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The global dynamics of isothermal chemical systems with critical nonlinearity

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Abstract
In this paper, we study the Cauchy problem of a cubic autocatalytic chemical reaction system

$$u_{1,t} = u_{1,xx} - u_1^\alpha u_2^\beta, \qquad u_{2,t} = du_{2,xx} + u_1^\alpha u_2^\beta$$

with non-negative initial data, where the exponents α, β satisfy $1 < \alpha, \beta < 2$, $\alpha + \beta = 3$ and the constant $d > 0$ is the Lewis number. Our purpose is to study the global dynamics of solutions under mild decay of initial data as $|x| \to \infty$. We show the exact large time behaviour of solutions which is universal.

Mathematics Subject Classification: 35K57, 35B40, 35Q99, 80A25

1. Introduction

In this paper, we study the Cauchy problem of the chemical reaction system

$$u_{1,t} = u_{1,xx} - u_1^\alpha u_2^\beta, \tag{1}$$

$$u_{2,t} = du_{2,xx} + u_1^\alpha u_2^\beta \tag{2}$$

with non-negative initial data

$$u_1(x, 0) = a_1(x), \quad u_2(x, 0) = a_2(x), \qquad a_1(x),\ a_2(x) \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R}) \tag{3}$$

of arbitrary size, where the exponents α, β satisfy $1 < \alpha, \beta < 2$, $\alpha + \beta = 3$ and the constant $d > 0$ is the Lewis number. Our purpose is to study the global dynamics of solutions under mild decay of initial data as $|x| \to \infty$.

The system (1)–(2) arises as a model for cubic autocatalytic chemical reactions of the type

$$\alpha A + \beta B \longrightarrow 3B$$

with isothermal reaction rate proportional to $u_1^\alpha u_2^\beta$. Here, u_1 is the concentration of reactant A, u_2 is the concentration of auto-catalyst B. The simple power rate is obtained by the usual assumption of exponential dependence on the temperature rise in flame systems. Such a system is also used to model thermal-diffusive combustion problems (see [7]). For recent works on similar systems with applications in mathematical biology (see [8]).

The system (1)–(2) on a bounded domain is well-studied in the literature, in particular, see Alikakos [1], Hollis *et al* [10], Martin and Pierre [12] and Masuda [13]. Among other things, the authors established boundedness, global existence and large time behaviour of solutions. For homogeneous Dirichlet or Neumann boundary conditions, the large time behaviour is that (u_1, u_2) converges to a constant vector (c_1, c_2) such that $c_1 \cdot c_2 = 0$ (see Masuda [13]).

More recently, Billingham and Needham [4,5] showed the existence of a travelling front, again for the case of $\alpha = 1$, $\beta = 2$, using a shooting argument and phase plane methods. In addition, the large time behaviour of solutions is also studied by formal methods and numerical computation by the authors.

Most recently, motivated by thermal-diffusion models with Arrhenius reactions [2, 14], Berlyand and Xin [3] and Bricmont *et al* [7] considered system (1)–(2) with initial data (3) that have sufficient polynomial decay at infinity for the case of $\alpha = 1$ and $\beta = 2$. The authors of [7], in particular Bricmont and Kupiainen [6], along with Goldenfeld *et al* [9], pioneered the application of the renormalization group (RG) to nonlinear parabolic equations. In [3,7], upper and lower bounds by self-similar profiles were established and, more importantly, the exact large time self-similarity of solutions with initial data decaying sufficiently fast as $|x| \to \infty$ was proved.

We will now give a very brief description of some underlying ideas on the application of the RG to nonlinear parabolic equations. The RG was first developed in physics in the study of quantum electrodynamics (QED) and quantum field theory (QFT) with an infinite number of degrees of freedom in dealing with difficulties such as the infinite electromagnetic mass of a point-like electron, the infinite energy density ground state (vacuum), etc. Its application in nonlinear parabolic equations involves, e.g., the study of the global existence or blow-up, asymptotic or spatial-temporal profiles of solutions when $t \to \infty$ for

$$u_t = u_{xx} + F(u, u_x, u_{xx}), \qquad u(x, 1) = \phi(x). \tag{4}$$

In many cases (4) exhibits some scaling laws and a certain universality, i.e. the existence of solutions of the form

$$u(x, t) \sim t^{-p/2} f^* \left(\frac{x}{\sqrt{t}} \right) \qquad \text{as } t \to \infty, \tag{5}$$

where p and f^* are independent of initial conditions or even the form of the equations. The RG method transforms the problem of large time limit into an iteration of a fixed time problem from 1 to a fixed time interval $L > 1$ followed by a suitable scaling that produces a new problem similar to the original one. The scale-invariant solution emerges then as a fixed point of a map in the space of initial data, the RG map and stability analysis becomes the analysis of the stability of the fixed point under the RG. For example, the form of (5) suggests that

$$u_L(x, t) = L^p u(Lx, L^2 t),$$

where u solves (4) with the initial data ϕ. The RG map is then $\mathrm{RG}(\phi)(x) = u_L(x, 1)$ and u_L satisfies

$$u_{L,t} = u_{L,xx} + F_L(u_L, u_{L,x}, u_{L,xx}),$$

with $F_L(a, b, c) = L^{2+p} F(L^{-p}a, L^{-1-p}b, L^{-2-p}c)$. Note that if $F = a^q$, then $F_L = L^{2+p-pq} a^q$. For the solutions that decay like the heat kernel, i.e. $p = 1$, $q = 3$ makes $F_L = F$ while if $q > 3$, F_L is smaller than F with a fixed proportion. This makes the nonlinearity of degree 3 critical (marginal in the RG literature) with respect to the heat equation. If we let

$\mathrm{RG}_{L^n,F_{L^n}}$, u_{L^n} denote the nth iteration of the RG map and the nth iterated solution, then we have

$$u_{L^n,t} = u_{L^n,xx} + F_{L^n}(u_{L^n}, u_{L^n,x}, u_{L^n,xx})$$

and the definition of RG maps implies

$$u(x,t) = t^{-p/2}\mathrm{RG}_{L^n,F_{L^n}}(\phi)\left(\frac{x}{\sqrt{t}}\right), \tag{6}$$

by letting $t = L^{2n}$.

Now the key is of course to show that there exists p such that

$$F_{L^n} \to F^*, \qquad \mathrm{RG}_{L^n,F_{L^n}}(\phi) \to f^*, \tag{7}$$

where $\mathrm{RG}_{L,F^*} f^* = f^*$ is the fixed point of the RG, corresponding to the scale-invariant equation

$$u_t = u_{xx} + F^*(u, u_x, u_{xx}). \tag{8}$$

That is the long-time behaviour is translated into a stability of the fixed point of RG map (more can be learned in [16]).

Now we give a detailed description of the main result of [7] in what follows. Suppose $(a_1, a_2) \in \mathcal{B} \times \mathcal{B}$, where $\mathcal{B}$ is the Banach space of continuous functions on $\mathbb{R}$ with the norm

$$\|f\| = \sup_{x\in\mathbb{R}} |f(x)|(1+|x|)^q, \qquad q > 1. \tag{9}$$

Let ϕ_d be the Gaussian

$$\phi_d = \frac{1}{\sqrt{4\pi d}} \mathrm{e}^{-x^2/4d}.$$

For $A > 0$, let ψ_A be the normalized ($\int \psi_A^2(x)\,\mathrm{d}\mu(x) = 1$, see below) first eigenfunction of the differential operator

$$\mathcal{L}_A = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} - \frac{1}{2}x\frac{\mathrm{d}}{\mathrm{d}x} - \frac{1}{2} + A^2\phi_d^2(x)$$

on $L^2(\mathbb{R}, \mathrm{d}\mu)$, with $\mathrm{d}\mu(x) = \mathrm{e}^{x^2/4}\,\mathrm{d}x$ and let E_A, which is positive for $A > 0$, be the corresponding eigenvalue, then the main result of [7] is as follows.

Theorem (Bricmont, Kupiainen and Xin). *Suppose $\alpha = 1$, $\beta = 2$, $(a_1, a_2) \in \mathcal{B} \times \mathcal{B}$ and $a_i \geqslant 0$, $a_i \not\equiv 0$, $i = 1, 2$. Let $A = \int_{\mathbb{R}} a_1(x) + a_2(x)\,\mathrm{d}x$, which is conserved in time. Then the system (1)–(2) has a unique global classical solution $(u_1(x,t), u_2(x,t)) \in \mathcal{B} \times \mathcal{B}\ \forall t \geqslant 0$. Moreover, there exists a $q(A)$, which is an increasing function of A and tending to ∞ as $A \to \infty$, such that if $q \geqslant q(A)$ in (9), there is a positive number B depending continuously on (a_1, a_2) such that*

$$\|t^{1/2+E_A}u_1(\sqrt{t}\cdot, t) - B\psi_A(\cdot)\| \to 0,$$
$$\|t^{1/2}u_2(\sqrt{t}\cdot, t) - A\phi_d(\cdot)\| \to 0$$

as $t \to \infty$.

Remark. As was pointed out in [7], the extra decay power E_A in time is due to the critical cubic nonlinearity of the system (1)–(2). In others word, the scaling law which works for non-cubic nonlinearity no longer works for cubic nonlinearity, and thus the appearance of the anomalous exponent E_A.

In this paper, following the work of [7] we continue to investigate the critical cubic nonlinearity (i.e. $\alpha + \beta = 3$) of the system (1)–(2). The main result of this work is the following theorem.

