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Classification of singular solutions of porous media equations with absorption

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We consider, for $m \in (0, 1)$ and $q > 1$, the porous media equation with absorption

$$u_t = \Delta u^m - u^q \quad \text{in } \mathbb{R}^n \times (0, \infty).$$

We are interested in those solutions, which we call singular solutions, that are non-negative, non-trivial, continuous in $\mathbb{R}^n \times [0, \infty) \setminus \{(0, 0)\}$, and satisfy $u(x, 0) = 0$ for all $x \neq 0$. We prove the following results. When $q \geqslant m + 2/n$, there does not exist any such singular solution. When $q < m + 2/n$, there exists, for every $c > 0$, a unique singular solution $u = u_{(c)}$, called the fundamental solution with initial mass c, which satisfies $\int_{\mathbb{R}^n} u(\cdot, t) \to c$ as $t \searrow 0$. Also, there exists a unique singular solution $u = u_\infty$, called the very singular solution, which satisfies $\int_{\mathbb{R}^n} u_\infty(\cdot, t) \to \infty$ as $t \searrow 0$.

In addition, any singular solution is either u_∞ or $u_{(c)}$ for some finite positive c, $u_{(c_1)} < u_{(c_2)}$ when $c_1 < c_2$, and $u_{(c)} \nearrow u_\infty$ as $c \nearrow \infty$.

Furthermore, u_∞ is self-similar in the sense that $u_\infty(x, t) = t^{-\alpha} w(|x| t^{-\alpha\beta})$ for $\alpha = 1/(q - 1)$, $\beta = \frac{1}{2}(q - m)$, and some smooth function w defined on $[0, \infty)$, so that

$$\int_{\mathbb{R}^n} u_\infty^{n\beta}(\cdot, t)$$

is a finite positive constant independent of $t > 0$.

1. Introduction

We are interested in singular solutions for the porous media equation with absorption

$$u_t = \Delta u^m - u^q \quad \text{in } \mathbb{R}^n \times (0, \infty). \tag{1.1}$$

Here, by a singular solution we always mean a non-negative and non-trivial solution which is continuous in $\mathbb{R}^n \times [0, \infty) \setminus \{(0, 0)\}$ and which satisfies

$$\lim_{t \searrow 0} \sup_{|x| > \varepsilon} u(x, t) = 0 \quad \forall \varepsilon > 0. \tag{1.2}$$

That is, we restrict our attention to solutions having an isolated singularity at $(0,0)$.

A singular solution is called a *fundamental solution* (FS) with mass $c \in (0,\infty)$ if

$$\lim_{t \searrow 0} \int_{|x|<1} u(x,t)\,\mathrm{d}x = c. \tag{1.3}$$

A singular solution is called a *very singular solution* (VSS) if

$$\lim_{t \searrow 0} \int_{|x|<1} u(x,t)\,\mathrm{d}x = \infty. \tag{1.4}$$

Denote by $\delta(\cdot)$ the Dirac delta function. Then (1.2) and (1.3) can be written in short as $u(\cdot,0) = c\delta(\cdot)$. Typical diffusion equations without absorption, such as the heat equation $u_t = \Delta u$, the porous media equation $u_t = \Delta u^m$, and the parabolic p-Laplacian equation $u_t = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$, admit only FSs (see [12]).

The purpose of this paper is to complete the following classification for all singular solutions of (1.1), under the assumption that $m > 0$ and $q > 1$:

(i) any singular solution of (1.1) is either an FS or a VSS;

(ii) when $q \geqslant m + 2/n$, (1.1) does not have any singular solution;

(iii) when $m < q < m + 2/n$, (1.1) admits a unique VSS, and, for each $c > 0$, a unique FS with initial mass c;

(iv) when $q \leqslant m$, (1.1) does not have any VSS but it admits, for each $c > 0$, a unique FS with initial mass c.

For $m = 1$, the above classification was made by Oswald [18] in 1988, following the pioneer works of Brezis and Friedman [1] on the existence and uniqueness of FSs, Brezis *et al*. [2] and Kamin and Peletier [10] on the discovery of VSSs and their existence and uniqueness, and Galaktionov *et al*. [8] on the stability of these singular solutions. Recently, Escobedo *et al*. [7] provided a rather complete description of the long-time behaviour of the solutions of the Cauchy problem of (1.1) (with $m = 1$), which relates to the singular solutions of (1.1).

The classification for $m > 1$ was given by Kamin *et al*. [15] in 1989, following the early works of Kamin and Peletier [11] on the existence of FSs, Peletier and Terman [19] on the existence of VSSs, and Kamin and Veron [14] on the uniqueness of VSSs.

In this paper, we shall prove the above classification for the case $m \in (0,1)$. In particular, we prove the following theorem.

THEOREM 1.1. *Assume that $m \in (0,1)$ and $q > 1$. Then the following hold:*

(i) *every singular solution of (1.1) is either an FS or a VSS;*

(ii) *when $q \geqslant m + 2/n$, (1.1) does not have any singular solution;*

(iii) *when $q < m + 2/n$, (1.1) admits a unique VSS, u_∞, and, for every $c > 0$, a unique FS, $u_{(c)}$, with initial mass c. In addition, $u_{(c_1)} < u_{(c_2)}$ for any $c_1 < c_2$ and $u_{(c)} \to u_\infty$ as $c \to \infty$.*

In this direction, there are results of Peletier and Zhao [21], who proved in 1990 that when $m \in ((1-2/n)_+, 1)$, (1.1) has both FS and VSS if $q \in (1, m+2/n)$ and no FS if $q \geqslant m+2/n$, and results of Leoni [16] who proved in 1996 that when $m \in (0,1)$ and $q > 1$, (1.1) has a *self-similar* VSS if and only if $m > (1-2/n)_+$ and $q \in (1, m+2/n)$. Here by 'self-similar' we mean that u has the form

$$u(x,t) = \left(\frac{\alpha}{t}\right)^{\alpha} w\left(|x|\left(\frac{\alpha}{t}\right)^{\alpha\beta}\right), \quad \alpha := \frac{1}{q-1}, \quad \beta := \frac{q-m}{2}, \tag{1.5}$$

where w, defined on $[0,\infty)$, solves

$$(w^m)'' + \frac{n-1}{r}(w^m)' + \beta r w' + w - w^q = 0 \quad \forall r > 0. \tag{1.6}$$

Note that the condition (1.2) is equivalent to

$$\lim_{r\to\infty} r^{1/\beta} w(r) = 0, \tag{1.7}$$

if u is given by (1.5).

Classification for singular solutions has also be given to the p-Laplacian evolution equation with absorption:

$$u_t = \operatorname{div}(|\nabla u|^{p-2}\nabla u) - u^q \quad \text{in } \mathbb{R}^n \times (0,\infty), \quad p > 1,\ q > 1.$$

The classification is the same as that for the porous media equation except the borderlines $q = m + 2/n$ and $q = m$ are replaced by $q = p - 1 + p/n$ and $q = p - 1$, respectively. For $p > 2$, the classification was given in 1992 by Kamin and Vazquez [13] following the works of Peletier and Wang [20] and Diaz and Saa [5] on, respectively, the existence and uniqueness of VSSs. The classification for $p \in (1,2)$ was recently given by the current authors in [4] after establishing the existence and uniqueness of self-similar VSSs in [3]. Recently, Leoni [17] considered an elliptic p-Laplacian equation originating from the study of $v_t = \nabla(|\nabla v|^{p-2}\nabla v)$.

Our paper is organized as follows. In § 2, we first show that a singular solution of (1.1) is either an FS or a VSS. Then we provide two upper bounds for any singular solution of (1.1), one of which has the form $At^{-\alpha}$ and the other the form $B(\sqrt{t}/|x|)^{\mu}$, $\mu := 2/(1-m)$. With these two bounds, we prove that (1.1) does not have any singular solution when $q \geqslant m + 2/n$ by using the fact that the integral

$$\int_{|x|\leqslant 1} \min\{At^{-\alpha}, B(\sqrt{t}/|x|)^{\mu}\}\,\mathrm{d}x$$

converges to zero as $t \to 0$ when $q > m + 2/n$, and is uniformly bounded in t when $q = m + 2/n$. Also, by taking the limit, as $\varepsilon \searrow 0$, of the solution of (1.1) with initial value $\min\{A\varepsilon^{-\alpha}, B(\sqrt{\varepsilon}/|x|)^{\mu}\}$, we show that if (1.1) has a singular solution, then (1.1) admits a *maximal self-similar* singular solution u^*; here, maximal means that $u^* \geqslant u$ for any singular solution u of (1.1).

In § 3, we establish the existence of the FS and VSS when $q \in (1, m+2/n)$. In fact, we show that an FS with initial mass c can be obtained as a limit of any sequence of solutions of (1.1) whose initial data approximate the measure $c\delta(\cdot)$. A VSS can be obtained as the limit, as $c \to \infty$, of an FS with initial mass c.

In § 4, we prove the uniqueness of singular solutions. We first show the uniqueness of a FS for the porous media equation without absorption:

$$u_t = \Delta u^m. \tag{1.8}$$

The proof relies on a blow-up technique and a scaling invariance $u \to u^h(x,t)$ of equation (1.8), where

$$u^h := hu(h^{1/n}x, h^k t), \quad k := m - 1 + \frac{2}{n}.$$

Then we establish the uniqueness of the FS for (1.1). From the existence proof in § 3, one derives that an FS of (1.1) is bounded by the unique FS of (1.8) with the same initial mass, which implies that the $L^1(\mathbb{R}^n)$ difference of any two FSs of (1.1) with the same initial mass approaches zero as $t \searrow 0$. The uniqueness then follows from a contraction principle which asserts that the $L^1(\mathbb{R}^n)$ difference of any two solutions of (1.1) is non-increasing in t.

To prove the uniqueness of VSS of (1.1), we show that any VSS is an upper bound of any FS, so u_∞, the limit of $u_{(c)}$ as $c \to \infty$, is the minimal VSS; i.e. $u_\infty \leqslant u$ for any VSS u. With this minimality and scaling invariance of (1.1), we show that u_∞ is self-similar. As both u^* and u_∞ are self-similar VSSs, the uniqueness of VSS then follows, provided that we have the uniqueness of the self-similar VSS.

