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The existence and non-existence of travelling front in high order auto-catalysis chemical reaction

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Abstract This article studies propagating wave fronts in an isothermal chemical reaction $A + nB \to (n+1)B$ involving two chemical species, a reactant A and an auto-catalyst B whose diffusion coefficients, D_A and D_B, are unequal due to different molecular weights and/or sizes. More accurate bounds v_* and v^* that depend on D_B/D_A, when the ratio is less than 1, are derived such that there is a unique travelling wave of every speed $v \geqslant v^*$ and there does not exist any travelling wave of speed $v < v_*$. The refined bounds for $D_B/D_A < 1$ case is compatible to what has been shown in earlier work for $D_B/D_A > 1$ when $n \geqslant 3$.

Keywords high order autocatalysis, travelling wave, minimum speed, reaction-diffusion

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1 Introduction

In this paper, we consider an isothermal autocatalytic chemical reaction step governed by the n-th order reaction relation

$$A + nB \to (n+1)B \qquad \text{with rate } kab^n \text{ and } n > 1 \text{ a positive integer.}$$

Here, $k > 0$ is the reaction rate, and a and b are the concentrations of reactant A and auto-catalyst B, respectively. A typical case is

$$A + 2B \to 3B \qquad \text{with rate } kab^2.$$

Well-documented in the literature, the cubic reaction relation has appeared in several important models of real chemical reactions, e.g., almost isothermal flames in the carbon-sulphide-oxygen reaction (Voronkov and Semenov [25]), iodate-arsenous acid reactions (Saul and Showwalter [22]), hydroxylamine-nitrate reactions (Gowland and Stedman [12]), as well as other applications (Aris et al. [3] and Sel'kov [23]).

Experimental observations demonstrate the existence of propagating chemical wave fronts in chemical systems for which cubic-catalysis forms a key step [16, 28]. These wavefronts, or travelling waves, arise due to the interaction of reaction and diffusion. Quite often when a quantity of auto-catalyst is added

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locally into an expanse of reactant, which is initially at uniform concentration, the ensuing reaction is observed to generate wavefronts which propagate outward from the initial reaction zone, consuming fresh reactant ahead of the wavefront as it propagates. This is the phenomenon to be addressed in this paper.

For this purpose, we consider a one-dimensional slab geometry and the following partial differential equations (PDEs) that govern mass concentration and molecular diffusion for the n-th order reaction scheme:

$$\frac{\partial a}{\partial t} = D_A \frac{\partial^2 a}{\partial x^2} - kab^n, \quad \frac{\partial b}{\partial t} = D_B \frac{\partial^2 b}{\partial x^2} + kab^n,$$

where D_A and D_B are the constant diffusion rates of A and B, respectively. Initial conditions, in accordance with the observed experiments, are

$$a(x,0) = a_0, \quad b(x,0) = g(x), \quad \forall\, x \in \mathbb{R},$$

where a_0 is a positive constant representing the initial uniform distribution of A and $g(x)$ is a non-negative function with compact support. It is not very difficult to derive from the PDEs that the solution has the following behavior at $x = \pm\infty$:

$$a(x,t) \to a_0, \quad b(x,t) \to 0 \qquad \text{as } |x| \to \infty, \quad \forall\, t \geqslant 0.$$

Introducing dimensionless parameters, dependent and independent variables

$$D = \frac{D_B}{D_A}, \quad \bar{a} = \frac{a}{a_0}, \quad \bar{b} = \frac{b}{a_0}, \quad \bar{t} = k a_0^n t, \quad \overline{x} = x\sqrt{\frac{k a_0^n}{D_A}}, \quad \bar{g} := \frac{g}{a_0},$$

and dropping the bars, the initial value problem takes the dimensionless form

$$\begin{cases} \dfrac{\partial a}{\partial t} = \dfrac{\partial^2 a}{\partial x^2} - ab^n, & x \in \mathbb{R}, \quad t > 0, \\ \dfrac{\partial b}{\partial t} = D\dfrac{\partial^2 b}{\partial x^2} + ab^n, & x \in \mathbb{R}, \quad t > 0, \\ a(x,0) = 1, \quad b(x,0) = g(x), & x \in \mathbb{R}, \quad t = 0. \end{cases} \tag{1.1}$$

Here D measures the rate of diffusion of the auto-catalyst relative to that of the reactant.