Theorem 1. *Suppose* $1 < \alpha,\ \beta < 2$ *and* $\alpha+\beta = 3$. *Consider initial data* $(a_1, a_2) \in \mathcal{B}\times\mathcal{B}$ *and* $a_i \geqslant 0,\ a_i \not\equiv 0,\ i = 1, 2$, *where* $q > 1$ *is fixed. Let* $A = \int_{\mathbb{R}} a_1(x) + a_2(x)\,\mathrm{d}x > 0$ *be the total mass, which is conserved in time. Then the system (1)–(2) has a unique global classical solution* $(u_1(x, t), u_2(x, t)) \in \mathcal{B} \times \mathcal{B}$ *for* $\forall t \geqslant 0$. *Furthermore, there exists* $B = B(A, d, \alpha, \beta) > 0$ *such that*

$$\|t^{1/2}(\log t)^{1/(\alpha-1)}u_1(\sqrt{t}\cdot, t) - B\phi_1(\cdot)\| \to 0, \tag{10}$$

$$\|t^{1/2}u_2(\sqrt{t}\cdot, t) - A\phi_d(\cdot)\| \to 0 \tag{11}$$

as $t \to \infty$.

Remark. Unlike the case of $\alpha = 1$ and $\beta = 2$, where the value of B is not explicit, we know the precise value of B. The exact expression of the constant B is as follows:

$$B = \left(\frac{4\pi d^{\beta/2}(\alpha + \beta/d)^{1/2}}{(\alpha - 1)A^{\beta}}\right)^{1/(\alpha-1)}. \tag{12}$$

Remark. The asymptotic of the solution is in a certain sense universal since the large time spatial-temporal profile is independent of the decay rate at $|x| = \infty$ and the size of the initial data except for its apparent dependence on the total mass. In addition, the constant B depends explicitly on the known parameters A, d, α and β.

Remark. The $\log t$ term in (10) is not out of a simple scaling argument, rather it is the by-product of the critical cubic nonlinearity and competition between diffusion and mass transfer. The power of $\log t$ is an anomalous exponent.

Remark. In order to understand intuitively why the asymptotic stated in theorem 1 holds, let us consider a more general case of $\alpha, \beta \geqslant 1$. If $\alpha + \beta > 3$, it can be shown as in [3] that both u_1 and u_2 go to zero like the solution of pure diffusion as $t \to \infty$. For $\alpha + \beta < 3$, one can apply the maximum principle and a simple *a priori* estimate to bound u_1 by $\bar{u}$, which is a solution of

$$\bar{u}_t = \bar{u}_{xx} - O\left(t^{-(\alpha+\beta-1)/2}\phi_d^{\beta}\left(\frac{x}{\sqrt{t}}\right)\right)\bar{u}.$$

Then, by applying the Feynman–Kac formula, one obtains

$$\bar{u} \leqslant \exp(-O(t^{(3-\alpha-\beta)/2})) \tag{13}$$

for $|x| \leqslant O(\sqrt{t})$. For $|x| \gg O(\sqrt{t})$, one gets diffusive behaviour, depending on the rate of decay, as $|x| \to \infty$ of the initial data. Next, by inserting the fast decay (13) for u_1 in (2), one shows the nonlinear term is insignificant and that u_2 diffuses to zero. Clearly, the critical case of $\alpha + \beta = 3$ is the most delicate and interesting one. In particular, instead of (13), one gets $\exp(-O(\log\log t))$, which, after detailed analysis, gives rise to (10).

Remark. In comparison of our result with that of Bricmont *et al* [7], it can be shown that

(i) For our case, only the minimum requirement on the decay of initial data as $|x| \to \infty$ is needed, the decay rate is independent of the total mass A, unlike the case in [7] where the decay rate is related to A. We believe our result is optimal. But our method fails for their case.
(ii) The appearance of $\log t$ indicates the analysis is more involved and subtle. In particular, it is well known in the scientific computation field that a scaling of $\log t$ is hardly detectable in computation.

(iii) The nonlinear dependence of u_1, in addition to the cubic nonlinearity, is solely responsible for the $\log t$ term arising, the power of which tends to infinity as $\alpha \to 1$. This may explain the appearance of the extra decay power of t in the limiting case of $\alpha = 1$.

Remark. To model chemical fronts (flames) propagating down a tube, one has to feed the system at one end of the tube or in an idealized situation at spatial infinity by specifying a non-zero value of u_1 or u_2. But when the feeding is turned off at a later stage of the reaction process and leaves the system to relax freely by itself, the decaying initial condition emerges. Therefore, the dynamics of the present case is vastly different from the case where travelling front solutions develop. In that case, the front solutions in general undergo transitions to chaos as the Lewis number d is sufficiently large or the order of autocatalytic reactions is high enough (see Sivashinsky [18] and Metcalf *et al* [15]). For a more detailed description, see [7]. But for the critical nonlinearity case studied in this work, we know the exact global dynamics of the problem, a rare treat in nonlinear problems. Our result is in spirit similar to the decay of turbulence results in fluid dynamics for the incompressible Navies–Stokes equations [11, 18]. However, for such problems, only decay rates of solutions in proper Sobolev norms are known.

The plan of the paper is as follows. In section 2, we derive *a priori* estimates on the solutions of the system (1)–(2) which are similar to the case of $\alpha = 1$, $\beta = 2$ as in [7] and that of [13] for a bounded domain. But the extra freedom of choice for α, β means the estimates are a bit more involved. These estimates directly imply the solution is a global, classical solution. In section 3, we derive key decay estimates using the maximum principle and a careful construction of a super-solution. In section 4, we use the RG method to find the exact large time dynamics of the solution. In particular, we give a rigorous proof that the RG map converges as is claimed.

Throughout the paper, $\|\cdot\|_p$ stands for the $L^p(\mathbb{R})$-norm of a function and $\|\cdot\|$ the norm defined in (9). Also, for simplicity of notation, we shall not distinguish generic constant C from line to line.

2. *A priori* estimates

In this section, we show that there is a uniform bound in time of the $L^p(\mathbb{R})$-norm, $1 \leqslant p \leqslant \infty$, for both u_1 and u_2. Then, it follows from the classical theory that the solutions (u_1, u_2) are smooth and exist globally in time.

First, we have the simple lemma as follows.

Lemma 1. *The solutions (u_1, u_2) satisfy the following L^1 estimates:*

$$\left(\int_{\mathbb{R}} (u_1 + u_2)\,\mathrm{d}x\right)(t) = \int_{\mathbb{R}} (a_1 + a_2)\,\mathrm{d}x, \qquad \left(\int_{\mathbb{R}} u_1\,\mathrm{d}x\right)(t) \leqslant \int_{\mathbb{R}} a_1\,\mathrm{d}x \qquad \textit{for } t \geqslant 0,$$
$$\left(\int_{\mathbb{R}} u_2\,\mathrm{d}x\right)(t) \geqslant \int_{\mathbb{R}} a_2\,\mathrm{d}x \qquad \textit{for } t \geqslant 0, \qquad \int_0^\infty \int_{\mathbb{R}} u_1^\alpha u_2^\beta\,\mathrm{d}x\,\mathrm{d}t < +\infty. \tag{14}$$

Proof. By integrating (1)–(2) over $\mathbb{R} \times [0, t]$, we get

$$\int_{\mathbb{R}} u_1(x,t)\,\mathrm{d}x = \int_{\mathbb{R}} a_1(x)\,\mathrm{d}x - \int_0^t \int_{\mathbb{R}} u_1^\alpha u_2^\beta\,\mathrm{d}x\,\mathrm{d}\tau, \tag{15}$$

$$\int_{\mathbb{R}} u_2(x,t)\,\mathrm{d}x = \int_{\mathbb{R}} a_2(x)\,\mathrm{d}x + \int_0^t \int_{\mathbb{R}} u_1^\alpha u_2^\beta\,\mathrm{d}x\,\mathrm{d}\tau. \tag{16}$$

Together, (15) and (16) give the result of lemma 1. QED

Next, using the maximum principle directly, we have the following result.

Lemma 2.

$$0 < u_1(x,t) \leqslant \|a_1\|_\infty, \qquad \forall t > 0,$$

$$0 < \underline{u}_2(x,t) \leqslant u_2(x,t), \qquad \forall t > 0,$$

where $\underline{u}_2$ *is a solution of*

$$\underline{u}_{2,t} = d\underline{u}_{2,xx}, \qquad \underline{u}_2|_{t=0} = a_2(x),$$

$$u_1(x,t) \leqslant \bar{u}_1(x,t), \qquad \forall t > 0,$$

where $\bar{u}_1$ *is a solution of*

$$\bar{u}_{1,t} = \bar{u}_{1,xx} - \bar{u}_1^\alpha \underline{u}_2^\beta, \qquad \bar{u}_1|_{t=0} = a_1(x).$$

The key estimates of this section are stated in the following lemma.

