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# Regularity of Solutions of the Cahn–Hilliard Equation with Non-constant Mobility

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**Abstract** In this paper, we study the regularity of solutions for two-dimensional Cahn–Hilliard equation with non-constant mobility. Basing on the $L^p$ type estimates and Schauder type estimates, we prove the global existence of classical solutions.

**Keywords** Cahn–Hilliard equation, regularity, Schauder estimates
**MR(2000) Subject Classification** 35B65, 35G25, 35K35

## 1 Introduction

In this paper, we study the Cahn–Hilliard equation

$$\frac{\partial u}{\partial t} + \operatorname{div}\left[m(u)(k\nabla\Delta u - \nabla A(u)\right] = 0,\ k > 0 \tag{1.1}$$

in a two-dimensional bounded domain $\Omega \subset \mathbb{R}^2$ with smooth boundary. On the basis of physical consideration, the equation (1.1) is supplemented by the zero mass flux boundary condition, the natural boundary condition

$$\left.\frac{\partial u}{\partial n}\right|_{\partial\Omega} = \left.\frac{\partial \Delta u}{\partial n}\right|_{\partial\Omega} = 0 \tag{1.2}$$

and the initial value condition

$$u(x, 0) = u_0(x), \quad x \in \Omega. \tag{1.3}$$

The Cahn–Hilliard equation was introduced to study several diffusive processes, such as phase separation in binary alloys, growth and dispersal in population; see for example [1–3]. In particular, in the two-dimensional case, it can be used as a model describing the spreading of an oil film over an solid surface; see [4]. In the past years, the Cahn–Hilliard equation with constant mobility was intensively studied, and there are many outstanding results concerning the existence, regularity and special properties of solutions; see for example [5–9]. In recent years, the equation with concentration dependent mobility has also caused much attention. For the one-dimensional case, results concerning the existence, regularity and other properties such as the nonnegativity, finite speed of propagation, etc, have been obtained by several authors; see

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for example, [10–14]. However, only a few papers have been devoted to the multi-dimensional case. It was Elliott and Garcke [15] who first studied the equation (1.1) in any space dimension, in which the existence of *weak solutions is established; see also [16–19, 22, 23]. Recently, Yin and Liu [20] discussed the regularity of solutions for the two-dimensional case with the rather restrictive small initial energy assumption.

This paper is a step further in the study of the regularity. The purpose is to remove the smallness restriction on the initial energy. The main result is as follows:

**Theorem** *Assume that* $m(s) \in C^1(R)$, $A(s) \in C^2(R)$, $\int_\Omega u_0(x)dx = 0$ *and*

$$0 < m_0 \le m(s) \le m_1 < +\infty, \quad |m'(s)| \le M_1,$$

$$H(s) = \int_0^s A(s)ds \ge C_1|s|^4 - C_2,\ |A'(s)| \le C_3|s|^2 + C_4,$$

*where* $M's$ *and* $C's$ *are positive constants. Assume also that the initial data is smooth with appropriate compatibility conditions. Then the problem* (1.1)–(1.3) *admits a unique classical solution.*

We note that a reasonable choice of $A(s)$ is the cubic polynomial, namely,

$$A(s) = \gamma_1 s^3 + \gamma_2 s^2 + \gamma_3 s + \gamma_4, \quad \gamma_1 > 0,$$

which corresponds to the so-called double-well potential

$$H(s) = \frac{1}{4}\gamma_1 s^4 + \frac{1}{3}\gamma_2 s^3 + \frac{1}{2}\gamma_3 s^2 + \gamma_4 s.$$

As for the proof of the theorem, the key step is to get a priori estimates on the Hölder norm of solutions. In [20], two of the authors have applied the Campanato framework to obtain the local Schauder type estimates, and finally obtain the Hölder norm estimates. In that derivation, the smallness of the initial energy is necessary. To drop out the smallness restriction, we first establish the $L^p$ type estimates, which is much more complicated as can be seen from our proof. Then we combine it with the technique used in [20] to obtain the Hölder norm estimates, and finally complete the existence proof following a now standard approach.