In the special case $D = 1$, the function $a + b$ satisfies a linear heat equation and can be solved explicitly, so the system is reduced to a single non-linear equation. For scalar equations, significant results are established and rich theories are available; see, for example, the works of Aronson and Weinberger [4], Chen and Guo [8], Fife and McLeod [10], Sattinger [21] and the excellent review paper by Xin [27] for detailed information on single equations. Primary concern of the present paper is the case $D \neq 1$, which arises when the chemical species involved have different molecular weights and/or sizes. In particular, enzyme reactions may involve large enzyme molecules and smaller substrate molecules, leading to significantly different rates of diffusion. The system (1.1) also arises in epidemiology (Bailey [5]), where a represents the population density of healthy individuals and b the population density of infected individuals; again, when healthy individuals are significantly more or lesser mobile than the infected, D is far away from unity. We note also the recent works by Ai and Huang [1,2], Hosono [17,18], Huang [20] and Guo and Tse [15] on travelling wave fronts of auto-catalysis reaction with decay, where existence is established for every speed.

The wave front propagating phenomenon corresponds to the following behavior of solutions to (1.1): After certain time of initiation, there are two wave fronts expanding towards $x = \pm\infty$ at a certain speed v. In between the two fronts, the reactant is consumed so $a \approx 0$; since each unit of reactant consumed produces exactly one unit of auto-catalyst, one can expect that $b \approx 1$ inside the wave front. Outside the wave front, the reactant mixture is basically unstirred, so $a \approx 1$ and $b \approx 0$. Focusing on the right front one expects that $(a(x,t), b(x,t)) = (\alpha(z), \beta(z))$, where $z = x - vt$, and (α, β) solves the following system:

$$\begin{cases} \alpha_{zz} + v\alpha_z = \alpha\beta^n, & \alpha \geqslant 0, \ \forall\, z \in \mathbb{R}, \\ D\beta_{zz} + v\beta_z = -\alpha\beta^n, & \beta \geqslant 0, \ \forall\, z \in \mathbb{R}, \\ \lim_{z\to\infty}(\alpha(z), \beta(z)) = (1, 0), \\ \lim_{z\to-\infty}(\alpha(z), \beta(z)) = (0, 1). \end{cases} \tag{1.2}$$

Here $v > 0$ is the constant travelling speed.

Travelling Wave Problem. Given $v > 0$, find $(\alpha, \beta) \in [C^2(\mathbb{R})]^2$ that satisfies (1.2).

In this paper we study the existence and non-existence of the travelling waves, which can be generated from the initial value problem (1.1) as being just described. One of the most important questions in the study of (1.2) is the existence of minimum speed travelling wave and good estimate of the minimum speed $v_{\min}$. In particular, for what range of v, in relation to D, does a travelling wave solution exist?

Due to the importance of travelling fronts in understanding the dynamics of (1.1), the study of (1.2) has a long history dating back to early 1990's, see [6,11]. But, it is only recently that significant progress has been made by Chen and Qi [9] on much better estimates of minimum speed when $n > 1$. Other important recent works are [19,26]. In particular, the following results are proved in [9].

Theorem 1. *Suppose $D < 1$ and $n \geqslant 2$. A unique (up to translation) travelling wave solution exists for (1.2) if $v \geqslant 4D/\sqrt{1+4D}$. On the other hand, there exists no solution for (1.2) if $v \leqslant D/\sqrt{K(n)}$, where $K(n)$ is a constant, which increases with n. In particular, $K(1) = 1/4$, $K(2) = 2$.*

Theorem 2. *Suppose $D \geqslant 1$ and $n \geqslant 1$. There exists a positive constant $v_{\min}$ such that (1.2) admits a travelling wave if and only if $v \geqslant v_{\min}$. In addition, $v_{\min}$ is bounded by*

$$\sqrt{\frac{D}{K(n)}} \leqslant v_{\min} \leqslant \sqrt{\frac{D}{K(n)}}\,\frac{1}{\sqrt{1-(1-\frac{1}{D})\frac{\sqrt{4K(n)+1}-1}{\sqrt{4K(n)+1}+1}}},$$

where $K(n)$ is the same constant as in Theorem 1.