Lemma 3. *The solutions* (u_1, u_2) *of (1)–(2) are uniformly bounded in time in* $L^p(\mathbb{R})$ *norm:*

$$\|u_1(\cdot,t)\|_{L^p} + \|u_2(\cdot,t)\|_{L^p} \leqslant C(a_1,a_2,p) < \infty, \qquad 1 \leqslant p < \infty, \tag{17}$$

where $c(a_1, a_2, p)$ *is a constant depending on the initial data and* p *is a positive integer.*

Proof. It is clear by lemmas 1 and 2 that we only need to prove the bound for u_2. By standard local existence theory, all $L^p(\mathbb{R})$-norm of (u_1, u_2) and the $L^2(\mathbb{R})$-norm of $(u_{1,x}, u_{2,x})$ are finite and continuous in time. Therefore, we can freely integrate by parts in what follows. By multiplying (2) by pu_2^{p-1} with $p \geqslant 2$, integrating over $\mathbb{R} \times R^+$, we get

$$\int_{\mathbb{R}} u_2^p \,\mathrm{d}x = \int_{\mathbb{R}} a_2^p \,\mathrm{d}x - d\int_0^t \int_{\mathbb{R}} p(p-1)u_{2,x}^2 u_2^{p-2} \,\mathrm{d}x\,\mathrm{d}\tau + \int_0^t \int_{\mathbb{R}} p u_1^\alpha u_2^{\beta+p-1} \,\mathrm{d}x\,\mathrm{d}\tau. \tag{18}$$

In addition, with the help of integration by parts, we have the identity

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}} (u_1^2 + u_1)u_2^p \,\mathrm{d}x \\
&\qquad = \int_{\mathbb{R}} (1+2u_1)(u_{1,xx} - u_1^\alpha u_2^\beta)u_2^p \,\mathrm{d}x + \int_{\mathbb{R}} (u_1^2+u_1)pu_2^{p-1}(du_{2,xx} + u_1^\alpha u_2^\beta)\,\mathrm{d}x \\
&\qquad = -2\int_{\mathbb{R}} u_{1,x}^2 u_2^p \,\mathrm{d}x - \int_{\mathbb{R}} (1+2u_1)u_{1,x}u_{2,x}pu_2^{p-1} - \int_{\mathbb{R}} (1+2u_1)u_1^\alpha u_2^\beta u_2^p \,\mathrm{d}x \\
&\qquad\quad -d\int_{\mathbb{R}} (1+2u_1)u_{1,x}u_{2,x}pu_2^{p-1} - d\int_{\mathbb{R}} (u_1^2+u_1)p(p-1)u_2^{p-2}u_{2,x}^2 \,\mathrm{d}x \\
&\qquad\quad + \int_{\mathbb{R}} (u_1^2+u_1)pu_2^{p-1}u_1^\alpha u_2^\beta \,\mathrm{d}x \\
&\qquad = I + II + III + IV + V + VI.
\end{aligned} \tag{19}$$

It is clear that

$$\begin{aligned}
I + II + IV &\leqslant (1 + 2\|a_1\|_\infty)(1+d)\int_{\mathbb{R}} |u_{1,x}u_{2,x}|pu_2^{p-1} - 2\int_{\mathbb{R}} u_{1,x}^2 u_2^p \,\mathrm{d}x \\
&\leqslant \int_{\mathbb{R}} u_{1,x}^2 u_2^p \,\mathrm{d}x + C(\|a_1\|_\infty, p)\int_{\mathbb{R}} u_{2,x}^2 u_2^{p-2} \,\mathrm{d}x - 2\int_{\mathbb{R}} u_{1,x}^2 u_2^p \,\mathrm{d}x \\
&\leqslant C(\|a_1\|_\infty, p)\int_{\mathbb{R}} u_{2,x}^2 u_2^{p-2} \,\mathrm{d}x - \int_{\mathbb{R}} u_{1,x}^2 u_2^p \,\mathrm{d}x.
\end{aligned}$$

Moreover,

$$III \leqslant -\int_{\mathbb{R}} u_1^{\alpha} u_2^{\beta} u_2^{p}\,\mathrm{d}x, \qquad V \leqslant 0$$

and

$$VI \leqslant (\|a_1\|_{\infty} + \|a_1\|_{\infty}^2) \int_{\mathbb{R}} p u_1^{\alpha} u_2^{\beta} u_2^{p-1}\,\mathrm{d}x.$$

Integrating (19) from 0 to t yields

$$\begin{aligned}\left(\int_{\mathbb{R}} (u_1^2 + u_1) u_2^p\,\mathrm{d}x\right)(t) &\leqslant \int_{\mathbb{R}} (a_1^2 + a_1) a_2^p\,\mathrm{d}x + C(p, a_1) \int_0^t \int_{\mathbb{R}} u_{2,x}^2 u_2^{p-2}\,\mathrm{d}x\,\mathrm{d}\tau \\ &\quad + (\|a_1\|_{\infty} + \|a_1\|_{\infty}^2) \int_0^t \int_{\mathbb{R}} u_1^{\alpha} u_2^{\beta} p u_2^{p-1}\,\mathrm{d}x\,\mathrm{d}\tau \\ &\quad - \int_0^t \int_{\mathbb{R}} u_1^{\alpha} u_2^{\beta} u_2^{p}\,\mathrm{d}x\,\mathrm{d}\tau - \int_0^t \int_{\mathbb{R}} u_{1,x}^2 u_2^p\,\mathrm{d}x\,\mathrm{d}\tau. \end{aligned} \tag{20}$$

The combination of (18) and (20) then gives

$$\begin{aligned}\int_{\mathbb{R}} u_2^p + \int_0^t \int_{\mathbb{R}} (u_{2,x}^2 u_2^{p-2} + u_{1,x}^2 u_2^p + u_1^{\alpha} u_2^{\beta+p})\,\mathrm{d}x\,\mathrm{d}\tau \\ \leqslant C(a_1, a_2, p) \left(1 + \int_0^t \int_{\mathbb{R}} u_1^{\alpha} u_2^{\beta+p-1}\,\mathrm{d}x\,\mathrm{d}\tau\right). \end{aligned} \tag{21}$$

For $p = 1$, we will proceed as in lemma 2.3 of [7]. First, letting $p = 1$ in (19) we obtain

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathbb{R}} (u_1^2 + u_1) u_2\,\mathrm{d}x &= -2 \int_{\mathbb{R}} u_{1,x}^2 u_2\,\mathrm{d}x - \int_{\mathbb{R}} (1 + 2u_1) u_{1,x} u_{2,x} - \int_{\mathbb{R}} (1 + 2u_1) u_1^{\alpha} u_2^{\beta+1}\,\mathrm{d}x \\ &\quad - d \int_{\mathbb{R}} (1 + 2u_1) u_{1,x} u_{2,x}\,\mathrm{d}x + \int_{\mathbb{R}} (u_1^2 + u_1) u_1^{\alpha} u_2^{\beta}\,\mathrm{d}x \\ &\leqslant -2 \int_{\mathbb{R}} u_{1,x}^2 u_2\,\mathrm{d}x + C(a_1, a_2) \int_{\mathbb{R}} (|u_{1,x} u_{2,x}| + u_1^{\alpha} u_2^{\beta}) - \int_{\mathbb{R}} u_1^{\alpha} u_2^{\beta+1}\,\mathrm{d}x \\ &\leqslant -2 \int_{\mathbb{R}} u_{1,x}^2 u_2\,\mathrm{d}x + C(a_1, a_2) \int_{\mathbb{R}} (\epsilon^{-1} |u_{1,x}|^2 + \epsilon |u_{2,x}|^2 + u_1^{\alpha} u_2^{\beta}) \\ &\quad - \int_{\mathbb{R}} u_1^{\alpha} u_2^{\beta+1}\,\mathrm{d}x \end{aligned} \tag{22}$$

for any $\epsilon > 0$. Now integrating the above inequality (22) from 0 to t gives

$$\begin{aligned}\int_{\mathbb{R}} (u_1^2 + u_1) u_2\,\mathrm{d}x + \int_0^t \int_{\mathbb{R}} (2u_{1,x}^2 u_2 + u_1^{\alpha} u_2^{\beta+1})\,\mathrm{d}x\,\mathrm{d}\tau \\ \leqslant C(a_1, a_2) \left(1 + \int_0^t \int_{\mathbb{R}} (\epsilon^{-1} |u_{1,x}|^2 + \epsilon |u_{2,x}|^2 + u_1^{\alpha} u_2^{\beta})\right)\,\mathrm{d}x\,\mathrm{d}\tau. \end{aligned} \tag{23}$$

With the help of (18) when $p = 2$ we obtain

$$2d \int_0^t \int_{\mathbb{R}} u_{2,x}^2\,\mathrm{d}x\,\mathrm{d}\tau \leqslant \int_{\mathbb{R}} a_2^2\,\mathrm{d}x + 2 \int_0^t \int_{\mathbb{R}} u_1^{\alpha} u_2^{\beta+1}\,\mathrm{d}x\,\mathrm{d}\tau. \tag{24}$$

Similarly by multiplying (1) by $2u_1$, integrating over $\mathbb{R} \times R^+$, we get

$$\int_{\mathbb{R}} u_1^2\,\mathrm{d}x = \int_{\mathbb{R}} a_1^2\,\mathrm{d}x - 2 \int_0^t \int_{\mathbb{R}} u_{1,x}^2\,\mathrm{d}x\,\mathrm{d}\tau - 2 \int_0^t \int_{\mathbb{R}} u_1^{\alpha+1} u_2^{\beta}\,\mathrm{d}x\,\mathrm{d}\tau, \tag{25}$$

which implies that

$$2 \int_0^t \int_{\mathbb{R}} u_{1,x}^2\,\mathrm{d}x\,\mathrm{d}\tau \leqslant \int_{\mathbb{R}} a_1^2\,\mathrm{d}x. \tag{26}$$