Finally, in § 5 we show that self-similar singular solutions of (1.1) are unique, thereby completing the proof of theorem 1.1.

For the reader's convenience, we list some special constants used in this paper:

$$\alpha = \frac{1}{q-1}, \qquad \beta = \frac{q-m}{2}, \qquad \mu = \frac{2}{1-m}, \qquad k = m - 1 + \frac{2}{n}.$$

Observe that $q < m + 2/n$ if and only if $n\beta < 1$. Also, $1 < q < m + 2/n$ and $m \in (0,1)$ imply that $m > (1 - 2/n)_+$ and $\mu > n$.

2. Bounds of singular solutions and non-existence results

As we need to work on both (1.1) and (1.8), it is convenient to write them in a single form,

$$u_t = \Delta u^m - \phi(u) \quad \text{in } \mathbb{R}^n \times (0,\infty). \tag{2.1}$$

LEMMA 2.1. *Let $m > 0$ and $\phi(\cdot)$ be a non-negative function defined on $[0,\infty)$. Assume that u is a singular solution of (2.1), i.e. a non-negative, non-trivial solution of (2.1) that is continuous in $\mathbb{R}^n \times [0,\infty) \setminus \{(0,0)\}$ and that satisfies (1.2). Then either (1.3) or (1.4) holds. In particular, taking $\phi(u) = u^q$ $(q > 0)$ we conclude that a singular solution of (1.1) must be either an FS or a VSS.*

Proof. The proof given below follows the same idea as that in [4, 15].

By (1.2), for every $\varepsilon > 0$, there exists $t_\varepsilon > 0$ such that $\sup_{|x|>1, t\in(0,t_\varepsilon]} u(x,t) \leqslant \varepsilon$. Multiplying (2.1) by $(u-\varepsilon)_+^\delta$ $(\delta > 0)$, integrating the resulting equation over $\mathbb{R}^n \times (\tau, t)$, $0 < \tau < t \leqslant t_\varepsilon$, and sending $\delta \to 0$, we then obtain, since ϕ is non-negative,

$$\int_{\mathbb{R}^n} (u-\varepsilon)_+(x,t)\,\mathrm{d}x \leqslant \int_{\mathbb{R}^n} (u-\varepsilon)_+(x,\tau)\,\mathrm{d}x, \quad 0 < \tau < t \leqslant t_\varepsilon.$$

Thus, $\int_{\mathbb{R}^n}(u(\cdot,t)-\varepsilon)_+$ is non-increasing in t, so that there exists $c_\varepsilon \in [0,\infty)\cup\{\infty\}$ such that

$$c_\varepsilon = \lim_{t\searrow 0}\int_{\mathbb{R}^n}(u-\varepsilon)_+(x,t)\,\mathrm{d}x = \lim_{t\searrow 0}\int_{|x|<1}(u-\varepsilon)_+(x,t)\,\mathrm{d}x.$$

Also, noting that c_ε is non-increasing in ε, and sending $\varepsilon \searrow 0$, we then find that there exists $c \in [0,\infty)\cup\{\infty\}$ such that

$$c = \lim_{\varepsilon\to 0} c_\varepsilon = \lim_{\varepsilon\searrow 0}\lim_{t\searrow 0}\int_{|x|<1}(u-\varepsilon)_+(x,t)\,\mathrm{d}x = \lim_{t\searrow 0}\int_{|x|<1} u(x,t)\,\mathrm{d}x.$$

Note that $c=0$ would imply, for all $\varepsilon>0$, that $c_\varepsilon = 0$, i.e. $\sup_{x\in\mathbb{R}^n} u(x,t) \leqslant \varepsilon$ for all $t \in (0,t_\varepsilon)$, which, by the maximum principle, implies that $u \leqslant \varepsilon$ for all $(x,t)\in\mathbb{R}^n\times(0,\infty)$, so that, as $\varepsilon>0$ is arbitrary, $u\equiv 0$. Since u is non-trivial, we must have $c \in (0,\infty)\cup\{\infty\}$. This completes the proof. □

Lemma 2.2. *Assume that* $m \in (0,1)$.

(i) *If* $q>1$ *and* u *is a singular solution of (1.1), then, for* $A := \alpha^\alpha$,

$$u(x,t) \leqslant At^{-\alpha} \quad \textit{in } \mathbb{R}^n\times(0,\infty). \tag{2.2}$$

(ii) *If* u *is a singular solution to (2.1) with* $\phi(\cdot)\geqslant 0$, *then, for* $B := [m(1+m)\mu]^{\mu/2}$,

$$u(x,t) \leqslant B(\sqrt{t}/|x|)^\mu \quad \textit{in } \mathbb{R}^n\times(0,\infty). \tag{2.3}$$

Proof. (i) For every $\varepsilon>0$, the function $A(t-\varepsilon)^{-\alpha}$ is a solution of (1.1) in $\mathbb{R}^n\times(\varepsilon,\infty)$. Since $u(\cdot,\varepsilon)$ is in $L^\infty(\mathbb{R}^n)$, comparing u with $A(t-\varepsilon)^{-\alpha}$ in $\mathbb{R}^n\times(\varepsilon,\infty)$ yields $u \leqslant A(t-\varepsilon)^{-\alpha}$ in $\mathbb{R}^n\times(\varepsilon,\infty)$. Sending $\varepsilon\searrow 0$ then gives (2.2).

(ii) Direct calculation shows that for any $\varepsilon>0$, the function $B(t+\varepsilon)^{\mu/2}(x_1-\varepsilon)^{-\mu}+\varepsilon$ is a sup-solution to $v_t-\Delta v^m = 0$ in $\{(x,t)\mid x_1>\varepsilon, t\geqslant 0\}$. Comparing this function with u in the domain $\{(x,t)\mid x_1>\varepsilon, t\geqslant 0\}$ then gives

$$u(x,t) \leqslant B(t+\varepsilon)^{\mu/2}(x_1-\varepsilon)^{-\mu}+\varepsilon \quad \text{for all } x_1>\varepsilon,\ t\geqslant 0.$$

Sending $\varepsilon\searrow 0$ yields $u(x,t)\leqslant Bt^{\mu/2}x_1^{-\mu}$ for all $x_1>0$ and $t\geqslant 0$. The assertion (2.3) then follows from the invariance of the equation for u under the rotation of x. □

Remark 2.3. If $\int_1^\infty \mathrm{d}s/\phi(s) < \infty$, then assertion (i) of the lemma can be stated as follows. Any singular solution of (2.1) satisfies the estimate

$$u(x,t)\leqslant a(t), \quad \text{where } t = \int_{a(t)}^\infty \frac{1}{\phi(s)}\,\mathrm{d}s. \tag{2.4}$$

Remark 2.4. If $m \in ((1-2/n)_+,1)$, then $\mu>n$, so that, for any $R>0$ and any singular solution u of (2.1),

$$\int_{|x|>R} u(x,t)\,\mathrm{d}x \leqslant Bt^{\mu/2}\int_{|x|>R}|x|^{-\mu}\,\mathrm{d}x \to 0 \quad \text{as } t\searrow 0.$$

Thus, (1.3) and (1.4) are equivalent to

$$\lim_{t\searrow 0}\int_{\mathbb{R}^n} u(x,t)\,\mathrm{d}x = c \quad \text{and} \quad \lim_{t\searrow 0}\int_{\mathbb{R}^n} u(x,t)\,\mathrm{d}x = \infty,$$

respectively.

With the upper bounds of singular solutions, we are ready to prove the non-existence of singular solutions of (1.1) when q is large.

Theorem 2.5. *Assume that $m \in (0,1)$, $q > 1$ and $q \geqslant m + 2/n$. Then (1.1) does not have any singular solution.*

Proof. Suppose for the contrary that (1.1) has a singular solution u. We consider three different cases.

Case 1 ($0 < m < 1 - 2/n$). In this case, $\mu < n$, so by applying (ii) of lemma 2.2 we obtain, for any $t > 0$,

$$\int_{|x|<1} u(x,t)\,\mathrm{d}x \leqslant Bt^{\mu/2}\int_{|x|<1} |x|^{-\mu}\,\mathrm{d}x \to 0 \quad \text{as } t \searrow 0.$$

But this contradicts lemma 2.1, which asserts that u satisfies either (1.3) or (1.4).

Case 2 ($m = 1 - 2/n > 0$ and $q > 1$). In this case $\mu = n$, so denoting by ω_n the area of unit sphere in $\mathbb{R}^n$ and applying lemma 2.2(i) for $|x| < R := t^{\alpha/n+1/2}$ and (ii) for $R < |x| < 1$ yields, for all small positive t,

$$\begin{aligned}\int_{|x|\leqslant 1} u(x,t)\,\mathrm{d}x &\leqslant \int_{|x|\leqslant R} At^{-\alpha}\,\mathrm{d}x + \int_{R<|x|<1} Bt^{n/2}|x|^{-n}\,\mathrm{d}x.\\ &\leqslant t^{n/2}\omega_n\{A/n + B(\alpha/n + 1/2)|\ln t|\} \to 0 \quad \text{as } t \searrow 0.\end{aligned}$$

Again, as in case (i), we get a contradiction.