It is obvious that the result for $D < 1$ is not satisfactory, while that of $D > 1$ is rather pleasing. In particular, in the limit of $D \to 1$ in Theorem 2, its result reduces to the classical one: (1.2) admits a solution if and only if $v \geqslant 1/\sqrt{2}$. Although the result for $D < 1$ in [9] seems to reach the limit of the method used for $n = 2$, more can be done for $n > 2$ and significant better results can be derived when $n \geqslant 3$.

The main purpose of the present paper is to establish more accurate bounds for $D < 1$ case, when $n \geqslant 3$, to make the result more compatible with that of $D > 1$ case.

The main result of the present paper is the following.

Theorem 1.1. *Suppose $D < 1$ and $n \geqslant 2$. For the travelling wave problem (1.2),*

(i) *there exists a unique (up to translation) solution if either*

$$v \geqslant \min\left(\sqrt{\frac{D}{K(n)}}, \sqrt{\frac{D}{K(n/2)}}\,\frac{1}{\sqrt{1+(\frac{1}{D}-1)\frac{\sqrt{4K(n/2)+1}-1}{\sqrt{4K(n/2)+1}+1}}}\right);$$

or, when $n \geqslant 3$, $v \geqslant \frac{9D}{\sqrt{49+63D}}$;

(ii) *there does not exist any solution if*

$$v \leqslant \sqrt{\frac{D}{K(n)}}\,\frac{1}{\sqrt{1+(\frac{1}{D}-1)\frac{\sqrt{4K(n)+1}-1}{\sqrt{4K(n)+1}+1}}},$$

where $K(n)$ is the same constant as in Theorem 1.

Clearly, the above result provides a pretty satisfying bound on the range of wave speeds. In particular, it shows that $v_{\min}(D) \propto D$ for small D. Furthermore, if $n \geqslant 4$, using the fact that $K(2) = 2$, one can deduce the following result:

Corollary 1.2. *Suppose $D < 1$ and $n \geqslant 4$. Then, there exists a unique travelling wave if $v \geqslant \frac{D}{\sqrt{1+D}}$.*

This result is compatible to the result of Theorem 2 when $D > 1$ and $n = 2$.

We note in passing the recent works to study the spatio-temporal profiles of L^1 initial values by Bricmont et al. [7] and the steady-state solutions by Shi and Wang [24].

The organization of this paper is as follows. In Section 2 preliminary analysis is presented, including some known results whose proof can be found in [9], for easy reference of readers. The proof of the main result is given in Section 3.

2 Preliminaries

2.1 A scalar equation

First we review the existence of travelling wave of unit speed to the equation

$$u_{zz} + u_z = ku(1-u)^n, \quad 0 \leqslant u \leqslant 1 \quad \text{on} \ \mathbb{R}, \qquad u(-\infty) = 0, \quad u(\infty) = 1. \tag{2.1}$$

Here $n \geqslant 1$ is a parameter and k is a positive constant. We seek upper bounds on k for the existence of a solution. Since a solution, if it exists, satisfies $u_z > 0$ on $\mathbb{R}$, we can write $u' = Q(u)$ and work on the (u, Q) phase plane. The resulting equation on the phase plane is

$$\begin{cases} QQ' + Q = ku(1-u)^n, \quad \forall\, u \in [0,1], \\ Q(0) = 0, \quad Q > 0 \quad \text{on} \ (0,1). \end{cases} \tag{2.2}$$

There is a one-to-one correspondence between solutions to (2.1) and solutions to (2.2) satisfying the additional requirement $Q(1) = 0$.