## 2 $L^p$-estimates

In this section, we establish the a priori $L^p$-estimates for the solutions of the problem (1.1)–(1.3). We begin with the interior $L^p$-estimates. Let $x_0$ and $t_0$ be fixed and denote by $B_R(x_0)$ the ball centers at the point $x_0$ with radius $R$, $S_R = B_R(x_0) \times (t_0 - R^4, t_0 + R^4)$. Let $u$ be a smooth solution. Denote also

$$(\nabla u)_R = \frac{1}{|S_R|}\iint_{S_R} \nabla u dx dt.$$

In addition, throughout this section, we set $n = 2$, just for the clarification of the dependence of the Sobolev embedding exponent on the spatial dimension.

**Lemma 2.1** *Let* $u$ *be a solution of the problem* (1.1)–(1.3). *If* $B_R(x_0) \subset \Omega$, *then for some* $p > 2$, *we have*

$$\left(\fint\!\!\fint_{S_{R/4}} (R^4|\nabla\Delta u|^2 + |\nabla u - (\nabla u)_{R/4}|^2)^{p/2} dx dt\right)^{1/p} \le C_1\left(\fint\!\!\fint_{S_R} (R^4|\nabla\Delta u|^2 + |\nabla u - (\nabla u)_R|^2) dx dt\right)^{1/2} + C_2,$$

*where* $C_1$, $C_2$ *are constants depending only on the known quantities, and the notation* $\fint\!\!\fint$ *denotes the average integral, namely*

$$\fint\!\!\fint_G = \frac{1}{|G|}\iint.$$

*Proof* First, we set

$$F(t) = \int_\Omega \left(\frac{k}{2}|\nabla u|^2 + H(u)\right) dx.$$

A direct calculation shows that

$$F'(t) = \int_\Omega (A(u) - k\Delta u)\frac{\partial u}{\partial t}dx = -\int_\Omega m(u)(k\nabla\Delta u - \nabla A(u))^2 dx \le 0.$$

It follows that $F(t) \le F(0)$ and hence from our assumption on $A(s)$ and $m(s)$, we have

$$\sup_{0<t<T}\int_\Omega |\nabla u|^2 dx \le C, \tag{2.1}$$

$$\sup_{0<t<T}\int_\Omega |u|^4 dx \le C. \tag{2.2}$$

Next, we multiply the equation (1.1) by $\Delta u$ and integrate the resulting relation over $\Omega$ to obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_\Omega |\nabla u|^2 dx + k\int_\Omega m(u)|\nabla\Delta u|^2 dx &= \int_\Omega m(u)\nabla A(u)\nabla\Delta u dx\\ &\le \frac{k}{2}\int_\Omega m(u)|\nabla\Delta u|^2 + C\int_\Omega |A'(u)|^2|\nabla u|^2 dx\\ &\le \frac{k}{2}\int_\Omega m(u)|\nabla\Delta u|^2 + C\int_\Omega |u|^4|\nabla u|^2 dx + C.\end{aligned}$$

By $\frac{\partial u}{\partial n}|_{\partial\Omega} = 0$, $\int_\Omega u_0(x)dx = 0$, and using the Gagliardo–Nirenberg inequality

$$\sup|u| \le C_1\left(\int_\Omega |\nabla\Delta u|^2 dx\right)^{1/10}\left(\int_\Omega |u|^4 dx\right)^{1/5} + C_2\left(\int_\Omega |u|^4 dx\right)^{1/4},$$

therefore,

$$\int_\Omega |u|^4|\nabla u|^2 dx \le C\sup|u|^4 \le C_1\left(\int_\Omega |\nabla\Delta u|^2 dx\right)^{2/5} + C_2.$$

Summing up, we obtain

$$\frac{d}{dt}\int_\Omega |\nabla u|^2 dx + \int_\Omega |\nabla\Delta u|^2 dx \le C,$$

and hence,

$$\iint_{Q_T} |\nabla\Delta u|^2 dxdt \le C. \tag{2.3}$$

Now, we choose a cut-off function $\chi(x)$, with support in $B_R(x_0)$ with $\chi(x) = 1$ in $B_{R/2}(x_0)$, $0 \le \chi(x) \le 1$, and $|\nabla\chi| \le \frac{C}{R}$, $|D^2\chi| \le \frac{C}{R^2}$. Let $g(t)$ be an arbitrary function in $C^\infty(-\infty, +\infty)$ with $0 \le g(t) \le 1$, $0 \le g'(t) \le \frac{C}{R^4}$, $g(t) = 1$ for $t \ge t_0 - (\frac{R}{2})^4$ and $g(t) = 0$ for $t < t_0 - R^4$.