Case 3 ($(1-2/n)_+ < m < 1$ and $q \geqslant m + 2/n$). In this case, $\mu > n$ so, from (2.3), $u(\cdot,t) \in L^1(\mathbb{R}^n)$ for all $t > 0$. In addition, as $q \geqslant m+2/n > 1$, and $u(\cdot,t)$ is uniformly bounded for every fixed $t > 0$, $u^q(\cdot,t) \in L^1(\mathbb{R}^n)$. Hence, integrating (1.1) over $\mathbb{R}^n$ yields that

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^n} u(x,t)\,\mathrm{d}x = -\int_{\mathbb{R}^n} u^q(x,t)\,\mathrm{d}x \quad \forall t > 0. \tag{2.5}$$

Define $e(t) = \int_{\mathbb{R}^n} u(x,t)\,\mathrm{d}x$ and denote by $R = R(t)$ the constant satisfying

$$B\omega_n(\mu-n)^{-1}t^{\mu/2}R^{n-\mu} = \tfrac{1}{2}e(t).$$

Then, by the estimate (2.3),

$$\begin{aligned}\int_{|x|<R} u(x,t)\,\mathrm{d}x &= \int_{\mathbb{R}^n} u(x,t)\,\mathrm{d}x - \int_{|x|\geqslant R} u(x,t)\,\mathrm{d}x\\ &\geqslant e(t) - \int_{|x|\geqslant R} Bt^{\mu/2}|x|^{-\mu}\,\mathrm{d}x = \tfrac{1}{2}e(t)\end{aligned}$$

by the definition of $R(t)$. Consequently, by Cauchy's inequality,

$$\int_{\mathbb{R}^n} u^q \geqslant \int_{|x|<R} u^q \geqslant \left(\int_{|x|<R} u\right)^q |B_R|^{1-q} \geqslant (\tfrac{1}{2}e(t))^q \left(\frac{\omega_n}{n} R^n\right)^{1-q}$$
$$= B_1(e(t))^{1+\nu_1} t^{-1-\nu_2}$$

by the definition of $R(t)$, where

$$\nu_1 = \frac{(q-1)\mu}{\mu - n} > 0 \quad \text{and} \quad \nu_2 = \frac{q-m-2/n}{k} \geqslant 0,$$

and B_1 is a positive constant depending only on m, q and n. It then follows from (2.5) that

$$\frac{\mathrm{d}}{\mathrm{d}t} e(t) \leqslant -B_1 e^{1+\nu_1}(t) t^{-1-\nu_2}$$

for all $t > 0$. Integrating this inequality over $(\tau, 1]$ then yields that

$$\frac{1}{\nu_1}\{e^{-\nu_1}(\tau) - e^{-\nu_1}(1)\} \leqslant -B_1 \int_\tau^1 t^{-1-\nu_2}\,\mathrm{d}t.$$

Sending $\tau \searrow 0$, we get a contradiction.

By summarizing cases 1–3, we then obtain the assertion of the theorem. □

Remark 2.6. In case 1 of the proof, we used only lemma 2.2(ii) and lemma 2.1, so when $m \in (0, 1-2/n)$ (and $n > 2$), problem (2.1) has no singular solution, provided that ϕ is non-negative. In particular, (1.8) has no singular solution when $m \in (0, 1-2/n)$ and $n > 2$.

Remark 2.7. Part of theorem 2.5 (the non-existence of FS in the case $(1-2/n)_+ < m < 1$) was first proved by Peletier and Zhao [21]. Their method is different from ours and it relies on taking appropriate test functions in the integral identity defining weak solutions of (1.1).

As another application of the upper bounds in lemma 2.2 for singular solutions, we now show that if (1.1) has a singular solution, then there exists a maximal singular solution, which has to be self-similar.

Lemma 2.8. *Let $m \in (0,1)$ and $q > 1$. Also assume that (1.1) has a singular solution. Then (1.1) admits a singular solution u^* having the following properties.*

(1) *$u^* \geqslant u$ for every singular solution u of (1.1); namely, u^* is the maximal singular solution of (1.1).*

(2) *u^* is self-similar; namely, there exists a smooth function $w(\cdot) : [0,\infty) \to [0,\infty)$ such that u^* has the form of (1.5).*

Proof. For each $\tau > 0$, let $u^{(\tau)}(x,t)$ be the solution of (1.1) in $\mathbb{R}^n \times (\tau, \infty)$ with the initial value

$$u^{(\tau)}(x,\tau) = \min\{A\tau^{-\alpha}, B(\sqrt{\tau}/|x|)^\mu\} \quad \text{on } \mathbb{R}^n \times \{t = \tau\}.$$

Then, as in the proof of lemma 2.2,

$$u^{(\tau)}(x,t) \leqslant \min\{At^{-\alpha}, B(\sqrt{t}/|x|)^{\mu}\} \quad \forall (x,t) \in \mathbb{R}^n \times [\tau, \infty). \tag{2.6}$$

Consequently, for any $\tau_1 > \tau_2 > 0$, $u^{(\tau_1)}(\cdot, \tau_1) \geqslant u^{(\tau_2)}(\cdot, \tau_1)$ so that, by comparison, $u^{(\tau_1)} \geqslant u^{(\tau_2)}$ in $\mathbb{R}^n \times [\tau_1, \infty)$. Hence, $\lim_{\tau \searrow 0} u^{(\tau)}$ exists for all $(x,t) \in \mathbb{R}^n \times (0, \infty)$. We denote this limit by u^*, which has to be a (weak) solution of (1.1), since it is the limit of a monotonic sequence of continuous functions and belongs to $L^\infty(R^n)$ by the upper bound (2.6). In addition, u^* is continuous on $R^n \times [0, \infty) \setminus (0, 0)$. It then follows that u^* satisfies (1.2).

To show that u^* is non-trivial, we need only to show that $u^* \geqslant u$ for any singular solution u of (1.1). In fact, from lemma 2.2 and a comparison principle, $u \leqslant u^{(\tau)}$ in $\mathbb{R}^n \times [s, \infty)$ for any $0 < \tau \leqslant s$. Thus, $u \leqslant u^*$ in $\mathbb{R}^n \times [s, \infty)$ for any $s > 0$; i.e. $u \leqslant u^*$ in $\mathbb{R}^n \times (0, \infty)$. Thus, u^* is non-trivial and is the maximal singular solution of (1.1) if (1.1) has a singular solution.

It remains to show that u^* is self-similar. From the construction of u^*, u^* is radially symmetric. Note that, for any $\lambda > 0$, the function $\mathcal{T}^\lambda(u^*) := \lambda^\alpha u^*(\lambda^{\alpha\beta} x, \lambda t)$ is a non-trivial and non-negative solution of (1.1) satisfying (1.2), so it is a singular solution of (1.1). Since u^* is maximal, $u^* \geqslant \mathcal{T}^\lambda(u^*)$ for all $\lambda > 0$. Observe that the operator $\mathcal{T}^\lambda$ preserves the order; namely, if $u_1 \leqslant u_2$, then $\mathcal{T}^\lambda(u_1) \leqslant \mathcal{T}^\lambda(u_2)$ for all $\lambda > 0$. We then obtain from $u^* \geqslant \mathcal{T}^\lambda(u^*)$ that $\mathcal{T}^\ell(u^*) \geqslant \mathcal{T}^\ell(\mathcal{T}^\lambda(u^*))$ for all $\ell > 0$ and $\lambda > 0$. In particular, taking $\lambda = 1/\ell$ and using $\mathcal{T}^\ell(\mathcal{T}^{1/\ell}(u^*)) = u^*$, we get $\mathcal{T}^\ell(u^*) \geqslant u^*$. Hence, $u^* = \mathcal{T}^\ell(u^*)$ for all $\ell > 0$. Thus, u is self-similar and can be written in the form of (1.5). This completes the proof. □

3. Existence of singular solutions

In the rest of this paper, we always assume that

$$q \in (1, m + 2/n), \quad m \in (0, 1). \tag{3.1}$$

We remark that the first condition implies that $1 < m + 2/n$, i.e. $m > (1 - 2/n)_+$.

First we establish the existence of the FSs for (2.1).

Theorem 3.1. *Assume that $\phi(\cdot) : [0, \infty) \to [0, \infty)$ is a continuous and non-decreasing function satisfying, for some positive constant C,*

$$0 \leqslant \phi(u) \leqslant Cu(1 + u^{q-1}) \quad \forall u \geqslant 0.$$

Let $c > 0$ be given and let $\{\varphi_j(\cdot)\}_{j=1}^\infty$ be a $c\delta$-sequence; namely, φ_j is continuous, non-negative,

$$\int_{\mathbb{R}^n} \varphi_j \, \mathrm{d}x = c \quad \textit{and} \quad \lim_{j \to \infty} \int_{|x| \geqslant r} \varphi_j(x) \, \mathrm{d}x = 0 \quad \forall r > 0.$$

Let u_j be the solution to (2.1) with initial data $u_j(\cdot, 0) = \varphi_j$. Then $\lim_{j \to \infty} u_j$ exists and is an FS of (2.1) with initial mass c.

In particular, taking $\phi(u) \equiv 0$ and $\phi(u) = u^q$, one concludes that, for every $c > 0$, both (1.1) and (1.8) have an FS with initial mass c.

Proof. Let u_j° be the corresponding solution with $\phi \equiv 0$. Since $\|u_j^{\circ}(\cdot,t)\|_{L^1(\mathbb{R}^n)} = c$ for all $t \geqslant 0$, it follows from Herrero and Pierre [9, theorem 2.2] that

$$0 \leqslant u_j^{\circ}(x,t) \leqslant M(p,n,c)\{t^{-1/k} + t^{\mu/2}\} \quad \forall t > 0, \tag{3.2}$$

where $k = m-1+2/n$. As $u_j \leqslant u_j^{\circ}$, $\{u_j\}$ is locally uniformly bounded. Consequently, by the regularity result [6] for locally bounded solutions of (2.1), the family $\{u_j\}_{j=1}^{\infty}$ is equicontinuous in any compact subset of $\mathbb{R}^n \times (0,\infty)$. Hence, we can find a function u and a subsequence, which we still denote by $\{u_j\}$, such that, as $j \to \infty$, $u_j \to u$ uniformly in any compact subset of $\mathbb{R}^n \times (0,\infty)$. The limit function u is necessarily a (weak) solution of (2.1) in $\mathbb{R}^n \times (0,\infty)$.