Lemma 2.1. *For each $n \geqslant 1$ and $k > 0$, there exists a unique solution $Q = Q(n, k; \cdot)$ to* (2.2)*. In addition, there exists a positive constant $K(n)$ such that $Q(n, k; 1) = 0$ if $k \in (0, K(n)]$ and $Q(n, K; 1) > 0$ if $k \in (K(n), \infty)$. Consequently,* (2.2) *admits a solution if and only if $k \in (0, K(n)]$. In addition, $K(n)$ is a strictly increasing function of n and $K(1) = \frac{1}{4}, K(2) = 2$.*

Proof. The existence of Q and K follows by the comparison principle. The exact value of $K(1)$ is calculated by a known fact that the function $u(1-u)$ is concave, so the minimum wave speed $v = 1$ satisfies $1 = 2\sqrt{K(1)}$; hence $K(1) = 1/4$. In the case $n = 2$, the exact solution is given by $Q = u(1-u)$, so $K(2) = 2$. We omit details, because it is a standard argument. □

2.2 Basic properties of travelling waves

Suppose (v, α, β) solves (1.2). Then $[\alpha_z + v\alpha + D\beta_z + v\beta]_z = 0$, so that $\alpha_z + D\beta_z + v(\alpha + \beta)$ is a constant function. Using the boundary conditions, we find that

$$\alpha_z + D\beta_z + v(\alpha + \beta - 1) = 0 \quad \text{on} \ \mathbb{R}.$$

With the new variable $w = \beta_z$, (1.2) is equivalent to the following third-order ODEs system

$$\begin{cases} \alpha_z = v(1 - \alpha - \beta) - Dw, \\ \beta_z = w, \\ w_z = -D^{-1}(\alpha\beta^n + vw), \\ \lim_{z\to\infty}(\alpha(z), \beta(z), w(z)) = (1, 0, 0), \\ \lim_{z\to-\infty}(\alpha(z), \beta(z), w(z)) = (0, 1, 0). \end{cases} \tag{2.3}$$

It is clear that in the (α, β, w) phase space, there are two equilibrium points: $(0, 1, 0)$ and $(1, 0, 0)$. The following are a few basic properties of travelling wave solutions. They can be proved by elementary computation, for details, see [9].

Proposition 2.2. *The systems* (1.2) *and* (2.3) *are equivalent. Any solution (α, β) to* (1.2) *or (α, β, w) to* (2.3) *has the following properties:*

1. $\alpha_z > 0 > \beta_z$ *in* $\mathbb{R}$;
2. $\alpha + \beta < 1$ *in* $\mathbb{R}$ *if* $D < 1$, $\alpha + \beta \equiv 1$ *if* $D = 1$, *and* $\alpha + \beta > 0$ *if* $D > 1$;

3. $v = \int_{-\infty}^{\infty} \alpha(z)\beta^n(z)\,dz > 0$;

4. *The equilibrium point* $(0,1,0)$ *of* (2.3) *is a saddle with a two-dimensional stable manifold and a one-dimensional unstable manifold. The eigenvalues and associated eigenvectors are*

$$\begin{aligned} &\lambda_1 = -vD^{-1}, && \mathbf{e}_{\lambda_1} = (0,-1,-\lambda_1)^{\mathrm{T}},\\ &\lambda_2 = -\frac{1}{2}(\sqrt{v^2+4}+v), && \mathbf{e}_{\lambda_2} = (\lambda_2(D\lambda_2+v),-1,-\lambda_2)^{\mathrm{T}},\\ &\lambda_3 = \frac{1}{2}(\sqrt{v^2+4}-v), && \mathbf{e}_{\lambda_3} = (\lambda_3(D\lambda_3+v),-1,-\lambda_3)^{\mathrm{T}}; \end{aligned}$$

5. *When* $n > 1$, *the equilibrium point* $(1,0,0)$ *is degenerate*; *it has a two-dimensional stable manifold and a one-dimensional center manifold. The eigenvalues and associated eigenvectors are*

$$\begin{aligned} &\mu_1 = -v, && \mathbf{e}_{\mu_1} = (1,0,0)^{\mathrm{T}},\\ &\mu_2 = -vD^{-1}, && \mathbf{e}_{\mu_2} = (0,1,-vD^{-1})^{\mathrm{T}},\\ &\mu_3 = 0, && \mathbf{e}_{\mu_3} = (1,-1,0)^{\mathrm{T}}. \end{aligned}$$