Combining (23)–(25) with sufficiently small ϵ gives

$$\int_0^t \int_{\mathbb{R}} (u_1^\alpha u_2^{\beta+1} + u_{1,x}^2 u_2)\,\mathrm{d}x\,\mathrm{d}\tau \leqslant C(a_1, a_2)\left(1 + \int_0^t \int_{\mathbb{R}} u_1^\alpha u_2^\beta\,\mathrm{d}x\,\mathrm{d}\tau\right). \qquad (27)$$

Clearly, a simple induction on p with the help of (18), (21), (27) and lemma 1 give the desired uniform bound for u_2. This completes the proof of the lemma. QED

Lemma 4. *The following estimates for the derivative of* (u_1, u_2) *of (1)–(2) hold if* $t \geqslant t_0 > 0$*:*

$$\|u_{1,x}(\cdot, t)\|_2 + \|u_{2,x}(\cdot, t)\|_2 \leqslant C(a_1, a_2) < \infty, \qquad (28)$$

where $t_0 > 0$ *is arbitrary.*

Proof. Once we have the L^p estimates for u_1, u_2 the derivatives' estimates follow by standard techniques. For the sake of completeness we state the details here. First, multiplying (1) by u_1 and integrating on $\mathbb{R}$ imply

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}} u_1^2\,\mathrm{d}x + \int_{\mathbb{R}} u_{1,x}^2\,\mathrm{d}x \leqslant 0.$$

Integrating the above inequality over $[0, t]$ gives

$$\int_{\mathbb{R}} u_1^2\,\mathrm{d}x + \int_0^t \int_{\mathbb{R}} u_{1,x}^2\,\mathrm{d}x\,\mathrm{d}\tau \leqslant \int_{\mathbb{R}} a_1^2\,\mathrm{d}x.$$

Similarly, we have

$$\int_{\mathbb{R}} u_2^2\,\mathrm{d}x + \int_0^t \int_{\mathbb{R}} u_{2,x}^2\,\mathrm{d}x\,\mathrm{d}\tau \leqslant \int_0^t \int_{\mathbb{R}} u_1^\alpha u_2^{\beta+1}\,\mathrm{d}x\,\mathrm{d}\tau + \int_{\mathbb{R}} a_2^2\,\mathrm{d}x \leqslant C(a_1, a_2)$$

by lemma 3. By Fubini's theorem, there exists $0 \leqslant t_1 \leqslant t_0$ such that

$$\left(\int_{\mathbb{R}} (u_{1,x}^2 + u_{2,x}^2)\,\mathrm{d}x\right)(t_1) \leqslant \frac{C(a_1, a_2)}{t_0}. \qquad (29)$$

Next, multiplying (1) by $u_{1,xx}$ and integrating over $\mathbb{R}$ yields

$$\begin{aligned}
-\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u_{1,x}\|_2^2 &= \int_{\mathbb{R}} u_{1,xx}^2\,\mathrm{d}x - \int_{\mathbb{R}} u_1^\alpha u_2^\beta u_{1,xx}\,\mathrm{d}x \\
&= \int_{\mathbb{R}} u_{1,xx}^2\,\mathrm{d}x + \alpha \int_{\mathbb{R}} u_1^{\alpha-1} u_2^\beta u_{1,x}^2\,\mathrm{d}x + \beta \int_{\mathbb{R}} u_1^\alpha u_2^{\beta-1} u_{1,x} u_{2,x}\,\mathrm{d}x. \qquad (30)
\end{aligned}$$

Similarly, we have

$$-\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u_{2,x}\|_2^2 = d\int_{\mathbb{R}} u_{2,xx}^2\,\mathrm{d}x - \beta \int_{\mathbb{R}} u_1^\alpha u_2^{\beta-1} u_{2,x}^2\,\mathrm{d}x - \alpha \int_{\mathbb{R}} u_1^{\alpha-1} u_2^\beta u_{1,x} u_{2,x}\,\mathrm{d}x. \qquad (31)$$

Adding (30) and (31), and integrating from t_1 to t then gives

$$\begin{aligned}
(\|u_{1,x}\|_2^2 + \|u_{2,x}\|_2^2)(t) &\leqslant (\|u_{1,x}\|_2^2 + \|u_{2,x}\|_2^2)(t_1) \\
&\quad + \alpha \int_{t_1}^t \int_{\mathbb{R}} u_1^{\alpha-1} u_2^\beta u_{1,x}^2\,\mathrm{d}x\,\mathrm{d}\tau + \beta \int_{t_1}^t \int_{\mathbb{R}} u_1^\alpha u_2^{\beta-1} |u_{1,x} u_{2,x}|\,\mathrm{d}x\,\mathrm{d}\tau \\
&\quad + \beta \int_{t_1}^t \int_{\mathbb{R}} u_1^\alpha u_2^{\beta-1} u_{2,x}^2\,\mathrm{d}x\,\mathrm{d}\tau + \alpha \int_{t_1}^t \int_{\mathbb{R}} u_1^{\alpha-1} u_2^\beta |u_{1,x} u_{2,x}|\,\mathrm{d}x\,\mathrm{d}\tau. \qquad (32)
\end{aligned}$$

We now derive bounds for each of the last four terms on the right-hand side of (32).

$$\begin{aligned}
\int_{t_1}^t \int_{\mathbb{R}} u_1^{\alpha-1} u_2^\beta u_{1,x}^2\,\mathrm{d}x\,\mathrm{d}\tau &\leqslant \|a_1\|_\infty^{\alpha-1} \int_{t_1}^t \int_{\mathbb{R}} u_2^\beta u_{1,x}^2\,\mathrm{d}x\,\mathrm{d}\tau \\
&\leqslant \|a_1\|_\infty^{\alpha-1} \left(\int_{t_1}^t \int_{\mathbb{R}} u_2^2 u_{1,x}^2\,\mathrm{d}x\,\mathrm{d}\tau\right)^{\beta-1} \left(\int_{t_1}^t \int_{\mathbb{R}} u_2 u_{1,x}^2\,\mathrm{d}x\,\mathrm{d}\tau\right)^{2-\beta} \qquad (33)
\end{aligned}$$

by Hölder's inequality. Similarly,

$$\int_{t_1}^{t}\int_{\mathbb{R}} u_1^{\alpha}u_2^{\beta-1}|u_{1,x}u_{2,x}|\,\mathrm{d}x\,\mathrm{d}\tau \leqslant \|a_1\|_\infty^{\alpha}\left(\int_{t_1}^{t}\int_{\mathbb{R}} u_2^{2(\beta-1)}u_{1,x}^2\,\mathrm{d}x\,\mathrm{d}\tau\int_{t_1}^{t}\int_{\mathbb{R}} u_{2,x}^2\,\mathrm{d}x\,\mathrm{d}\tau\right)^{1/2}. \tag{34}$$

If $0 < 2(\beta-1) < 1$, then by Hölder's inequality we find

$$\int_{t_1}^{t}\int_{\mathbb{R}} u_2^{2(\beta-1)}u_{1,x}^2\,\mathrm{d}x\,\mathrm{d}\tau \leqslant \left(\int_{t_1}^{t}\int_{\mathbb{R}} u_2u_{1,x}^2\,\mathrm{d}x\,\mathrm{d}\tau\right)^{2(\beta-1)}\left(\int_{t_1}^{t}\int_{\mathbb{R}} u_{1,x}^2\,\mathrm{d}x\,\mathrm{d}\tau\right)^{3-2\beta}. \tag{35}$$

If $1 = 2(\beta-1)$, the right-hand side of (34) is already bounded by $C(a_1, a_2)$ by lemma 3.

If $1 < 2(\beta-1) < 2$, again by a simple application of Hölder's inequality, we get

$$\int_{t_1}^{t}\int_{\mathbb{R}} u_2^{2(\beta-1)}u_{1,x}^2\,\mathrm{d}x\,\mathrm{d}\tau \leqslant \left(\int_{t_1}^{t}\int_{\mathbb{R}} u_2^2u_{1,x}^2\,\mathrm{d}x\,\mathrm{d}\tau\right)^{1/p}\left(\int_{t_1}^{t}\int_{\mathbb{R}} u_2u_{1,x}^2\,\mathrm{d}x\,\mathrm{d}\tau\right)^{(p-1)/p}, \tag{36}$$

where $p = 1/(2(\beta-1)-1)$. Continuing the procedure,

$$\begin{aligned}\int_{t_1}^{t}\int_{\mathbb{R}} u_1^{\alpha}u_2^{\beta-1}u_{2,x}^2\,\mathrm{d}x\,\mathrm{d}\tau &\leqslant \|a_1\|_\infty^{\alpha}\int_{t_1}^{t}\int_{\mathbb{R}} u_2^{(\beta-1)}u_{2,x}^2\,\mathrm{d}x\,\mathrm{d}\tau\\ &\leqslant \|a_1\|_\infty^{\alpha}\left(\int_{t_1}^{t}\int_{\mathbb{R}} u_2u_{2,x}^2\,\mathrm{d}x\,\mathrm{d}\tau\right)^{\beta-1}\left(\int_{t_1}^{t}\int_{\mathbb{R}} u_{2,x}^2\,\mathrm{d}x\,\mathrm{d}\tau\right)^{2-\beta}.\end{aligned} \tag{37}$$