Now we show that u is a fundamental solution of (2.1) with initial mass c. First of all, by Fatou's lemma,

$$\int_{\mathbb{R}^n} u(x,t)\,\mathrm{d}x \leqslant \liminf_{j\to\infty} \int_{\mathbb{R}^n} u_j(x,t)\,\mathrm{d}x \leqslant c \quad \forall t > 0.$$

Next, we show that, for any $\delta > 0$,

$$\lim_{t\searrow 0} \int_{|x|<\delta} u(x,t)\,\mathrm{d}x \geqslant c. \tag{3.3}$$

For this purpose, let $\{\tilde{\varphi}_j(x)\}$ be a sequence such that $\tilde{\varphi}_j$ is continuous and compactly supported in $\{x; |x| \leqslant \delta_j\}$ with $\lim_{j\to\infty} \delta_j \to 0$, that $\tilde{\varphi}_j \leqslant \varphi_j$, and that $\int_{|x|<\delta_j} \tilde{\varphi}_j \to c$ as $j \to \infty$. Since $\{\varphi_j\}$ is a c δ-sequence, such a $\{\tilde{\varphi}_j\}$ exists.

Now let $\tilde{u}_j$ be the solution of (2.1) with initial data $\tilde{\varphi}_j$. Note that the function $B[\sqrt{t}/(x_1-\delta_j)]^{\mu}$ is a solution to $v_t = \Delta v^m$ in $\{(x,t) \mid t \geqslant 0, x_1 > \delta_j\}$. Comparing this function with $\tilde{u}_j$ in $\{(x,t) \mid t \geqslant 0, x_1 \geqslant \delta_j\}$ then yields $\tilde{u}_j \leqslant B[\sqrt{t}/(x_1-\delta_j)]^{\mu}$. By rotational invariance, we then obtain

$$\tilde{u}_j(x,t) \leqslant B[\sqrt{t}/(|x|-\delta_j)]^{\mu} \quad \forall t \geqslant 0 \text{ and } |x| \geqslant \delta_j.$$

Now we estimate the total mass of $\tilde{u}_j(\cdot,t)$. From the differential equation, we have

$$\int_{\mathbb{R}^n} \tilde{u}_j(x,t)\,\mathrm{d}x = \int_{\mathbb{R}^n} \tilde{\varphi}_j(x)\,\mathrm{d}x - \int_0^t \int_{\mathbb{R}^n} \phi(\tilde{u}_j)\,\mathrm{d}x\,\mathrm{d}\tau.$$

As $u_j^{\circ} \geqslant u_j \geqslant \tilde{u}_j$, we can use the L^{∞} bound of u_j° in (3.2) and the assumption that $0 \leqslant \phi(s) \leqslant Cs[1+s^{q-1}]$ to obtain

$$\begin{aligned}\int_0^t \int_{\mathbb{R}^n} \phi(\tilde{u}_j) &\leqslant \biggl(\sup_{\tau\in(0,t)} \int_{\mathbb{R}^n} \tilde{u}_j(x,\tau)\,\mathrm{d}x\biggr)\biggl(\int_0^t \sup_{x\in\mathbb{R}^n} C[1+(u_j^{\circ})^{q-1}]\,\mathrm{d}t\biggr) \\ &\leqslant c\int_0^t C[1+Mt^{-(q-1)/k} + Mt^{(q-1)\mu/2}] \\ &\leqslant \hat{M}(m,n,c,q)t^{(m+2/n-q)/k} \quad \forall t \in (0,1].\end{aligned}$$

Thus, for any fixed $t > 0$ and large j such that $\delta_j < \delta$,

$$\begin{aligned}\int_{|x|\leqslant\delta} u_j(x,t)\,\mathrm{d}x &\geqslant \int_{|x|\leqslant\delta} \tilde{u}_j\,\mathrm{d}x \\ &\geqslant \int_{\mathbb{R}^n} \tilde{u}_j(x,t)\,\mathrm{d}x - \int_{|x|>\delta} Bt^{\mu/2}(|x|-\delta_j)^{-\mu}\,\mathrm{d}x \\ &\geqslant \int_{\mathbb{R}^n} \tilde{\varphi}_j(x)\,\mathrm{d}x - \hat{M}t^{(m+2/n-q)/k} - Bt^{\mu/2}\int_{|x|\geqslant\delta}(|x|-\delta_j)^{-\mu}\,\mathrm{d}x.\end{aligned}$$

As $\mu > n$ and $1 < q < m + 2/n$, by sending $j \to \infty$, we then obtain

$$\int_{|x|<\delta} u(x,t)\,\mathrm{d}x \geqslant c - M(m,q,n,c)\{t^{(m+2/n-q)/k} + t^{\mu/2}\}.$$

By sending $t \searrow 0$, we then obtain (3.3).

Finally, we show that u satisfies (1.2). Multiplying the difference of the differential equation (1.8) satisfied by u_j^{o} and $b(x,t) := B(t+\varepsilon)^{\mu/2}(x_1-\varepsilon)^{-\mu}$ by $\{(u_j^0)^m - b^m(x,t)\}_+^\delta$, integrating the resulting equation over $\{(x,\tau) \mid x_1 > \varepsilon, 0 < \tau < t\}$ and sending $\delta \searrow 0$ we obtain

$$\int_{x_1>\varepsilon} \{u_j^{\mathrm{o}}(x,t) - b(x,t)\}_+\,\mathrm{d}x \leqslant \int_{x_1>\varepsilon} \{u_j^{\mathrm{o}}(x,0) - b(x,0)\}_+\,\mathrm{d}x.$$

As $u_j^{\mathrm{o}}(\cdot,0) = \varphi_j$ and $\int_{|x|\geqslant\varepsilon} \varphi_j(x)\,\mathrm{d}x \to 0$ as $j \to \infty$, by sending $j \to \infty$ and using $u \leqslant \limsup u_j^{\mathrm{o}}$, we obtain $\{u(x,t) - b(x,t)\}_+ = 0$ in $\{x \mid x_1 > \varepsilon\}$, so that $u(x,t) \leqslant b(x,t)$ if $x_1 > \varepsilon$. By sending $\varepsilon \to 0$ and using the rotational invariance, we then obtain the estimate $u(x,t) \leqslant B(\sqrt{t}/|x|)^\mu$. This shows that u satisfies (1.2). Hence, u is an FS with initial mass c.

As we shall show later, FSs are unique, so the whole sequence $\{u_j\}$ converges to u. □

Next we establish the existence of VSS of (2.1).

THEOREM 3.2. *Assume (3.1). Let ϕ be as in theorem 3.1 and satisfy $\int_1^\infty \mathrm{d}s/\phi(s) < \infty$. Then (2.1) has a VSS u_∞ which is the limit, as $c \to \infty$, of the FS of (2.1) with initial mass c. In particular, taking $\phi(u) = u^q$ one concludes that (1.1) admits a VSS.*

Proof. Let $\zeta(\cdot)$ be any non-negative continuous function on $\mathbb{R}^n$ satisfying

$$\int_{\mathbb{R}^n} \zeta(x)\,\mathrm{d}x = 1.$$

Define $\varphi_j^c = cj^n\zeta(jx)$ for all real $c > 0$ and all integer $j \geqslant 1$. Then $\{\varphi_j^c\}_{j=1}^\infty$ is a $c\delta$-sequence, and we can apply theorem 3.1 to obtain an FS $u_{(c)}$ of (2.1). Since, for each j, φ_j^c is monotonically increasing in c, so is the limit $u_{(c)}$. Also, by lemma 2.2(ii) and remark 2.3, $u_{(c)}$ satisfies the estimates (2.3) and (2.4). Therefore, $u_\infty(x,t) = \lim_{c\to\infty} u_{(c)}(x,t)$ exists, and satisfies also the estimates (2.3) and (2.4). By the local equicontinuity of $\{u_{(c)}\}$ (since they are locally uniformly bounded),

u_∞ is a (weak) solution of (2.1). Also, the estimate (2.3) for $u = u_\infty$ shows that u_∞ satisfies (1.2). Finally, since $u \geqslant u_{(c)}$ for every c,

$$\liminf_{t \searrow 0} \int_{|x|<1} u_\infty(x,t)\,\mathrm{d}x \geqslant \lim_{t \searrow 0} \int_{|x|<1} u_{(c)}(x,t)\,\mathrm{d}x = c \quad \forall c > 0.$$

Thus, u_∞ satisfies (1.4). That is, u_∞ is a VSS of (2.1). □

REMARK 3.3. In the particular case of $\phi(u) = u^q$, $q > 1$, construction of both FS and VSS has been carried out in [21].

4. Uniqueness of singular solutions

In this section, we prove the uniqueness of singular solutions of (1.1) under (3.1). To do this, we first show the uniqueness of the FS for (1.8), then the uniqueness of the FS for (1.1), and finally the uniqueness of VSS for (1.1).

4.1. Uniqueness of the FS for $u_t = \Delta u^m$

THEOREM 4.1. *Assume $m \in ((1 - 2/n)_+, 1)$. Then, for every $c > 0$, (1.8) has a unique FS with initial mass c. It is given by*

$$E_c(x,t) := G t^{-1/k} \{D(c) + (|x| t^{-1/(nk)})^2\}^{-1/(1-m)},$$

where

$$k = m - 1 + 2/n, \qquad G = (\mu m n k)^{\mu/2}, \qquad \mu = 2/(1-m),$$

and $D(c)$ is the unique constant such that $\int_{\mathbb{R}^n} G[D(c) + |x|^2]^{-\mu/2}\,\mathrm{d}x = c$.

Proof. The assertion that E_c is an FS follows by direct verification.

We now prove the uniqueness. Assume that u is any fundamental solution of (1.8) with initial mass c. We want to show that $u = E_c$. We divide the proof into three steps.

STEP 1. Consider the family $\{u^h\}_{h>0}$ where $u^h(x,t) = h u(h^{1/n} x, h^k t)$.