2.3 New setting—A non-autonomous 2 × 2 system

Different from earlier works in [6, 11], here we shall use a transformation to turn the third-order autonomous system (2.3) into a second-order non-autonomous system, using $u := 1-\beta$ as the independent variable. This is allowed since for the solution of interest, $\beta_z < 0$, so $z \to 1-\beta(z)$ has an inverse. To make the resulting system as simple as possible, we also scale the other variables. Hence, we introduce

$$u = 1-\beta, \quad A = \frac{D\alpha}{v^2}, \quad y = \frac{vz}{D}, \quad \kappa := \frac{D}{v}.$$

The system of differential equations (1.2) becomes

$$\begin{cases} u_{yy} + u_y = A(1-u)^n, & \text{on } \mathbb{R},\\ A_y = \kappa^2(u+u_y) - DA, & \text{on } \mathbb{R}. \end{cases}$$

Since $u_y > 0$ for the solution of interest, we can use u as the independent variable. Introducing $P(u) = u_y$, we have an equivalent system of second order non-autonomous (singular) ODEs,

$$\begin{cases} PP' = A[1-u]^n - P, & \forall\, u \in [0,1],\\ PA' = \kappa^2[P+u] - DA, & \forall\, u \in [0,1],\\ P(u) > 0,\ A(u) > 0, & \forall\, u \in (0,1),\\ P(0) = 0,\ A(0) = 0. \end{cases} \tag{2.4}$$

Lemma 2.3. *For every* $D > 0$ *and* $\kappa > 0$, (2.4) *admits a unique solution. In addition,*

$$P(u) = \lambda u + O(u^2), \quad A(u) = \lambda(1+\lambda)u + O(u^2) \quad \text{as } u \searrow 0, \tag{2.5}$$

where

$$\lambda := \frac{1}{2}(\sqrt{4\kappa^2+D^2} - D), \quad \text{the only positive root to } \lambda(\lambda+D) = \kappa^2.$$

Furthermore, $A'(u) > 0$ *for all* $u \in [0,1)$ *and there are only two possible cases*:

(a) $P(1) > 0$; *there does not exist any travelling wave solution to* (1.2).

(b) $P(1) = 0$; *there exists a travelling wave solution to* (1.2), *unique up to translation.*

Proof. The proof can be found in [9]. □

In the sequel, we shall estimate upper and lower bounds of A/u, so Lemma 2.1 can be applied to generating upper and lower bounds of $v_{\min}$.

3 Proof of Theorem 1.1

In this section we prove Theorem 1.1 through a series of lemmas.

Lemma 3.1. *Suppose $D < 1$. Then $\kappa^2 u < A < \frac{\kappa^2}{D}u$ on $(0,1)$. Consequently, when $\kappa^2/D \leqslant K(n)$, i.e., $v > \sqrt{\frac{D}{K(n)}}$, there exists a travelling wave solution to* (1.2).

Proof. A direct calculation shows that

$$P[A-\kappa^2 u]' = \kappa^2(P+u) - DA - \kappa^2 P = \kappa^2(1-D)u - D(A-\kappa^2 u) > -D(A-\kappa^2 u), \quad \forall\, u \in (0,1).$$

Since $A = \lambda(1+\lambda)u + O(u^2) > \kappa^2 u$, for all sufficiently small positive u, Gronwall's inequality gives $A > \kappa^2 u$ on $(0,1)$. Similarly,

$$\begin{aligned} P\left(A-\frac{\kappa^2}{D}u\right)' &= \kappa^2(P+u) - DA - \frac{\kappa^2}{D}P = -\frac{(1-D)\kappa^2}{D}P - D\left(A-\frac{\kappa^2}{D}u\right) \\ &< -D\left(A-\frac{\kappa^2}{D}\right)u, \quad \forall\, u \in (0,1). \end{aligned}$$

Again, by using the asymptotic expression of A for u small and Gronwall's inequality one obtains $A < \frac{\kappa^2}{D}u$ on $(0,1)$.