$$\begin{aligned}\int_{t_1}^{t}\int_{\mathbb{R}} u_1^{\alpha-1}u_2^{\beta}|u_{1,x}u_{2,x}|\,\mathrm{d}x\,\mathrm{d}\tau &\leqslant \|a_1\|_\infty^{\alpha-1}\left(\int_{t_1}^{t}\int_{\mathbb{R}} u_2^{\beta}u_{1,x}^2\,\mathrm{d}x\,\mathrm{d}\tau\int_{t_1}^{t}\int_{\mathbb{R}} u_2^{\beta}u_{2,x}^2\,\mathrm{d}x\,\mathrm{d}\tau\right)^{1/2}\\ &\leqslant \|a_1\|_\infty^{\alpha-1}\left(\int_{t_1}^{t}\int_{\mathbb{R}} u_2^2u_{2,x}^2\,\mathrm{d}x\,\mathrm{d}\tau\right)^{(\beta-1)/2}\left(\int_{t_1}^{t}\int_{\mathbb{R}} u_2u_{2,x}^2\,\mathrm{d}x\,\mathrm{d}\tau\right)^{(2-\beta)/2}\\ &\quad\times\left(\int_{t_1}^{t}\int_{\mathbb{R}} u_2^2u_{1,x}^2\,\mathrm{d}x\,\mathrm{d}\tau\right)^{1/2\sigma}\left(\int_{t_1}^{t}\int_{\mathbb{R}} u_2u_{1,x}^2\,\mathrm{d}x\,\mathrm{d}\tau\right)^{(\sigma-1)/2\sigma},\end{aligned} \tag{38}$$

where $\sigma = 1/(\beta-1)$. Combining (32)–(38), and (21) and (27) in lemma 3, we obtain

$$(\|u_{1,x}\|_2^2 + \|u_{2,x}\|_2^2)(t) \leqslant (\|u_{1,x}\|_2^2 + \|u_{2,x}\|_2^2)(t_1) + C(a_1, a_2).$$

Hence, by (29),

$$(\|u_{1,x}\|_2^2 + \|u_{2,x}\|_2^2)(t) \leqslant C(a_1, a_2)$$

for all $t \geqslant t_0$. QED

Proposition 1. *The system (1)–(2) has a unique classical solution satisfying, for all $t > 0$,*

$$\|u_1\|_{L^p} + \|u_2\|_{L^p} \leqslant C(a_1, a_2), \qquad 1 \leqslant p \leqslant \infty, \tag{39}$$

where the constant depends only on the initial data $(a_1, a_2) \in (L^1(\mathbb{R}) \cap L^\infty(\mathbb{R}))^2$.

Proof. The bound for u_1 follows directly from lemmas 1 and 2. For u_2, the classical theory of local existence yields the bound for small t, say $t \leqslant t_0$, where $t_0 > 0$. For $t > t_0$, we can use the Sobolev imbedding, (21) for $p = 2$ and lemma 4 to get

$$\|u_2\|_\infty \leqslant C(a_1, a_2).$$

This completes the proof of the proposition. QED

3. Sharp decay estimates

The purpose of this section is to prove sharp decay estimates for (u_1, u_2) in time. We assume from now on that the initial data $(a_1, a_2) \in \mathcal{B} \times \mathcal{B}$ with $q > 1$ being a fixed constant. To put the solutions in the right scale we let

$$\tilde{u}_1(x,t) = \sqrt{t}u_1(\sqrt{t}x, t), \qquad \tilde{u}_2(x,t) = \sqrt{t}u_2(\sqrt{t}x, t).$$

Proposition 2. *The solutions (u_1, u_2) of (1)–(2) satisfies the bounds*

$$\|\tilde{u}_1\|(t) \leqslant C(a_1, a_2)[\log(1+t)]^{-1/(\alpha-1)}, \tag{40}$$

$$\|\tilde{u}_2\|(t) \leqslant C(a_1, a_2), \tag{41}$$

where $\|\cdot\|$ stands for the norm in (9).

Remark. We note that as a direct consequence of (40) and (41) we have

$$\begin{aligned} \|u_1\|_\infty &\leqslant C(a_1, a_2)(1+t)^{-1/2}[\log(1+t)]^{-1/(\alpha-1)}, \\ \|u_2\|_\infty &\leqslant C(a_1, a_2)(1+t)^{-1/2}. \end{aligned} \tag{42}$$

We first prove a preliminary lemma for local time.

Lemma 5. *For any $a \in \mathcal{B}$ with $q > 1$, there exists a $C = (a, t, d) < \infty$ such that if u is a solution of*

$$u_t = du_{xx}, \qquad u|_{t=0} = a(x),$$

then for $t > 0$

$$\|u\|(t) \leqslant C(a, t, d).$$

Proof. Let

$$s = \log(t+1), \qquad y = \frac{x}{\sqrt{t+1}}, \qquad u(x,t) = \frac{v(y,s)}{\mathrm{e}^{s/2}}.$$

Then, it is easy to verify that v satisfy, for $s > 0$,

$$v_s = v_{yy} + \tfrac{1}{2}v + \tfrac{1}{2}yv_y.$$

Let $w(y,s) = B\mathrm{e}^{As}/\sqrt{1+y^2}^q$ to have that

$$\begin{aligned} w_s - w_{yy} - \tfrac{1}{2}w - \tfrac{1}{2}yw_y &= B\frac{((A - 1/2 + q/2)y^4 + (-1 - q/2 - q^2 + 2A)y^2 + A - 1/2 + q)}{(1+y^2)^{q/2+2}}\mathrm{e}^{As} \\ &\geqslant 0, \end{aligned}$$

if A is so chosen say $A = (1 + q/2 + q^2)/2$. Since $a \in \mathcal{B}$ with $q > 1$, if we let $B = \|a\|$ then we will have $w(y,0) \geqslant v(y,0)$ and hence $u(x,t) = v(y,s)/\mathrm{e}^{s/2} \leqslant w(y,s)/\mathrm{e}^{s/2}$, which proves the lemma.

Lemma 6. *The estimate (40) holds.*

Proof. Again from the scaling properties of the linear parabolic equation and the fact that our equations are critical nonlinear, we let

$$s = \log(t+T), \qquad y = \frac{x}{\sqrt{t+T}},$$

$$u_1(x,t) = \frac{v_1(y,s)}{\mathrm{e}^{s/2}s^{1/(\alpha-1)}}, \qquad u_2(x,t) = \frac{v_2(y,s)}{\mathrm{e}^{s/2}}$$

where $T > 1$ is fixed. Then, it is easy to verify that $v_1,\ v_2$ satisfy, for $s \geqslant s_0 = \log T > 0$,

$$v_{1,s} = v_{1,yy} + \frac{1}{2}v_1 + \frac{1}{2}yv_{1,y} + \frac{v_1}{(\alpha-1)s} - \frac{v_1^\alpha v_2^\beta}{s},$$
$$v_{2,s} = dv_{2,yy} + \frac{1}{2}v_2 + \frac{1}{2}yv_{2,y} + \frac{v_1^\alpha v_2^\beta}{s^{\alpha/\alpha-1}}.$$

According to lemma 5, we have $\|u_1\|(t) \leqslant C(a_1, t, 1)$. And by proposition 1 we have that both u_1 and u_2 are bounded. Therefore u_2 is a sub-solution of

$$u_t = du_{xx} + \|u_1\|_\infty^\alpha \|u_2\|_\infty^{\beta-1} u, \qquad u|_{t=0} = a_2(x),$$

which implies that $w = \mathrm{e}^{-\|u_1\|_\infty^\alpha \|u_2\|_\infty^{\beta-1} t} u$ is a solution of

$$w_t = dw_{xx}, \qquad w|_{t=0} = a_2(x).$$

Hence we would have by lemma 5 that $\|u_1\|(1) + \|u_2\|(1) \leqslant C(a_1, a_2)$ and clearly that $u_2(x, 1)$ is strictly positive. Therefore, without loss of generality we may assume that $a_2(x) > 0$ if necessary shifting t from 0 to 1.

Set

$$\mathcal{N}(v) = v_s - v_{yy} - \frac{1}{2}v - \frac{1}{2}yv_y - \frac{v}{(\alpha-1)s} + \frac{v^\alpha v_2^\beta}{s}.$$

We now try to construct a super-solution for $\mathcal{N}$ and apply the maximum principle for parabolic equations with Cauchy data. By the maximum principle if there exists a function v which has the property that $\mathcal{N}(v) \geqslant 0$ on $\mathbb{R} \times (s_0, \infty)$ and $v(y, s_0) \geqslant v_1(y, s_0)$ on $\mathbb{R}$, then $v(y, s) \geqslant v_1(y, s)$ on $\mathbb{R} \times (s_0, \infty)$. In other words, such a v is a super-solution of v_1. To construct such a super-solution, we let

$$v = A\mathrm{e}^{-y^2/4} + AB\frac{(1+y^2)^{-k/2}}{s},$$

where $k > 1$ is arbitrary and $A,\ B > 0$ are constants to be fixed later. Detailed calculations show that if $k > 1$,

$$\begin{aligned}
\mathcal{N}(v) &= -\frac{A\mathrm{e}^{-y^2/4}}{(\alpha-1)s} \\
&\quad + \frac{\frac{((k-1)/2)BAy^4 + (-k^2-k/2-1)BAy^2 + (k-1/2)BA}{s} + \frac{-\alpha BAy^4/(\alpha-1) - 2\alpha BAy^2/(\alpha-1) - \alpha BA/(\alpha-1)}{s^2}}{(1+y^2)^{(k/2)+2}} \\
&\quad + \frac{(A\mathrm{e}^{-y^2/4} + AB/(1+y^2)^{k/2}s)^\alpha v_2^\beta}{s} \\
&\geqslant \frac{A}{s}\left(-\frac{\mathrm{e}^{-y^2/4}}{\alpha-1} \right. \\
&\quad + B\frac{((k-1)/2)y^4 + (-k^2-k/2-1)y^2 + (k-1/2) + \frac{-\alpha y^4/(\alpha-1) - 2\alpha y^2/(\alpha-1) - \alpha/(\alpha-1)}{s}}{(1+y^2)^{(k/2)+2}} \\
&\quad \left. + A^{\alpha-1}\mathrm{e}^{-\alpha y^2/4} v_2^\beta \right) \\
&= \frac{A}{s}\left(I + \frac{B}{(1+y^2)^{(k/2)+2}} II + III \right).
\end{aligned} \tag{43}$$