Direct calculation shows that each u^h is a solution of (1.8). In addition, from remark 2.4,

$$\int_{\mathbb{R}^n} u^h(x,t)\,\mathrm{d}x = \int_{\mathbb{R}^n} u(y, h^k t)\,\mathrm{d}y = c \quad \text{for all } h > 0,\ t > 0.$$

Furthermore, using lemma 2.2(ii), $u^h(x,t) \leqslant B(\sqrt{t}/|x|)^\mu$. Hence, by the regularity of solutions of (1.8), the family $\{u^h(\cdot,1)\}_{h>0}$ is equicontinuous in any bounded domain of $\mathbb{R}^n$, so that there exists a sequence $\{h_j\}_{j=1}^\infty$ satisfying $h_j \searrow 0$ as $j \to \infty$ and a function u^{o} such that $u^{h_j}(\cdot,1) \to u^{\mathrm{o}}(\cdot)$ uniformly in any compact subset of $\mathbb{R}^n$. As $\mu > n$ and $u^h(\cdot,1) \leqslant B|x|^{-\mu}$, the Lebesgue dominated convergence theorem then gives that $u^{h_j}(\cdot,1) \to u^{\mathrm{o}}$ in $L^1(\mathbb{R}^n)$.

Let $v(x,t)$ be the solution to (1.8) in $\mathbb{R}^n \times (1,\infty)$ with initial data $v(\cdot,1) = u^{\mathrm{o}}$. Then, as both u^{h_j} and v are solutions of (1.8), the contraction principle shows that, for all $t \geqslant 1$,

$$\int_{\mathbb{R}^n} |u^{h_j}(\cdot,t) - v(\cdot,t)| \leqslant \int_{\mathbb{R}^n} |u^{h_j}(\cdot,1) - v(\cdot,1)| \to 0 \quad \text{as } j \to \infty. \tag{4.1}$$

STEP 2. Denote

$$e^h(t) = \int_{\mathbb{R}^n} |u^h(\cdot,t) - E_c(\cdot,t)|,$$

for each $h > 0$. By the contraction principle, $e^h(t)$ is a non-increasing function of t. Also, for any $h > 0$, by the scaling invariance, $E_c = E_c^h$ and

$$\begin{aligned} e^h(t) &= \int_{\mathbb{R}^n} |u^h(\cdot,t) - E_c^h(\cdot,t)| \\ &= h\int_{\mathbb{R}^n} |u(h^{1/n}x, h^k t) - E(h^{1/n}x, h^k t)|\,\mathrm{d}x \\ &= \int_{\mathbb{R}^n} |u(y, h^k t) - E_c(y, h^k t)|\,\mathrm{d}y = e^1(h^k t). \end{aligned}$$

Thus $e^h(t)$ is non-increasing in both t and h. Since the initial mass of u and E_c is c, $e^h(t)$ is bounded by $2c$ for all h and t. It then follows that $\lim_{h\searrow 0} e^h(t)$ exists, and

$$\lim_{h\searrow 0} e^h(1) = \lim_{h\searrow 0} e^1(h^k) = \lim_{h\searrow 0} e^1(2h^k) = \lim_{h\searrow 0} e^h(2).$$

Denote the limit by e^{o}. Then, in view of (4.1) and the definition of e^h, we obtain

$$\begin{aligned} e^{\mathrm{o}} &= \lim_{j\to\infty} e^{h_j}(1) = \lim_{j\to\infty}\int_{\mathbb{R}^n} |u^{h_j}(\cdot,1) - E_c(\cdot,1)| = \int_{\mathbb{R}^n} |v(\cdot,1) - E_c(\cdot,1)| \\ &= \lim_{j\to\infty} e^{h_j}(2) = \int_{\mathbb{R}^n} |v(\cdot,2) - E_c(\cdot,2)|. \end{aligned}$$

STEP 3. We show that $e^{\mathrm{o}} = 0$. Suppose for the contrary that $e^{\mathrm{o}} > 0$. We define $\bar{u}$ and $\underline{u}$ as the solution of (1.8) in $\mathbb{R}^n \times (1,\infty)$ with initial data

$$\bar{u}(\cdot,1) := \max\{v(\cdot,1), E_c(\cdot,1)\}, \qquad \underline{u}(\cdot,1) := \min\{v(\cdot,1), E_c(\cdot,1)\}.$$

The comparison principle then gives $\bar{u} \geqslant \max\{v, E_c\} \geqslant \min\{v, E_c\} \geqslant \underline{u}$ in $\mathbb{R}^n \times [1,\infty)$. Since $v(\cdot,2) \not\equiv E_c(\cdot,2)$ and $\int_{\mathbb{R}^n} E_c(\cdot,2) = \int_{\mathbb{R}^n} v(\cdot,2) = c$, $\bar{u} \not\equiv \max\{v, E_c\}$ for $t > 1$. It then follows that

$$\begin{aligned} \int_{\mathbb{R}^n} [\bar{u}(\cdot,2) - \underline{u}(\cdot,2)] &> \int_{\mathbb{R}^n} [\max\{v(\cdot,2), E_c(\cdot,2)\} - \min\{v(\cdot,2), E_c(\cdot,2)\}] \\ &= \int_{\mathbb{R}^n} |v(\cdot,2) - E_c(\cdot,2)| \\ &= e^{\mathrm{o}}. \end{aligned}$$

On the other hand, by the contraction principle [1],

$$\int_{\mathbb{R}^n} |\bar{u}(\cdot,2) - \underline{u}(\cdot,2)| \leqslant \int_{\mathbb{R}^n} |\bar{u}(\cdot,1) - \underline{u}(\cdot,1)| = \int_{\mathbb{R}^n} |v(\cdot,1) - E_c(\cdot,1)| = e^{\mathrm{o}}.$$

Hence we obtain a contradiction. This contradiction shows that $e^{\mathrm{o}} = 0$.

As $e^1(t)$ is non-increasing in t, $0 = e^{\mathrm{o}} = \lim_{t\searrow 0} e^1(t)$ then implies that $e^1(t) = 0$ for all $t > 0$. Thus, $u \equiv E_c$. The proof is completed. □

4.2. Uniqueness of the FS of $u_t = \Delta u^m - u^q$

THEOREM 4.2. *Assume (3.1). Then for any given $c > 0$, (1.1) admits a unique fundamental solution u_c with initial mass c. In addition, u_c is monotonic in c.*

Proof. We need only to prove the uniqueness of the FS. The following proof follows an idea of [13].

Let v be an FS of (1.1) with initial mass c. We first show that $v \leqslant E_c$. In fact, for every $\tau > 0$, let $v_{(\tau)}$ be the solution to (1.8) for $t > \tau$ with initial value $v_{(\tau)} = v$ on $\{t = \tau\}$. Then, by comparison, $v_{(\tau)} \geqslant v$ for all $t > \tau$, so that, when $\tau_1 \leqslant \tau_2$, $v_{(\tau_1)} \geqslant v_{(\tau_2)}$ for all $t > \tau_2$, i.e. $\{v_{(\tau)}\}_{\tau>0}$ is monotonic decreasing in τ. Consequently, the limit function $w = \lim_{\tau \searrow 0} v_{(\tau)}$ exists.

By the upper bound for singular solutions (lemma 2.2) and local regularity of solutions of (1.8), we know that, for any $t > 0$, $v_{(\tau)}(\cdot, t) \to w$, as $\tau \searrow 0$, uniformly in any compact set of $\mathbb{R}^n$ and in $L^1(\mathbb{R}^n)$. As $\int_{\mathbb{R}^n} v_{(\tau)}(x, t)\,\mathrm{d}x$ is a constant equal to $\int_{\mathbb{R}^n} v(x, \tau)\,\mathrm{d}x$, which, by remark 2.4, approaches c as $\tau \to 0$, we conclude that $\int_{\mathbb{R}^n} w(\cdot, t)\,\mathrm{d}x = c$ for all t. Thus, w is an FS of (1.8) with initial mass c. By uniqueness, $w = E_c$. Consequently, $v \leqslant \lim_{\tau \searrow 0} v_{(\tau)} = E_c$.

Let u_1 and u_2 be any two FSs of (1.1) with initial mass c. Then $u_i \leqslant E_c$ for $i = 1, 2$, so that by contraction principle, for any $t > s > 0$,

$$\begin{aligned}\int_{\mathbb{R}^n} |u_1(\cdot, t) - u_2(\cdot, t)| &\leqslant \int_{\mathbb{R}^n} |u_1(\cdot, s) - u_2(\cdot, s)| \\ &\leqslant \int_{\mathbb{R}^n} \{|u_1(\cdot, s) - E_c(\cdot, s)| + |E_c(\cdot, s) - u_2(\cdot, s)|\} \\ &= \int_{\mathbb{R}^n} \{[E_c(\cdot, s) - u_1(\cdot, s)] + [E_c(\cdot, s) - u_2(\cdot, s)]\}.\end{aligned}$$

Sending $s \searrow 0$ we conclude that $u_1(\cdot, t) = u_2(\cdot, t)$, since all the integrals $\int E_c(\cdot, s)$, $\int u_1(\cdot, s)$, and $\int u_2(\cdot, s)$ approach c as $s \searrow 0$. This completes the proof of the theorem. □

4.3. Uniqueness of VSS for $u_t = \Delta u^m - u^q$

THEOREM 4.3. *Assume (3.1). Then problem (1.1) has a unique VSS.*

Proof. Let u_∞ be the VSS established in theorem 3.2. We first show that u_∞ is the minimal VSS. Since u_∞ is the limit of $u_{(c)}$, we need only show that any VSS of (1.1) is an upper bound of any FS of (1.1).

For this purpose, let u be any VSS of (1.1). Let also $c > 0$ be any fixed constant. Since u satisfies (1.4), for all sufficiently small $\tau > 0$, there exists a non-negative continuous function $\varphi_\tau(\cdot)$ defined on $\mathbb{R}^n$ such that (i) $\varphi_\tau(\cdot) \leqslant u(\cdot, \tau)$ and (ii) $\int_{\mathbb{R}^n} \varphi_\tau(x)\,\mathrm{d}x = c$. As $u(\cdot, \tau) \leqslant B(\sqrt{\tau}/|x|)^\mu$, we have

$$\limsup_{\tau \searrow 0} \int_{|x|>\varepsilon} \varphi_\tau(x, \tau)\,\mathrm{d}x \leqslant \limsup_{\tau \searrow 0} \int_{|x|>\varepsilon} B\tau^{\mu/2}|x|^{-\mu}\,\mathrm{d}x = 0 \quad \forall \varepsilon > 0.$$

Thus $\{\varphi_\tau\}_{\tau>0}$ is a $c\delta$-family. Consequently, from theorem 3.1 and theorem 4.2, $\lim_{\tau \searrow 0} u^\tau \to u_{(c)}$, where u^τ is the solution of (1.1) with initial value $u^\tau(\cdot, 0) = \varphi_\tau$.