One can show that $P(u) \leqslant Q(n,\kappa^2/D;u)$ for all $u \in (0,1)$ by first using an asymptotic expansion at $u=0$ for $0 < u \leqslant \epsilon$ and then a comparison principle for the differential equation in $(\epsilon,1)$.

It then follows from Lemma 2.1 that when $\kappa^2/D \leqslant K(n)$, we must have $P(1) \leqslant Q(n,\kappa^2/D;1) = 0$, i.e., there exists a solution to the travelling wave problem. □

To establish better estimate of range of the speed v for the existence of a solution, we need to find an upper bound of A. More specifically, we are not satisfied with the estimate in Lemma 3.1, since when D is very small, it is not sufficient to show that $v_{\min} = O(D)$. Hence, we seek another bound.

Lemma 3.2. *Suppose $D < 1$. Then $A(u)(1-u)^{n/2} \leqslant \lambda[P(u)+u], \forall\, u \in [0,1)$.*

Proof. When $u=0$, the two sides are equal. Computation shows, in $(0,1]$,

$$\begin{aligned} &P[(1-u)^{n/2}A - \lambda(P+u)]' \\ &\quad= (1-u)^{n/2}[\kappa^2(P+u) - DA] - \frac{1}{2}nPA(1-u)^{n/2-1} - \lambda A(1-u)^n \\ &\quad\leqslant -[D+\lambda(1-u)^{n/2}][A(1-u)^{n/2} - \lambda(P+u)] + (P+u)[(\kappa^2-\lambda^2)(1-u)^{n/2} - \lambda D] \\ &\quad= -[D+\lambda(1-u)^{n/2}][A(1-u)^{n/2} - \lambda(P+u)] - \lambda D(P+u)[1-(1-u)^{n/2}] \\ &\quad\leqslant -[D+\lambda(1-u)^{n/2}][A(1-u)^{n/2} - \lambda(P+u)]. \end{aligned}$$

Here we have dropped the term $\frac{1}{2}nP(1-u)^{n/2-1}$ in the first inequality and used $\kappa^2 = \lambda(\lambda+D)$ in the second inequality. The assertion of the lemma thus follows from the Gronwall's inequality. □

Lemma 3.3. *Suppose $D < 1$. Then $A \geqslant \lambda(P+u)$ and $\lambda u(1-u)^n \leqslant P \leqslant \lambda u$ in $[0,1]$.*

Proof. It is easy to show that

$$\begin{aligned} P[A-\lambda(P+u)]' &= \kappa^2(P+u) - DA - \lambda A(1-u)^n \\ &= -[D+\lambda(1-u)^n][A-\lambda(P+u)] + (P+u)[\kappa^2 - \lambda^2(1-u)^n - \lambda D] \\ &> -[D+\lambda(1-u)^n][A-\lambda(P+u)] \qquad \text{in } (0,1). \end{aligned}$$

When $u > 0$ but very small, Taylor expansion at $u=0$ shows $A > \lambda(P+u)$. It follows from Gronwall's inequality that $A \geqslant \lambda(P+u)$ in $[0,1]$. To show the results concerning P, we calculate that, with η and δ two positive constants,

$$[P - \eta u(1-u)^\delta]' = -1 + \frac{A(1-u)^n}{P} - \eta(1-u)^\delta + \eta\delta u(1-u)^{\delta-1}.$$

Let $\delta = 0$, an easy computation shows, by letting $A = \lambda(1+\lambda)u + \mu_1 u^2$, $P = \lambda u + \mu_2 u^2$ around $u = 0$, that when $D < 1$, $\mu_1 > 0$ and $-n\lambda < \mu_2 < 0$. Hence, $P - \lambda u < 0$ when $0 < u \ll 1$. If $P - \lambda u = 0$ at some positive value $u = u_0$, then $(P - \lambda u)' \geqslant 0$ at the same point. But, using the result of Lemma 3.2, we have, at $u = u_0$,

$$[P - \lambda u]' \leqslant -1 + \lambda(1-u)^{n/2} + \frac{\lambda u(1-u)^{n/2}}{P} - \lambda \leqslant -1 + \lambda(1-u)^{n/2} + (1-u)^{n/2} - \lambda < 0.$$

A contradiction. This proves $P \leqslant \lambda u$ in $[0,1]$.