Now for $s > s_1(k) = 4\alpha/(k-1)(\alpha-1) > 0$ since $k > 1$, we have

$$(k+1)(1+y^2)^2 \geqslant II \geqslant \frac{k-1}{4}y^4 + c_1(k)y^2 + c_2(k),$$

where $c_1(k), c_2(k)$ are bounded constants. Therefore, there exists some $y_1(k) > 0$ such that

$$II \geqslant \frac{k-1}{8}(1+y^2)^2, \qquad |y| \geqslant y_1(k). \tag{44}$$

Hence for $|y| \geqslant y_1(k), s > s_1(k)$ we have from (43) and (44) that

$$\begin{aligned} \mathcal{N}(v) &\geqslant \frac{A}{s}\left(-\frac{e^{-y^2/4}}{\alpha-1} + \frac{B(k-1)(1+y^2)^2}{8(1+y^2)^{(k/2)+2}}\right) \\ &\geqslant 0, \end{aligned} \tag{45}$$

provided that we choose $B = B_1(k)$ large enough. For example, let

$$B_1(k) = \sup_{|y|\geqslant y_1(k)} \frac{8e^{-y^2/4}(1+y^2)^{k/2}}{(\alpha-1)(k-1)}.$$

Now with $B = B_1(k)$ fixed and $|y| < y_1(k), s > s_1(k) = \log T$ we have from $\underline{u}_2(x,t) \leqslant u_2(x,t)$ $\forall t > 0$ in lemma 2 that

$$\begin{aligned} v_2(y,s) &= e^{s/2}u_2(x,t) \geqslant e^{s/2}\underline{u}_2(x,t), \\ &= e^{s/2}\int_{\mathbb{R}} \frac{\phi((x-z)/\sqrt{t})a_2(z)}{\sqrt{t}\,dz}, \\ &= \frac{e^{s/2}}{\sqrt{4\pi t}}\int_{\mathbb{R}} e^{-(x-z)^2/4t}a_2(z)\,dz, \\ &= \frac{\sqrt{t+T}}{\sqrt{4\pi t}}\int_{\mathbb{R}} e^{-(\sqrt{1+(T/t)}y-\sqrt{(1/t)}z)^2/4}a_2(z)\,dz, \\ &\geqslant l_2(k,a_2), \end{aligned} \tag{46}$$

for $a_2 > 0$ and $\lim_{t\to\infty}\int_{\mathbb{R}} e^{-(\sqrt{1+(T/t)}y-\sqrt{(1/t)}z)^2/4}a_2(z)\,dz = \int_{\mathbb{R}} e^{-y^2/4}a_2(z)\,dz > 0$ for $|y| < y_1(k)$.

Hence for $|y| < y_1(k), s > s_1(k) = \log T$ we have from (44) and (46) that

$$\begin{aligned} \mathcal{N}(v) &\geqslant \frac{A}{s}\left(-\frac{e^{-y^2/4}}{\alpha-1} - \frac{B(k+1)(1+y^2)^2}{(1+y^2)^{(k/2)+2}} + A^{\alpha-1}e^{-\alpha y^2/4}l_2(k,a_2)^\beta\right), \\ &\geqslant 0, \end{aligned}$$

provided that A is chosen sufficiently large.

It follows from our assumption on initial data that we can let $k = q$ and the above construction that the estimate (40) holds by the choices of A, B, y_1 and s_1. QED

To prove proposition 2, we only need to show the following lemma.

Lemma 7. *There exists $C = C(a_1, a_2)$ such that*

$$\|u_2\|_\infty \leqslant C(a_1,a_2)(1+t)^{-1/2}.$$

Proof. Consider the equation for u_2. Write the equation in integral form:

$$\begin{aligned} u_2(x,t) = {} & \frac{1}{\sqrt{4\pi\,dt}}\int_{\mathbb{R}} e^{-y^2/(4\,dt)}a_2(x-y)\,dy \\ & + \int_0^t\int_{\mathbb{R}} \frac{1}{\sqrt{4\pi\,ds}}e^{-y^2/(4\,ds)}(u_1^\alpha u_2^\beta)(x-y,t-s)\,dy\,ds. \end{aligned}$$

Taking the L^∞-norm then yields

$$\|u_2\|_\infty \leqslant \frac{c\|a_2\|}{(2+t)^{1/2}} + C\int_0^t s^{-1/2}\|u_1^\alpha u_2^\beta\|_1(t-s)\,ds. \tag{47}$$

Now,

$$\|u_1^\alpha u_2^\beta\|_1 \leqslant \|u_1\|_\infty^\alpha \|u_2^\beta\|_1 \leqslant C(2+t)^{-\alpha/2}\|u_2^{\beta-1}\|_\infty \|u_2\|_1 \leqslant C(2+t)^{-\alpha/2} \tag{48}$$

and

$$\int_0^t s^{-1/2}(2+t-s)^{-\alpha/2}\,\mathrm{d}s \leqslant C(2+t)^{-\min(1,(\alpha-1))/2} = C(2+t)^{-(\alpha-1)/2}.$$

Thus,

$$\|u_2\|_\infty \leqslant C(2+t)^{-(\alpha-1)/2}. \tag{49}$$

We can use (49) to improve (48) into

$$\|u_1^\alpha u_2^\beta\|_1 \leqslant C(2+t)^{-(2\alpha-1)/2},$$

which, when inserted into (47), gives

$$\|u_2\|_\infty \leqslant C(2+t)^{-(\alpha-1)}. \tag{50}$$

We have now from $\|u_2\|_\infty < C(a_1, a_2)$ and (47) the improved estimate (49) and (50). And a finite number of iterations of this process gives the desired estimate since each time the nonlinear part would improve its decay rate by $(\alpha-1)/2$ until it reaches $\frac{1}{2}$. This completes the proof of the lemma. QED

Proof of proposition 2. By the bound of u_1 and lemma 7, $u_2 \leqslant \bar{u}_2$, where $\bar{u}_2$ is the solution of the linear heat equation

$$\bar{u}_{2,t} = d\bar{u}_{2,xx} + C(2+t)^{-1}\log^{-\alpha/(\alpha-1)}(2+t)\bar{u}_2,$$
$$\bar{u}_2|_{t=0} = a_2.$$

If we let $w(x,t) = \int_0^t \mathrm{e}^{-C(2+\tau)^{-1}\log^{-\alpha/(\alpha-1)}(2+\tau)\,\mathrm{d}\tau} u(x,t)$ then w would be the solution of

$$w_t = dw_{xx},$$
$$w|_{t=0} = a_2.$$

The bound in (41) then follows from the classical theory since the integrating factor is uniformly bounded from below and above. This ends the proof of proposition 2. QED

4. The RG method and asymptotics

In this section, we prove theorem 1. The method we use is the RG method. We refer the reader to the pioneering work of Bricmont *et al* [6] and the references therein for a thorough and stimulating discussion of its application to a wide range of PDE problems. Our approach here is close in spirit to the one used in [7].

Define for $n \in N$ and for $L > 1$ large

$$u_i^n = L^n u_i(L^n x, L^{2n} t), \qquad t \in [1, L^2], \quad i = 1, 2. \tag{51}$$

(u_1^n, u_2^n) satisfy the system (1)–(2) with initial data (when $t = 1$):

$$a_i^n = L^n u_i(L^n x, L^{2n}), \qquad i = 1, 2. \tag{52}$$

The decay estimates we get in section 3 imply

$$\|a_1^n\| \leqslant C(L)n^{-1/(\alpha-1)} \qquad \text{and} \qquad \|a_1^n\|^\alpha \|a_2^n\|^\beta \leqslant C(L)n^{-\alpha/(\alpha-1)}. \tag{53}$$

We shall consider the RG map $(a_1, a_2) \to (a_1^1, a_2^1)$ defined in $\mathcal{B} \times \mathcal{B}$ with

$$a_i^1 = Lu_i(Lx, L^2), \qquad i = 1, 2 \quad \text{inductively}, \tag{54}$$

where (u_1, u_2) solve (1)–(2) with initial data (a_1, a_2). That is,

$$a_1^1 = L[e^{(L^2-1)\Delta} a_1(Lx)] - Ln_1(Lx, L^2), \tag{55}$$

$$a_1^2 = L[e^{d(L^2-1)\Delta} a_2(Lx)] + Ln_2(Lx, L^2), \tag{56}$$

where Δ represents the differential operator d^2/dx^2 and

$$n_1(x,t) = \int_1^t \int_{\mathbb{R}} G(x-y, t-s) u_1^\alpha(y,s) u_2^\beta(y,s)\, dy\, ds, \tag{57}$$

$$n_2(x,t) = \int_1^t \int_{\mathbb{R}} G_d(x-y, t-s) u_1^\alpha(y,s) u_2^\beta(y,s)\, dy\, ds \tag{58}$$

with G and G_d being corresponding Gaussians of $u_t = u_{xx}$ and $u_t = du_{xx}$, respectively.