Also, by comparison, we have $u^\tau(\cdot,\cdot) \leqslant u(\cdot,\cdot+\tau)$ in $\mathbb{R}^n\times(0,\infty)$. It then follows that $u_{(c)}(\cdot,t)\leqslant u$. Thus, every VSS of (1.1) is an upper bound of every FS. Consequently, u_∞ is the minimal VSS.

Next we show that u_∞ is self-similar. Since $u_{(c)}$ is unique, it must be radially symmetric. As u_∞ is the limit of $u_{(c)}$, as $c\to\infty$, so is u_∞.

Now, following the same proof for the self-similarity of u^* in the proof of lemma 2.8, we can show that u_∞ is scaling invariant; namely, $u_\infty = \mathcal{T}^\ell(u_\infty)$ for every $\ell>0$. Thus, u_∞ is a self-similar solution of (1.1).

As u^* is the maximal VSS and u_∞ is the minimal VSS, we conclude that all VSS coincide with $u^*=u_\infty$, by the following theorem on the uniqueness of self-similar VSS of (1.1), whose proof will be given in the next section. □

THEOREM 4.4. *Assume (3.1). Then the ordinary differential equation (1.6) has a unique non-trivial, non-negative solution w satisfying (1.7) and $w'(0)=0$.*

5. The self-similar solution

In this section we prove theorem 4.4. We need only to consider the initial-value problem (1.6) with the initial value

$$w'(0)=0, \qquad w(0)=a. \tag{5.1}$$

For each $a\geqslant 0$, (1.6), (5.1) has a unique solution $w(r;a)$ (if we write $w^q=|w|^{q-1}w$ and $w^m=|w|^{m-1}w$) and the solution is continuous differentiable in a. Since $a\geqslant 1$ implies that $w'\geqslant 0$ in its existence interval, we need only consider the case $a\in(0,1)$. For $a\in(0,1)$, if we denote by $(0,R(a))$ the maximal existence interval where $w>0$, then $w'<0$ in $(0,R(a))$ and either $R(a)=\infty$ and $\lim_{r\to\infty}w(r;a)=0$ or $R(a)<\infty$ and $w(R(a);a)=0$. For more details, see [16], where Leoni proved by a shooting argument that there exists at least an $a^*\in(0,1)$ such that the solution to (1.6), (5.1) satisfies (1.7). In this section we shall prove that such an a^* is unique.

5.1. A monotonicity lemma

Observe that $w=w(r;a)$ satisfies

$$(w^m)' + \beta r w = r^{1-n}\int_0^r \rho^{n-1}[n\beta - 1 + w^{q-1}]w\,\mathrm{d}\rho. \tag{5.2}$$

It follows that, near the origin, the solution has the following expansion:

$$w^m = a^m - (a-a^q)r^2/(2n) + O(r^4) \quad \text{as } r\searrow 0. \tag{5.3}$$

To study the behaviour of the solution $w(r;a)$, we introduce a function J, defined by

$$J(r;a) := \{r^\mu w(r;a)\}^m.$$

A substitution of $w^m = r^{-m\mu}J$ into (1.6) gives

$$\begin{aligned} r^2J'' + [n-1-2m\mu]rJ' + m\mu(\mu-n)J \\ + [\beta/m]rJ^{1/m-1}J' - [\tfrac{1}{2}(q-1)\mu]J^{1/m} - r^{\mu(1-q)}J^{q/m} = 0. \end{aligned} \tag{5.4}$$

In addition, a differentiation in a gives, for $J_a := \partial J/\partial a$,

$$\begin{aligned}\mathcal{L}(J_a) := {} & r^2 J_a'' + [n-1-2m\mu] r J_a' + m\mu(\mu - n) J_a \\ & + \beta(1-m)m^{-2} r J^{1/m-2} J' J_a + \beta m^{-1} r J^{1/m-1} J_a' \\ & - [(q-1)\mu/2m] J^{1/m-1} J_a - q m^{-1} r^{\mu(1-q)} J^{q/m-1} J_a = 0. \end{aligned} \tag{5.5}$$

LEMMA 5.1. *If $J' > 0$ in a finite interval $(0, r_1)$, then $\mu a J_a > r J'$ in $(0, r_1]$ and $w_a > 0$ on $[0, r_1]$.*

Proof. Applying the differential operator $r(\mathrm{d}/\mathrm{d}r)$ to (5.4) and using the identity $r[r^2 J'']' = r^2 [rJ']''$, one obtains

$$\mathcal{L}(rJ') = \mu(1-q) r^{\mu(1-q)} J^{q/m} < 0 \quad \text{in } (0, R(a)).$$

In the interval $(0, r_1)$, write $J_a = C(r) r J'$. Using the expansion (5.3), we have, for all r sufficiently small,

$$C(r) = (\mu a)^{-1} \left\{ 1 + \frac{q-1}{2mn} a^{q-m} r^2 + O(r^4) \right\}.$$

It then follows that $C(0) = (\mu a)^{-1}$, and $C' > 0$ near the origin. Now, from the differential equation for $J_a = C(r) r J'$, we have

$$r^2 C''[rJ'] + C'[\cdots] + C\mathcal{L}(rJ') = 0.$$

As $\mathcal{L}(rJ') < 0$ and $rJ' > 0$ in $(0, r_1)$, we know that C' cannot attain its first zero in $(0, r_1)$. Thus, $C' > 0$ in $(0, r_1)$. Consequently, $J_a = C r J' > (\mu a)^{-1} r J' > 0$ in $(0, r_1)$.

It remains to show that $J_a > 0$ at r_1. For a later application, here we provide an elaborated proof.

Let $r_0 = \min\{1, r_{1/2}\}$ and let ψ be the solution to $\mathcal{L}(\psi) = 0$ in $(0, R(a))$ with the initial values $\psi(r_0) = 0$ and $\psi'(r_0) = 1$. Then $\psi > 0$ in $(r_0, r_1]$, since between any two zeros of ψ there is a zero of J_a. Set $k_0 = C' r J'|_{r=r_0} > 0$ and $c_0 = C|_{r=r_0}$.

Now consider the function $\varphi = J_a - k_0 \psi$. We have $\mathcal{L}(\varphi) = 0$ in $(0, R(a))$. In addition, at $r = r_0$, $\varphi = J_a = c_0 r J'$ and $\varphi' = \{C' r J' + C(rJ')' - k_0 \psi'\}|_{r=r_0} = c_0 (rJ')'$. Therefore, writing $\varphi = \hat{C}(r) r J'$, we have that $\hat{C}(r_0) = c_0$, $\hat{C}'(r_0) = 0$, and $\hat{C}$ satisfies the same equation as that for C. As $\hat{C}''(r_0) > 0$ (from the differential equation), one sees that $\hat{C}' > 0$ in (r_0, r_1). Therefore, $\varphi = \hat{C}(r) r J' > 0$ in $[r_0, r_1)$. Consequently, $J_a \geqslant k_0 \psi > 0$ in $(r_0, r_1]$. This completes the proof of the lemma. □

For convenience, we define

$$\begin{aligned} \mathcal{A} &= \{a \in (0,1) \mid \text{there exists } R_1(a) \in (0, R(a)) \text{ such that } J'(R_1(a); a) = 0\}, \\ \mathcal{B} &= \left\{ a \in (0,1) \;\middle|\; J'(\cdot; a) > 0 \text{ in } (0, \infty), \lim_{r \to \infty} J(r; a) < \infty \right\}, \\ \mathcal{C} &= \left\{ a \in (0,1) \;\middle|\; J'(\cdot; a) > 0 \text{ in } (0, \infty), \lim_{r \to \infty} J(r; a) = \infty \right\}. \end{aligned}$$

Since $J' > 0$ near the origin, if $a \in (0,1)$ is not in $\mathcal{A}$, then $J' > 0$ in $(0, R(a))$, which implies $R(a) = \infty$, so that $a \in \mathcal{B} \cup \mathcal{C}$. Thus, $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ are disjoint to each other and $\mathcal{A} \cup \mathcal{B} \cup \mathcal{C} = (0,1)$.

5.2. Characterization of the set $\mathcal{A}$

LEMMA 5.2. *Let $a \in (0,1)$. The following statements are equivalent:*

(i) $a \in \mathcal{A}$;

(ii) *there exists $R_1 \in (0, R(a))$ such that $J' > 0$ in $(0, R_1(a))$, $J''(R_1(a); a) < 0$ and $J' < 0$ in $(R_1(a), R(a))$;*

(iii) $\sup_{r\in(0,R(a))} J < J^* = [2m(\mu - n)/(q-1)]^{m/(1-m)}$;

(iv) *there exists $r_1 \in (0, R(a))$ such that $\int_0^{r_1} \rho^{n-1}(\beta n - 1 + w^{q-1}) w \, \mathrm{d}\rho < 0$;*

(v) $R(a) < \infty$ *and* $(w^m)'(R(a); a) < 0$;

(vi) $R(a) < \infty$.

Proof. (i) $\Rightarrow$ (ii). Let $(0, R_1(a))$ be the maximal interval where $J' > 0$. Since $a \in \mathcal{A}$, $R_1(a) < R(a)$ and $J'(R_1(a); a) = 0$. We claim that $J''(R_1(a); a) < 0$. In fact, if $J''(R_1(a); a) = 0$, then by differentiating (5.4) with respect to r and evaluating the resulting equation at $r = R_1(a)$, we obtain $J'''(R_1(a); a) < 0$. This contradicts the fact that $J' > 0$ in $(0, R_1)$. Thus, $J''(R_1(a); a) < 0$.