Multiplying the equation (1.1) by $g(t)\nabla\cdot[\chi^4(\nabla u - (\nabla u)_R)]$ and then integrating the resulting relation over $(t_0 - R^4, t) \times B_R(x_0)$, we have

$$\begin{aligned}&\int_{t_0-R^4}^t\int_{B_R(x_0)}\frac{\partial u}{\partial t}g(t)\nabla\cdot[\chi^4(\nabla u - (\nabla u)_R]dxdt\\ &\quad + \int_{t_0-R^4}^t\int_{B_R(x_0)} g(t)\mathrm{div}[m(u)(k\nabla\Delta u - \nabla A(u))]\nabla\cdot[\chi^4(\nabla u - (\nabla u)_R)]dxdt = 0.\end{aligned}$$

Integrating by parts, we obtain

$$\begin{aligned}&\int_{t_0-R^4}^t \frac{1}{2}\frac{d}{dt}\int_{B_R(x_0)} g(t)\chi^4|\nabla u - (\nabla u)_R|^2 dxdt - \int_{t_0-R^4}^t g'(t)\int_{B_R(x_0)}\chi^4|\nabla u - (\nabla u)_R|^2 dxdt\\ &\quad + \int_{t_0-R^4}^t\int_{B_R(x_0)} g(t)[m(u)(k\nabla\Delta u - \nabla A(u))]\nabla[\nabla\chi^4(\nabla u - (\nabla u)_R) + \chi^4\Delta u]dxdt = 0,\end{aligned}$$

that is,

$$\begin{aligned}&\frac{1}{2}\int_{B_R(x_0)} g(t)\chi^4|\nabla u - (\nabla u)_R|^2 dx + \int_{t_0-R^4}^t\int_{B_R(x_0)} kg(t)m(u)\chi^4|\nabla\Delta u|^2 dxdt\\ &\quad = -\int_{t_0-R^4}^t\int_{B_R(x_0)} 4kg(t)m(u)\chi^3\nabla\Delta u\cdot D^2u\nabla\chi dxdt\end{aligned}$$

$$
\begin{aligned}
&-\int_{t_0-R^4}^{t}\int_{B_R(x_0)} kg(t)m(u)\nabla\Delta u D^2\chi^4(\nabla u-(\nabla u)_R)dxdt\\
&+\int_{t_0-R^4}^{t}\int_{B_R(x_0)} g(t)m(u)A'(u)\nabla u\cdot\nabla\Delta u\chi^4 dxdt\\
&+\int_{t_0-R^4}^{t}\int_{B_R(x_0)} 4g(t)m(u)A'(u)\chi^3\nabla u\cdot D^2u\nabla\chi dxdt\\
&+\int_{t_0-R^4}^{t}\int_{B_R(x_0)} g(t)m(u)A'(u)\nabla u D^2\chi^4(\nabla u-(\nabla u)_R)dxdt\\
&+\int_{t_0-R^4}^{t} g'(t)\int_{B_R(x_0)} \chi^4|\nabla u-(\nabla u)_R|^2dxdt.
\end{aligned}
$$