If we denote $\mathcal{B}^+ = \{f \in \mathcal{B} \mid f \geqslant 0\}$. The map S_L from $\mathcal{B}^+ \to \mathcal{B}^+$ defined as

$$(S_L f)(x) = Lf(Lx)$$

is a bounded operator. In fact, we have the following result.

Lemma 8. *Suppose* $L \geqslant 1$. *Then,*

(a) $\|S_L\| \leqslant L$,
(b) $\|e^{\mu(t-s)\Delta}\| \leqslant e^{c(t-s)}$, *for* $\mu = 1,\ d, c < \infty$.

Proof. Since the above statements can be easily verified by the direct calculations, we omit the proof here. For each step, we write

$$a_1(x) = A_1\phi(x) + b_1,$$
$$a_2(x) = A_2\phi_d(x) + b_2.$$

where $A_i = \int_{\mathbb{R}} a_i(x)\, dx,\ i = 1, 2$ and $\int_{\mathbb{R}} b_i(x)\, dx = 0,\ i = 1, 2, \phi(x) = \phi_1(x)$.

In order to estimate the change of coefficients A_i and b_i, we make some preliminary analyses. First, we look at the equation for u_2,

$$u_2(x,t) = e^{d(t-1)\Delta} a_2 + n_2(x,t) = u_{20}(x,t) + n_2(x,t). \tag{59}$$

Consider the ball

$$B_R = \{u_2 : |||u_2||| \equiv \sup_{t\in[1,L^2]} \|u_2(\cdot,t)\| \leqslant R\|a_2\|\}.$$

Suppose $u_2 \in B_R$. First, we have

$$\|u_{10}(\cdot,s)\| \leqslant \|e^{(t-1)\Delta} a_1\| \leqslant C(L)\|a_1\|$$

and

$$\|u_{20}(\cdot,s)\| \leqslant \|e^{(t-1)\Delta} a_2\| \leqslant C(L)\|a_2\|$$

by lemma 8. Next, by lemma 2,

$$\|u_1(\cdot,s)\|_\infty \leqslant \|a_1\|_\infty \leqslant C\|a_1\|$$

and lemma 8(b), we find

$$|||n_1||| \leqslant \sup_{t\in[1,L^2]} \int_1^t ds \int_{\mathbb{R}} G(x-y,t-s)(c\|a_1\|)^\alpha (R\|a_2\|)^\beta\, dy$$
$$\leqslant C(L)R^\beta \|a_1\|^\alpha \|a_2\|^\beta \leqslant C(L,R)\epsilon\|a_2\|,$$

where $\epsilon = \|a_1\|^\alpha \|a_2\|^{\beta-1}$. Similarly,

$$|||n_2||| \leqslant C(L,R))\epsilon\|a_2\|.$$

In consequence,

$$\|Ln_i(L\cdot, L^2)\| = \|S_L n_i(\cdot, L^2)\| \leqslant C(L,R)\epsilon\|a_i\|, \qquad i = 1, 2.$$

Therefore, the right-hand side of (59) defines a map from B_R to B_R if $R > 0$ is sufficiently large and ϵ sufficiently small. A closer look reveals it is a contraction map. Hence, there is a unique fixed point in B_R.

For the RG map, we have the relations

$$\begin{aligned} A_1^1 &= A_1 - \int_{\mathbb{R}} Ln_1(Lx, L^2)\,\mathrm{d}x, \qquad A_2^1 = A_2 + \int_{\mathbb{R}} Ln_2(Lx, L^2)\,\mathrm{d}x, \\ b_1^1 &= \mathrm{e}^{(L^2-1)\Delta} b_1 - Ln_1(Lx, L^2) + (A_1 - A_1^1)\phi, \\ b_2^1 &= \mathrm{e}^{d(L^2-1)\Delta} b_2 + Ln_2(Lx, L^2) + (A_2 - A_2^1)\phi_d, \end{aligned} \tag{60}$$

where the major terms in n_1, n_2 are given by

$$\begin{aligned} w(y,s) &= u_1^\alpha(y,s)u_2^\beta(y,s) - (A_1\phi(y))^\alpha(A_2\phi_d(y))^\beta \\ &= (u_{10} - n_1)^\alpha(u_{20} + n_2)^\beta - (A_1\phi)^\alpha(A_2\phi_d)^\beta \\ &= (u_{10} - n_1)^\alpha(u_{20} + n_2)^\beta - (u_{10} - n_1)^\alpha(A_2\phi_d)^\beta + (u_{10} - n_1)^\alpha(A_2\phi_d)^\beta \\ &\quad - (A_1\phi)^\alpha(A_2\phi_d)^\beta \\ &= (u_{10} - n_1)^\alpha[(A_2\phi_d + \mathrm{e}^{d(s-1)\Delta}b_2 + n_2)^\beta - (A_2\phi_d)^\beta] \\ &\quad + (A_2\phi_d)^\beta[(A_1\phi + \mathrm{e}^{(s-1)\Delta}b_1 - n_1)^\alpha - (A_1\phi)^\alpha] \\ &= (u_{10} - n_1)^\alpha\beta(A_2\phi_d + \theta_1(\mathrm{e}^{d(s-1)\Delta}b_2 + n_2))^{\beta-1}(\mathrm{e}^{d(s-1)\Delta}b_2 + n_2) \\ &\quad + (A_2\phi_d)^\beta\alpha(A_1\phi + \theta_2(\mathrm{e}^{(s-1)\Delta}b_1 - n_1))^{\alpha-1}(\mathrm{e}^{(s-1)\Delta}b_1 - n_1), \end{aligned}$$

where $0 \leqslant \theta_i \leqslant 1$, $i = 1, 2$, such that by the help of estimates on n_1, n_2 above

$$\begin{aligned} \|w(.,s)\| &\leqslant \beta\|u_1\|_\infty^\alpha\|(A_2\phi_d + \theta_1(\mathrm{e}^{d(s-1)\Delta}b_2 + n_2))^{\beta-1}\|_\infty\|(\mathrm{e}^{d(s-1)\Delta}b_2 + n_2)\| \\ &\quad + \alpha\|(A_2\phi_d)\|^\beta\|(A_1\phi + \theta_2(\mathrm{e}^{(s-1)\Delta}b_1 - n_1))^{\alpha-1}\|_\infty\|(\mathrm{e}^{(s-1)\Delta}b_1 - n_1)\| \\ &\leqslant C\|a_1\|^\alpha(A_2 + \|b_2\| + C(L,R)\epsilon\|a_2\|)^{\beta-1}\cdot(\|b_2\| + C(L,R)\epsilon\|a_2\|) \\ &\quad + CA_2^\beta(A_1 + \|b_1\| + C(L,R)\epsilon\|a_2\|)^{\alpha-1}\cdot(\|b_1\| + C(L,R)\epsilon\|a_2\|). \end{aligned} \tag{61}$$

Now for any initial data $(a_1, a_2) \in \mathcal{B}\times\mathcal{B}$, estimates in the previous section imply that if we consider the solution (u_1, u_2) after large enough time T, then $a_1 = u_1(\cdot, T)$ will be very small while $a_2 = u_2(\cdot, T))$ is preserved in $\mathcal{B}\times\mathcal{B}$ and the iterated RG map will be contractive for some $R = R(a_1, a_2)$. Therefore, for the rest of this section, we will fix R and take $\epsilon = \|a_1\|^\alpha\|a_2\|^{\beta-1}$ as small as necessary.

Lemma 9. *Suppose $L \geqslant 1$. There exists $\epsilon_0(L) > 0$ such that if $\|a_1\|^\alpha\|a_2\|^{\beta-1} < \epsilon \leqslant \epsilon_0(L)$, the following results hold:*

(a) $|A_i^1 - A_i| \leqslant C(L)\epsilon\|a_2\|$, $i = 1, 2$,
(b) $\|b_2^1\| \leqslant L^{-2\delta}\|b_2\| + C(L)\epsilon\|a_2\|$,
(c) $|A_1^1 - A_1 + \gamma A_1^\alpha A_2^\beta| \leqslant \eta$, *where*

$$\gamma = \frac{2\log L}{4\pi d^{\beta/2}(\alpha + \beta/d)^{1/2}}$$

and

$$\begin{aligned} \eta &= \eta(\|a_1\|, \|a_2\|, \|b_1\|, \|b_2\|, A_1, A_2, \epsilon) \\ &= C(L)(\|a_1\|^\alpha(A_2 + \|b_2\| + \epsilon\|a_2\|)^{\beta-1}\cdot(\|b_2\| + \epsilon\|a_2\|)) \\ &\quad + C(L)(A_2^\beta(A_1 + \|b_1\| + \epsilon\|a_2\|)^{\alpha-1}\cdot(\|b_1\| + \epsilon\|a_2\|)), \end{aligned}$$

(d) $\|b_1^1\| \leqslant L^{-2\delta}\|b_1\| + C(L)\epsilon\|a_2\|$,

where $\delta > 0$ *is independent of* L.

Proof. (a) Follows directly from our previous analysis since $|A_i^1 - A_i| = \int_{\mathbb{R}} Ln_i(Lx, L^2)\,\mathrm{d}x$.

For part (b), by our previous analysis on $Ln_2(Lx, L^2)$, (a) on $A_2^1 - A_2$, and the classical theory for heat equation $u_t = u_{xx}$ of bounded initial data $b_2(x)$ with $\int_{\mathbb{R}} b_2(x)\,\mathrm{d}x = 0$, we get (b).