Next we show that $J' < 0$ in $(R_1(a), R(a))$. In fact, if this is not true, then there exists $R_2(a) \in (R_1(a), R(a))$ such that $J'(R_2(a); a) = 0$ and $J' < 0$ in $(R_1(a), R_2(a))$. By evaluating (5.4) at $r = R_1(a)$ with $J'(R_1; a) = 0$, $J''(R_1; a) < 0$, and at $r = R_2$ with $J'(R_2; a) = 0$ and $J''(R_2; a) \geqslant 0$, and using the definition $r^{m\mu} w^m = J$, we obtain

$$[(q-1)\mu/2 + w^{q-1}) J^{1/m-1}|_{r=R_1(a)} < m\mu(\mu - n) \leqslant [(q-1)\mu/2 + w^{q-1}) J^{1/m-1}|_{r=R_2(a)}. \quad (5.6)$$

But this is impossible, since $J(R_1; a) > J(R_2; a)$ and $w(R_1; a) > w(R_2; a)$. Hence, $J' < 0$ in $(R_1(a), R(a))$.

(ii) $\Rightarrow$ (iii). Note that the maximum of J is obtained at $r = R_1(a)$, so the assertion follows from the first inequality in (5.6).

(iii) $\Rightarrow$ (iv). Assume for the contrary that $\int_0^r \rho^{n-1}(\beta n - 1 + w^{q-1}) w \, \mathrm{d}\rho \geqslant 0$ for all $r \in (0, R(a))$. Then from (5.2), for all $r \in (0, R(a))$, $(w^m)' + \beta r w \geqslant 0$. Upon integrating this inequality over $(0, r)$ we obtain

$$w(r; a) \geqslant \{a^{m-1} + \beta r^2/(m\mu)\}^{-1/(1-m)} \quad \forall r \in (0, R(a)).$$

It then follows that $R(a) = \infty$ and that

$$\hat{J} := \liminf_{r\to\infty} J = \liminf_{r\to\infty} (r^\mu w)^m > 0.$$

Note that either $J' > 0$ in $(0, \infty)$ or, if J' changes sign, then $a \in \mathcal{A}$, so that $J' < 0$ in $(R_1(a), \infty)$. In either case, we have $\lim_{r\to\infty} J = \hat{J}$ and $\liminf_{r\to\infty} |rJ'| = 0$.

Let $\{r_j\}_{j=1}^\infty$ be a sequence such that $\lim_{j\to\infty} r_j = \infty$ and $\lim_{j\to\infty}(rJ')|_{r=r_j} = 0$. We claim that $\{r_j\}$ can be selected such that, in addition, $\lim_{j\to\infty}(r^2 J'')|_{r=r_j} = 0$.

In fact, if $|rJ'|$, which is positive for all large r, oscillates infinitely many times, then one can select $\{r_j\}$ to be local minimum points of $|rJ'|$ so that $0 = (rJ')' = rJ'' + J'$ on $\{r_j\}$. That is, $\lim_{j\to\infty}(r^2J'')|_{r=r_j} = -\lim(rJ')|_{r=r_j} = 0$.

If $|rJ'|$ does not oscillate infinitely many times, then $|rJ'|$ eventually monotonically decreases to zero, so that one can select $\{r_j\}$ along which $r(r|J'|)'$ approaches zero; namely, $r^2J'' = r(rJ')' - rJ'$ approaches zero along the sequence $\{r_j\}$.

Now by evaluating (5.4) at r_j and sending $j \to \infty$, we obtain $\hat{J} = J^*$, contradicting the assumption that $\sup_{r\in(0,R(a))} J < J^*$.

(iv) $\Rightarrow$ (v). Since the function $z = \beta n - 1 + w^{q-1}$ is strictly decreasing in $(0, R(a))$, $\int_0^{r_1} \rho^{1-n} zu < 0$ implies that $z < 0$ for all $r \in [r_1, R(a))$. It then follows that, for some $\delta > 0$, $\int_0^r \rho^{n-1}(\beta n - 1 + w^{q-1})w \leqslant -\delta$ in $[r_1, R(a))$. Consequently, from (5.2),

$$(w^m)' \leqslant -\beta rw - \delta r^{1-n} \quad \forall r \in [r_1, R(a)). \tag{5.7}$$

If $n = 1$, then $(w^m)' < -\delta$ in $(r_1, R(a))$ and assertion (v) is trivially true.

If $n \geqslant 2$, then $2/n \in (0, 1]$, so by using the inequality

$$\beta rw + \delta r^{1-n} \geqslant (\beta rw)^{1-2/n}(r^{1-n}\delta)^{2/n} = \beta^{1-2/n}\delta^{2/n}w^{1-2/n}r^{-1},$$

we obtain from (5.7) that, for all $r \in (r_1, R(a))$,

$$-(w^m)' \geqslant \beta^{1-2/n}\delta^{2/n}r^{-1}w^{m(1-\nu)}, \tag{5.8}$$

where $\nu = 2(\mu - n)/(mn\mu) > 0$. Multiplying both sides of (5.8) by $w^{m(\nu-1)}$ and integrating over $[r_1, r)$, $r < R(a)$, one immediately concludes that $R(a) < \infty$. In addition, it follows from (5.7) that $(w^m)'(R(a); a) < 0$.

(v) $\Rightarrow$ (vi) is trivially true.

(vi) $\Rightarrow$ (i) is also trivially true since $w(R(a); a) = 0$ implies that $J = (r^\mu w)^m$ has an interior maximum in $(0, R(a))$. This completes the proof of the lemma. □

THEOREM 5.3. *There exists $a_* \in ((1 - \beta n)^{1/(q-1)}, 1]$ such that $\mathcal{A} = (0, a_*)$.*

Proof. When $a \in (0, (1 - \beta n)^{1/(q-1)}]$, $w(r; a) < a$ for all $r \in (0, R(a))$, so that lemma 5.2(iv) holds. It then follows that $a \in \mathcal{A}$. Thus, $(0, (1 - \beta n)^{1/(q-1)}] \in \mathcal{A}$.

For any given $\hat{a} \in \mathcal{A}$, since $J''(R_1(\hat{a}); \hat{a}) < 0$, an implicit function theorem then yields that the $J'(R_1; a) = 0$ has a local unique C^1 solution $R_1 = R_1(a)$ in a neighbourhood of $\hat{a}$. Thus, $\mathcal{A}$ is open and $R_1(a)$ is C^1 in $\mathcal{A}$. Furthermore, defining $m(a) = J(R_1(a); a)$, we have

$$\frac{\mathrm{d}}{\mathrm{d}a}m(a) = J'\frac{\mathrm{d}}{\mathrm{d}a}R_1 + J_a = J_a > 0 \quad \text{for all } a \in \mathcal{A}.$$

To finish the proof, we need only show that, if $(a_1, a_2) \in \mathcal{A}$ and $a_1 > 0$, then $a_1 \in \mathcal{A}$.

In fact, by the continuous dependence of initial data,

$$\sup_{r\in(0,R(a_1))} J(r; a_1) \leqslant \limsup_{a\searrow a_1} m(a) < m(\tfrac{1}{2}(a_1 + a_2)) < J^*,$$

so that, by lemma 5.2(iii), $a_1 \in \mathcal{A}$. This completes the proof of the theorem. □

5.3. Characterization of the set $\mathcal{C}$

LEMMA 5.4. *Let $a \in (0,1)$. Then $a \in \mathcal{C}$ if and only if $\sup_{r\in(0,R(a))} J > J^*$.*

Proof. The 'only if' part follows from the definition of $\mathcal{C}$. Now, if $\sup_{r\in(0,R(a))} J > J^*$, then, by lemma 5.2(iii), $a \notin \mathcal{A}$, and so $a \in \mathcal{C} \cup \mathcal{B}$. However, if $\hat{J} := \lim_{r\to\infty} J$ is finite, then one can find a sequence $\{r_j\}$ along which rJ' and r^2J'' approach zero. This implies, from equation (5.4), that $\hat{J} = J^*$. But this contradicts the assumption that $\sup_{r\in(0,R(a))} J > J^*$. □

THEOREM 5.5. *There exists $a^* \in (0,1)$ such that $\mathcal{C} = (a^*, 1)$. In addition, for every $a \in \mathcal{C}$, there exists $k(a) > 0$ such that*

$$\lim_{r\to\infty} r^{1/\beta} w(r;a) = k(a).$$

Furthermore, $k(a)$, as a function of $a \in (a^*, 1)$, is positive, continuous, strictly increasing, and

$$\lim_{a\searrow a^*} k(a) = 0, \qquad \lim_{a\nearrow 1} k(a) = \infty.$$

Proof.

STEP 1. We first show that $\mathcal{C}$ is open and non-empty. Since $a \in \mathcal{C}$ if and only if $\sup_{r\in(0,R(a))} J(r;a) > J^*$, by the continuous dependence of initial data, $\mathcal{C}$ is open. Also, as $\lim_{a\nearrow 1} w(r;a) = w(r;1) \equiv 1$ uniformly in any compact subset of $[0,\infty)$, $\lim_{a\nearrow 1} J((2J^*)^{1/(m\mu)};a) = 2J^*$, so that $(1-\varepsilon, 1) \subset \mathcal{C}$ for some sufficiently small positive ε.

As $\mathcal{A} = (0, a_*)$, $[a_*, 1) \in \mathcal{B} \cup \mathcal{C}$, so that $J' > 0$ for all $r \in (0,\infty)$ and all $a \in [a_*, 1)$. Consequently, by lemma 5.1, $J_a > 0$ for all $r \in (0,\infty)$ and all $a \in [a_*, 1)$. This implies that $\mathcal{C} = (a^*, 1)$, where $a^* = \inf\{a \geqslant a_* \mid \lim_{r\to\infty} J(r;a) > J^*\}$.

As a by-product, $\mathcal{B} = [a_*, a^*] = \{a \mid J(r;a) \nearrow J^* \text{ as } r \to \infty\}$.

STEP 2. We now study the behaviour of the solution $w(\cdot;a)$ for $a \in \mathcal{C}$. For simplicity, we write $w(r;a)$ and $J(r;a)$ as $w(r)$ and $J(r)$, respectively.