Using the Hölder inequality, we obtain

$$
\begin{aligned}
&\frac{1}{2}\int_{B_R(x_0)} g(t)\chi^4|\nabla u-(\nabla u)_R|^2dx+\int_{t_0-R^4}^{t}\int_{B_R(x_0)} kg(t)m(u)\chi^4|\nabla\Delta u|^2dxdt\\
&\leq\int_{t_0-R^4}^{t} g'(t)\int_{B_R(x_0)} \chi^4|\nabla u-(\nabla u)_R|^2dxdt\\
&\quad+\frac{1}{8}\int_{t_0-R^4}^{t}\int_{B_R(x_0)} kg(t)m(u)\chi^4|\nabla\Delta u|^2dxdt\\
&\quad+C\int_{t_0-R^4}^{t}\int_{B_R(x_0)} kg(t)m(u)\chi^2|\nabla\chi|^2|D^2u|^2dxdt\\
&\quad+\frac{1}{8}\int_{t_0-R^4}^{t}\int_{B_R(x_0)} kg(t)m(u)\chi^4|\nabla\Delta u|^2dxdt\\
&\quad+C\int_{t_0-R^4}^{t}\int_{B_R(x_0)} kg(t)m(u)(4\chi|D^2\chi|+12|\nabla\chi|^2)^2|\nabla u-(\nabla u)_R|^2dxdt\\
&\quad+\frac{1}{8}\int_{t_0-R^4}^{t}\int_{B_R(x_0)} kg(t)m(u)\chi^4|\nabla\Delta u|^2dxdt\\
&\quad+C\int_{t_0-R^4}^{t}\int_{B_R(x_0)} g(t)m(u)|A'(u)|^2\chi^4|\nabla u|^2dxdt\\
&\quad+\frac{1}{8}\int_{t_0-R^4}^{t}\int_{B_R(x_0)} g(t)\chi^4|D^2u|^2dxdt\\
&\quad+C\int_{t_0-R^4}^{t}\int_{B_R(x_0)} g(t)m^2(u)|A'(u)|^2\chi^2|\nabla u|^2|\nabla\chi|^2dxdt\\
&\quad+C\int_{t_0-R^4}^{t}\int_{B_R(x_0)} g(t)\chi^4(|D^2\chi|+|\nabla\chi|^2)^2|\nabla u-(\nabla u)_R|^2dxdt\\
&\quad+\int_{t_0-R^4}^{t}\int_{B_R(x_0)} g(t)m^2(u)|A'(u)|^2|\nabla u|^2dxdt,
\end{aligned}
$$

where $D^2u$ denotes the Hessian matrix of $u$.

Taking (2.1)–(2.3) into account, we have

$$
\begin{aligned}
&\int_{B_R(x_0)} g(t)\chi^4|\nabla u-(\nabla u)_R|^2dx+\int_{t_0-R^4}^{t}\int_{B_R(x_0)} kg(t)m(u)\chi^4|\nabla\Delta u|^2dxdt\\
&\leq\frac{C}{R^4}\int_{t_0-R^4}^{t}\int_{B_R(x_0)} \chi^4|\nabla u-(\nabla u)_R|^2dxdt+C,
\end{aligned}
$$

and consequently

$$
\begin{aligned}
&\sup_{t_0-(R/2)^4<t<t_0+(R/2)^4} \int_{B_{R/2}(x_0)} |\nabla u - (\nabla u)_{R/2}|^2 dx \\
&\quad \le \sup_{t_0-(R/2)^4<t<t_0+(R/2)^4} \int_{B_{R/2}(x_0)} |\nabla u - (\nabla u)_R|^2 dx \\
&\quad \le \frac{C}{R^4} \int_{t_0-R^4}^{t} \int_{B_R(x_0)} |\nabla u - (\nabla u)_R|^2 dxdt + C
\end{aligned}
\tag{2.4}
$$

and

$$
\iint_{S_{R/2}(x_0)} |\nabla \Delta u|^2 dxdt \le \frac{C}{R^4} \int_{t_0-R^4}^{t} \int_{B_R(x_0)} |\nabla u - (\nabla u)_R|^2 dxdt + C. \tag{2.5}
$$

By a variant of the Sobolev–Poincaré inequality, $\forall\, \varepsilon > 0$, we have

$$
\begin{aligned}
&\iint_{S_R} |\nabla u - (\nabla u)_R|^2 dxdt \\
&\quad = \int_{t_0-R^4}^{t} \left[ \int_{B_R} |\nabla u - (\nabla u)_R|^2 dx \right]^{2/(n+2)+n/(n+2)} dt \\
&\quad \le C \sup_{t\in I_R} \left( \int_{B_R} |\nabla u - (\nabla u)_R|^2 dx \right)^{2/(n+2)} \int_{t_0-R^4}^{t} \left[ \int_{B_R} |\nabla u - (\nabla u)_R|^2 dx \right]^{n/(n+2)} dt \\
&\quad \le C \sup_{t\in I_R} \left( \int_{B_R} |\nabla u - (\nabla u)_R|^2 dx \right)^{2/(n+2)} \int_{t_0-R^4}^{t} \int_{B_R} |\nabla \Delta u|^{2n/(n+2)} dxdt + C \\
&\quad \le \varepsilon R^4 \sup_{t\in I_R} \int_{B_R} |\nabla u - (\nabla u)_R|^2 dx + C(\varepsilon) R^{-8/n} \left( \int_{t_0-R^4}^{t} \int_{B_R} |\nabla \Delta u|^{2n/(n+2)} dxdt \right)^{(n+2)/n} + C.
\end{aligned}
$$