(c) Follows from our estimates in (61) and some elementary calculation.

(d) Follows from the same argument as for (b). QED

Applying the lemma inductively, we have

$$|A_2^{n+1} - A_2^n| \leqslant C(L)\epsilon_n\|a_2^n\|,$$
$$|A_1^{n+1} - A_1^n| \leqslant C(L)\|a_1^n\|^\alpha((A_2^n)^{\beta-1} + \|b_2^n\|^{\beta-1} + \epsilon_n^{\beta-1}\|a_2^n\|^{\beta-1})(\|b_2^n\| + \epsilon_n\|a_2^n\|)$$
$$+C(L)(A_2^n)^\beta((A_1^n)^{\alpha-1} + \|b_1^n\|^{\alpha-1} + (\epsilon_n\|a_2\|)^{\alpha-1}) \cdot (\|b_1^n\| + \epsilon_n\|a_2^n\|),$$
$$\|b_2^{n+1}\| \leqslant L^{-2\delta}\|b_2^n\| + C(L)\epsilon_n\|a_2^n\|,$$
$$\|b_1^{n+1}\| \leqslant L^{-2\delta}\|b_1^n\| + C(L)\epsilon_n\|a_2^n\|, \qquad n = 1, 2, \ldots,$$

where $\epsilon_n = \|a_1^n\|^\alpha\|a_2^n\|^{\beta-1}$. We show A_i^n, $\|b_i^n\|$, $i = 1, 2$ converge as $n \to \infty$. Let

$$A^n = A_1^n + A_2^n, \qquad a^n = \|a_1^n\| + \|a_2^n\|, \qquad b^n = \|b_1^n\| + \|b_2^n\|, \qquad n = 1, 2, \ldots.$$

Then,

$$0 \leqslant A^{n+1} \leqslant A^n + C(L)\epsilon_n a^n,$$
$$b^{n+1} \leqslant L^{-2\delta}b^n + C(L)\epsilon_n a^n, \qquad n = 1, 2, \ldots.$$

By using the fact that

$$\epsilon_n = \|a_1^n\|^\alpha\|a_2^n\|^{\beta-1} \leqslant \frac{C}{(\log L^{2n})^{\alpha/(\alpha-1)}} \leqslant C(L)n^{-\alpha/(\alpha-1)}$$

and a^n is uniformly bounded, we deduce that A^n converges as $n \to \infty$. Since $\{A_2^n\}$ is an increasing sequence, this means both A_1^n and A_2^n converge as $n \to \infty$. As a matter of fact, we know from the evolution of the L^1-norm, as deduced from (53) and lemma 1, of (u_1, u_2) that $A_1^n \to 0$ and $A_2^n \to A$ as $n \to \infty$.

As to b^n, we have

$$\begin{aligned} b^{n+1} &\leqslant L^{-2\delta}b^n + C(L)n^{-\alpha/(\alpha-1)} \\ &\leqslant L^{-2\delta}(L^{-2\delta}b^{n-1} + C(L)(n-1)^{-\alpha/(\alpha-1)}) + C(L)n^{-\alpha/(\alpha-1)} \leqslant \cdots \\ &\leqslant L^{-2n\delta}b^1 + C(L)\sum_{l=0}^{n-1} L^{-2l\delta}(n-l)^{-\alpha/(\alpha-1)} \\ &\leqslant C(L)n^{-\alpha/(\alpha-1)} \to 0 \qquad \text{as } n \to \infty. \end{aligned}$$

To calculate the exact limit of A_1^n, we first note $\|a_1^n\| \leqslant C(L)n^{-1/(\alpha-1)}$ and therefore, $A_1^n \leqslant C(L)n^{-1/(\alpha-1)}$. Then from (c) of lemma 9 and the above bound for b^n we get

$$\begin{aligned} |A_1^{n+1} - A_1^n + \gamma(A_1^n)^\alpha(A_2^n)^\beta| &\leqslant C(L)\{n^{-\alpha/(\alpha-1)} \times n^{-\alpha/(\alpha-1)} + ((A_1^n)^{\alpha-1} + n^{-\alpha})n^{-\alpha/(\alpha-1)}\} \\ &\leqslant C(L)(n^{-2\alpha/(\alpha-1)} + (A_1^n)^{\alpha-1}n^{-\alpha/(\alpha-1)} + n^{-\alpha^2/(\alpha-1)}) \\ &\leqslant C(L)n^{-(2\alpha-1)/(\alpha-1)}. \end{aligned}$$

Here we use the fact that $1 < \alpha < 2$. Hence, the difference equation which A_1^n satisfies takes the form

$$A_1^{n+1} = A_1^n - \gamma(A_1^n)^\alpha(A_2^n)^\beta + O(n^{-(2\alpha-1)/(\alpha-1)}).$$

Taking into account that $A_2^n \to A$ as $n \to \infty$, we can write it as

$$A_1^{n+1} = A_1^n - \gamma(A_1^n)^\alpha(A)^\beta + O((A_1^n)^\alpha) + O(n^{-(2\alpha-1)/(\alpha-1)}).$$

Using the fact that the difference equation of the form with $\Gamma > 0$

$$y^{n+1} = y^n - \Gamma(y^n)^\alpha + O(n^{-(2\alpha-1)/(\alpha-1)})$$

has the maximum principle if the positive sequence $\{y_n\}$ is sufficiently small, and $y_n = cn^{-1/(\alpha-1)}$ is a sub-solution if $\Gamma c^{\alpha-1} < 1/(\alpha-1)$ but a super-solution if $\Gamma c^{\alpha-1} > 1/(\alpha-1)$, we get $y_n n^{1/(\alpha-1)}$ has a positive finite limit c satisfying $\Gamma c^{\alpha-1} = 1/(\alpha-1)$ as $n \to \infty$. Therefore, it is also true for A_1^n.

$$A_1^n n^{1/(\alpha-1)} \to B_L \qquad \text{as } n \to \infty,$$

where

$$B_L = \left(\frac{1}{(\alpha-1)\gamma A^\beta}\right)^{1/(\alpha-1)}.$$

Proof of theorem 1. It is clear that theorem 1 follows directly from the limits

$$A_1^n(2n\log L)^{1/(\alpha-1)} \to B \qquad \text{and} \qquad A_2^n \to A \qquad \text{as } n \to \infty. \qquad \text{QED}$$

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References

[1] Alikakos N 1979 L^p bounds of solutions of the reaction–diffusion systems *Commun. Partial Diff. Eqns* **4** 827–68

[2] Avrin J D 1990 Qualitative theory for a model of laminar flames with arbitrary non-negative initial data *J. Diff. Eqns* **84** 290–308

[3] Berlyand L and Xin J 1995 Large time asymptotics of solutions to a model combustion system with critical nonlinearity *Nonlinearity* **8** 161–78

[4] Billingham J and Needham D J 1991 The development of traveling waves in quadratic and cubic autocatalysis with unequal diffusion rates: I. Permanent form travelling waves *Phil. Trans. R. Soc.* A **334** 1–24

[5] Billingham J and Needham D J 1991 The development of traveling waves in quadratic and cubic autocatalysis with unequal diffusion rates: II. An initial-value problem with an immobilized or nearly immobilized autocatalyst *Phil. Trans. R. Soc.* A **336** 497–539

[6] Bricmont J, Kupiainen A and Lin G 1994 Renormalization group and asymptotics of solutions of nonlinear parabolic equations *Commun. Pure Appl. Math* **47** 893–922

[7] Bricmont J, Kupiainen A and Xin J 1996 Global large time self-similarity of a thermal-diffusive combustion system with critical nonlinearity *J. Diff. Eqns* **130** 9–35

[8] Feireisl E, Hilhorst D, Mimura M and Weidenfeld R On some reaction–diffusion systems with nonlinear diffusion arising in biology *Preprint*

[9] Goldenfeld N, Martin O, Oono Y and Liu F 1990 Anomalous dimensions and the renormalization group in a nonlinear diffusion process *Phys. Rev. Lett.* **64** 1361–4

[10] Hollis S, Martin R and Pierre M 1987 Global existence and boundedness in reaction–diffusion systems *SIAM J. Math. Anal.* **18** 744–61

[11] Kato T 1984 Strong L^p-solution of the Navier–Stokes equation in R^m, with application to weak solutions *Math. Z.* **187** 471–80

[12] Martin R and Pierre M 1992 Nonlinear reaction–diffusion systems *Nonlinear Equations in the Applied Sciences* ed W F Ames and C Rogers (Boston, MA: Academic)

[13] Masuda K 1983 On the global existence and asymptotic behavior of solutions of reaction–diffusion equations *Hokkaido Math. J.* **12** 360–70

[14] Matkowsky B J and Sivashinsky G I 1979 An asymptotic derivation of two models in flame theory associated with the constant density approximation *SIAM J. Appl. Math.* **37** 686–99

[15] Metcalf M J, Merkin J H and Scott S K 1994 Oscillating wave fronts in isothermal chemical systems with arbitrary powers of autocatalysis *Proc. R. Soc.* A **447** 155–74

[16] Nishiura Y *Mathematics of Patterns* at press

[17] Schonbek M E 1985 L^2 decay for weak solutions of the Navier–Stokes equation *Arch. Rational Mech. Anal.* **88** 209–22

[18] Sivashinsky G I 1983 Instability, pattern formation and turbulence in flames *Ann. Rev. Fluid Mech.* **15** 179–99