A similar argument shows $P \geqslant \lambda u(1-u)^n$ in $[0,1]$. □

Lemma 3.4. *Suppose $D < 1$. Then $A > \lambda(1+\lambda)u$ on $(0,1)$. Consequently, when $\lambda(1+\lambda) > K(n)$, i.e.,*

$$v < \sqrt{\frac{D}{K(n)}}\,\frac{1}{\sqrt{1 + (\frac{1}{D} - 1)\frac{\sqrt{4K(n)+1}-1}{\sqrt{4K(n)+1}+1}}},$$

there is no travelling wave solution to (1.2).

Proof. A direct calculation gives that

$$\begin{aligned} P[A - \lambda(1+\lambda)u]' &= -D[A - \lambda(1+\lambda)u] + \frac{D-1}{D+\lambda}[P - \lambda u] \\ &> -D[A - \lambda(1+\lambda)u], \quad \forall\, u \in (0,1), \end{aligned}$$

since by Lemma 3.3, $P < \lambda u$. In addition, $A = \lambda(1+\lambda)u + O(u^2) > \lambda(1+\lambda)u$, for all sufficiently small positive u, Gronwall's inequality gives $A > \lambda(1+\lambda)u$ on $(0,1)$.

One can show that $P(u) > Q(n, \lambda(1+\lambda); u)$ for all $u \in (0,1)$ by using now the standard procedure.

It then follows from Lemma 2.1 that when $\lambda(1+\lambda) > K(n)$, we must have $P(1) \geqslant Q(n, \lambda(1+\lambda); 1) > 0$, i.e., there does not exist any solution to the travelling wave problem. □

Lemma 3.5. *Suppose $D < 1$ and $n \geqslant 3$. If $\lambda \leqslant 7/9$, then there exists $\eta > \lambda$ such that*

$$P < \eta u(1-u) \quad \text{in } (0,1). \tag{3.1}$$

Proof. For any $\eta > \lambda$, (3.1) holds for $u \in (0, \epsilon)$. If it is violated initially at $u_0 > 0$, then at this point $I \equiv [P - \eta u(1-u)]' \geqslant 0$. But, a direct computation shows

$$\begin{aligned} [P - \eta u(1-u)]' &= -1 + \frac{A(1-u)^n}{P} - \eta(1-u) + \eta u \\ &\leqslant -1 + \lambda(1-u)^{n/2} + \frac{\lambda}{\eta}(1-u)^{n/2-1}\eta(1-u) + \eta u \end{aligned}$$

by Lemma 3.2. It is clear that the right-hand side, which we denote by $G(u)$, satisfies $G(0) < 0$, and $G(1) < 0$ when $\eta < 1$ and with equality if $\eta = 1$. It is also easy to see that it is a decreasing function of n. Therefore, we only need to prove the lemma when $n = 3$. Elementary computation shows that when $\lambda \leqslant 3/4$, and $\eta = \lambda$, G is a decreasing function on $(0,1)$. Hence, there exists $\eta > \lambda$ but close to it, which makes $G(u) < 0$ in $(0,1)$. Therefore, with such a choice of η, (3.1) holds for $u \in (0,1)$.

When $3/4 < \lambda \leqslant 7/9$, there exists a local maximum and local minimum in $(0,1)$ when $\lambda = \eta$. By computation, G takes negative value at the local maxima. Again, a choice of $\eta > \lambda$ but close to it, will make $G(u) < 0$ in $(0,1)$. This proves the lemma. □

Proof of Theorem 1.1. The non-existence follows directly from Lemma 3.4. The existence is a simple consequence of Lemmas 3.2, 3.3 and 3.5. In particular, the combination of Lemmas 3.2 and 3.3 shows

$$A(u)(1-u)^{n/2} \leqslant \lambda(P+u) \leqslant \lambda(1+\lambda)u.$$

Hence, the conclusion of the theorem follows from a simple comparison with the single equation (2.2). □

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