Let $\tau = \ln r$. As w is positive, we can write

$$w(e^\tau) = w(1)\exp\biggl(-\int_0^\tau \Lambda(\hat{\tau})\,\mathrm{d}\hat{\tau}\biggr).$$

Since $w' < 0$ and $J' = r^{m\mu-1}[r(w^m)' + m\mu w^m) > 0$ for all $r > 0$, we have $0 < \Lambda(\tau) < \mu$ for all $\tau \in (-\infty, \infty)$.

By substituting this transformation into (1.6) and using the relations

$$r\frac{\mathrm{d}}{\mathrm{d}r} = \frac{\mathrm{d}}{\mathrm{d}\tau}, \qquad r^2\frac{\mathrm{d}^2}{\mathrm{d}r^2} = \frac{\mathrm{d}^2}{\mathrm{d}\tau^2} - \frac{\mathrm{d}}{\mathrm{d}\tau} \quad \text{and} \quad r^2 w^{1-m} = J^{1/m-1},$$

we obtain, on denoting $\mathrm{d}/\mathrm{d}\tau$ by a superposed dot,

$$\dot{\Lambda} = F(\Lambda, \tau) := m\Lambda^2 + (2-n)\Lambda + m^{-1}[1 - \beta\Lambda - w^{q-1}]J^{1/m-1}.$$

Here we consider Λ as an unknown function whereas $w = w(e^\tau)$ and $J = J(e^\tau)$ as known functions of τ.

Since $a \in \mathcal{C}$, as $\tau \nearrow \infty$, $w \searrow 0$ and $J \nearrow \infty$. It then follows that, for any $\varepsilon > 0$, there exists $\tau_\varepsilon > 0$ such that $F(\Lambda, \tau) > 0$ for all $\Lambda \in (0, (1-\varepsilon)/\beta]$, $\tau > \tau_\varepsilon$ and $F(\Lambda, \tau) < 0$ for all $\Lambda \in ((1+\varepsilon)/\beta, \mu]$, $\tau > \tau_\varepsilon$. It then follows from an invariant region argument that

$$\lim_{\tau \to \infty} \Lambda(\tau) = \frac{1}{\beta}.$$

STEP 3. Next, we show that, as $\tau \to \infty$, Λ approaches $1/\beta$ exponentially fast, with an exponent of at least

$$\nu = \tfrac{1}{2} \min\left\{\frac{q-1}{\beta}, 2 - \frac{1-m}{\beta}\right\}.$$

Consider the function

$$\Lambda^-(\tau) = \frac{1}{\beta}\left[1 - \frac{e^{\nu(T-\tau)}}{2}\right]$$

defined on $[T, \infty)$. We want to show that Λ^- is a subsolution to $\dot{\Lambda} = F(\Lambda, \tau)$ in $[T, \infty)$ provided that T is sufficiently large.

First let T be large enough that $w^{q-1}(e^T) < \frac{1}{4}$ and $\Lambda(\tau) > 1/(2\beta)$ for all $\tau > T$. Then

$$w^{q-1}(e^\tau) = w^{q-1}(e^T) \exp\left(-(q-1)\int_T^\tau \Lambda\right) < \tfrac{1}{4} e^{\nu(T-\tau)} \quad \text{for all } \tau \geqslant T.$$

Next, taking a larger T if necessary, we assume that $\Lambda(\tau) \leqslant 1/\beta + \frac{1}{2}(\mu - 1/\beta)$ for all $\tau \geqslant T$. Then

$$\begin{aligned} J^{1/m-1}(e^\tau) &= J^{1/m-1}(e^T) \exp\left\{\int_T^\tau (2 - (1-m)\Lambda)\right\} \\ &\geqslant J^{1/m-1}(e^T) e^{\nu(\tau - T)} \quad \forall \tau \geqslant T. \end{aligned}$$

Thus, $\{1 - \beta\Lambda^- - w^{q-1}(e^\tau)\} J^{1/m-1}(e^\tau) \geqslant \frac{1}{4} J^{1/m-1}(e^T)$ for all $\tau \geqslant T$. Consequently, for all $\tau \geqslant T$,

$$\frac{d}{d\tau}\Lambda^- - F(\Lambda^-, \tau) \leqslant \frac{\frac{1}{2}\nu + n}{\beta} - \frac{1}{4m} J^{1/m-1}(e^T) < 0 \quad \forall \tau > T$$

if we take T large enough, since $J(e^T) \to \infty$ as $T \to \infty$.

Comparing $\Lambda(\tau)$ to $\Lambda^-(\tau)$ in $[T, \infty)$, we find that

$$\Lambda(\tau) > \Lambda^-(\tau) = \frac{1}{\beta}(1 - \tfrac{1}{2} e^{\nu(T-\tau)}) \quad \text{in } [T, \infty).$$

In a similar manner, we can show that

$$\Lambda(\tau) \leqslant \Lambda^+(\tau) = \frac{1}{\beta}\left[1 + \frac{1}{2}\left(\mu - \frac{1}{\beta}\right) e^{\nu(T-\tau)}\right].$$

Therefore, $|\Lambda - 1/\beta| \leqslant \mu e^{\nu(T-\tau)}$. Consequently, as $r \to \infty$,

$$\begin{aligned} r^{1/\beta} w(r) = w(1) \exp\left\{-\int_0^{\ln r} \left(\Lambda - \frac{1}{\beta}\right)\right\} &\to w(1) \exp\left\{-\int_0^\infty \left(\Lambda(\tau) - \frac{1}{\beta}\right) d\tau\right\} \\ &=: k(a). \end{aligned}$$

Since $(w^m)_a = r^{-m\mu} J_a > 0$ in $(0, \infty)$, we then know that $k(\cdot)$ is positive, continuous and non-decreasing on $(a^*, 1)$.

Since $\Lambda(\tau)$ approaches $1/\beta$ exponentially fast, we have, as $r \to \infty$, $r(r^{1/\beta} w)' = 0(r^{-\nu})$.

Recall from lemma 5.1 that $J_a \geqslant 1/(\mu a) r J'$, which implies that

$$w_a \geqslant \frac{1}{\mu a}[\mu w + r w'],$$

i.e.

$$(r^{1/\beta} w)_a \geqslant \frac{1}{\mu a}[(\mu - 1/\beta) r^{1/\beta} w + r(r^{1/\beta} w)'].$$

Hence, for any $a^* < a_1 < a_2$,

$$\begin{aligned} k(a_2) - k(a_1) &= \lim_{r\to\infty} \int_{a_1}^{a_2} r^{1/\beta} w_a \, \mathrm{d}a \\ &\geqslant \lim_{r\to\infty} \int_{a_1}^{a_2} \frac{\mu - 1/\beta}{\mu a} r^{1/\beta} w \, \mathrm{d}a \\ &= \frac{\mu - 1/\beta}{\mu a} \int_{a_1}^{a_2} k(a) \, \mathrm{d}a. \end{aligned}$$

Thus, $k(\cdot)$ is strictly increasing.

Now if $\lim_{a \searrow a^*} k(a) > 0$, we can derive that $\sup_{r>0} r^{m\mu} w^m(r; a^*) > J^*$, which would imply that $a^* \in \mathcal{C}$, contradicting the definition of a^*. Hence, $\lim_{a \searrow a^*} k(a) = 0$.

Finally, one can show that, if $K = r^{1/\beta} w$ obtains a local maximum, say, at $r = r_1$, then at $r = r_1$, $K' = 0$ and $K'' < 0$ (if $K'' = 0$ then one can get $K''' < 0$, which contradicts $K' > 0$ in $(0, r_1)$), i.e. $\beta r w' + w = 0$ and $r^2 w'' \leqslant (1 + \beta)/(\beta^2) w$. Substituting this information into (1.6) then yields

$$K(r_1; a) < K^* := \{m\beta^{-2}(q - n\beta)\}^{1/(q-m)}.$$

Hence, if the value of K exceeds K^*, then K increases monotonically thereafter. It then follows from $w(r; 1) \equiv 1$ that $\lim_{a \nearrow 1} k(a) = \infty$. This completes the proof of the theorem. □

5.4. Characterization of the set $\mathcal{B}$

THEOREM 5.6. *$\mathcal{B} = \{a_*\} = \{a^*\}$ and $J(r; a^*) \nearrow J^*$ as $r \nearrow \infty$.*

Proof. From the previous discussion, we know that $\mathcal{B} = [a_*, a^*]$, and that, for all $a \in \mathcal{B}$, $J_a > 0$ for all $r > 0$ and $J(r; a) \nearrow J^*$ as $r \nearrow \infty$. It remains to show that $a_* = a^*$.

We claim that if $a \in \mathcal{B}$, then $\lim_{r\to\infty} J_a(r; a) = \infty$.

To show this, we use the independent variable $\tau = \ln r$. Note that $rJ' = \dot{J}$ vanishes as $\tau \to \infty$. The linear operator $\mathcal{L}$ in (5.5) takes the form

$$\mathcal{L}(\phi) = \ddot{\phi} + [b + o(1)]\dot{\phi} - [c + o(1)]\phi,$$

for τ sufficiently large, where $o(1) \to 0$ as $\tau \to \infty$, b is a certain constant and $c = 2(\mu - n)$. As $c > 0$, it is easy to see that the solution to $\mathcal{L}(\phi_1) = 0$ in (T, ∞) with initial value $\phi_1(T) = 0$, $\dot{\phi}_1(T) = 1$ with T large enough will have the property that $\phi_1 \to \infty$ exponentially fast as $\tau \to \infty$.

Note that the function ψ constructed in the proof of lemma 5.1 is positive in (r_0, ∞). As J_a and ψ are linearly independent, one of them will be unbounded. Since $J_a \geqslant k_0 \psi$, we then know that $J_a \to \infty$ as $r \to \infty$.

Finally, we show that $a^* = a_*$. In fact, if $a^* > a_*$, then, by Fatou's lemma,

$$0 = \lim_{r\to\infty} (J(r; a^*) - J(r; a_*)) = \lim_{r\to\infty} \int_{a_*}^{a^*} J_a(r; a)\,\mathrm{d}a \geqslant \int_{a_*}^{a^*} \liminf_{r\to\infty} J_a(r; a)\,\mathrm{d}a = \infty,$$

which is impossible. This completes the proof of the theorem. □

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