < \infty$ would imply $v(x_2) = 0 > v'(x_2)$, we see that $x_2 = \infty$. In addition,

$$\liminf_{x\to\infty} \frac{v'(x)}{v(x)} \geqslant -\hat{\mu} > -\mu.$$

It then follows from Lemma 4.8 (in particular (4.8)) that we must have $c \in \mathcal{B}$. This completes the proof of the lemma. □

4.7. Proof of Theorem 1

Since $[0, \infty) = \mathcal{A} \cup \mathcal{B} \cup \mathcal{C}$ and $\mathcal{A}$ and $\mathcal{B}$ are open and disjoint non-empty sets, $\mathcal{C}$ is non-empty. When $c \in \mathcal{C}$, the solution of (2.1) is a solution of (1.2). This completes the proof of Theorem 1.

Acknowledgments

The authors would like to thank Xuefeng Wang and Xiaoqiang Zhao for stimulating discussions. Chen is partially supported by NSF grant DMS-1008905 and hundred experts program of Shanxi province; Qi by NSF grant DMS-1312644 and Zhang by NSFC No. 61374089.

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