By virtue of this inequality, (2.4) and (2.5), we have

$$
\begin{aligned}
&\iint_{S_{R/4}} (R^4 |\nabla \Delta u|^2 + |\nabla u - (\nabla u)_{R/4}|^2) dxdt \\
&\quad \le \iint_{S_{R/4}} R^4 |\nabla \Delta u|^2 dxdt + \iint_{S_{R/4}} |\nabla u - (\nabla u)_R|^2 dxdt \\
&\quad \le C_1 \iint_{S_{R/2}} |\nabla u - (\nabla u)_{R/2}|^2 dxdt + C_2 \\
&\quad \le \varepsilon R^4 \sup_{t\in I_{R/2}} \int_{B_{R/2}} |\nabla u - (\nabla u)_R|^2 dx \\
&\qquad + C(\varepsilon) R^{-8/n} \left( \int_{t_0-(R/2)^4}^{t} \int_{B_{R/2}} |\nabla \Delta u|^{2n/(n+2)} dxdt \right)^{(n+2)/n} + C \\
&\quad \le \varepsilon \iint_{S_R} |\nabla u - (\nabla u)_R|^2 dxdt \\
&\qquad + C(\varepsilon) R^{-8/n-4} \left( \int_{t_0-R^4}^{t} \int_{B_R} \left[ R^4 |\nabla \Delta u|^2 + |\nabla u - (\nabla u)_R|^2 \right]^{n/(n+2)} dxdt \right)^{(n+2)/n} + C \\
&\quad \le \varepsilon \iint_{S_R} |\nabla u - (\nabla u)_R|^2 dxdt \\
&\qquad + C(\varepsilon) R^{-8/n-4} \left( ⨌_{S_R} \left[ R^4 |\nabla \Delta u|^2 + |\nabla u - (\nabla u)_R|^2 \right]^{n/(n+2)} dxdt \cdot R^{4+n} \right)^{(n+2)/n} + C \\
&\quad \le \varepsilon \iint_{S_R} |\nabla u - (\nabla u)_R|^2 dxdt
\end{aligned}
$$

$$+ C(\varepsilon)R^{n+4}\left(\iint\!\!\!\!\!\!-_{S_R}\left[R^4|\nabla\Delta u|^2+|\nabla u-(\nabla u)_R|^2\right]^{n/(n+2)}dxdt\right)^{(n+2)/n}+C.$$

Then from a lemma of the Gehring type, Proposition 1.3 in [21], we see that for some $p>2$, there holds

$$\left(\iint\!\!\!\!\!\!-_{S_{R/4}}(R^4|\nabla\Delta u|^2+|\nabla u-(\nabla u)_{R/4}|^2)^{p/2}dxdt\right)^{1/p}$$
$$\leq C_1\left(\iint\!\!\!\!\!\!-_{S_R}(R^4|\nabla\Delta u|^2+|\nabla u-(\nabla u)_R|^2)dxdt\right)^{1/2}+C_2.$$

The proof is complete.

Now, let us turn to the $L^p$-estimates near the boundary. Let $(x_0,t_0)\in\partial\Omega\times(0,T)$ be fixed and set $\Omega_R(x_0)=B_R(x_0)\cap\Omega$.

**Lemma 2.2** *Let $u$ be a solution of the problem* (1.1)–(1.3)*. If $(x_0,t_0)\in\partial\Omega\times(0,T)$, then we have*

$$\left(\iint\!\!\!\!\!\!-_{\Omega_{R/4}}(R^4|\nabla\Delta u|^2+|\nabla u|^2)^{p/2}dxdt\right)^{1/p}\leq C_1\left(\iint\!\!\!\!\!\!-_{\Omega_R}(R^4|\nabla\Delta u|^2+|\nabla u|^2)dxdt\right)^{1/2}+C_2,$$

*where $C_1$, $C_2$ are constants depending only on the known quantities.*

*Proof* The main difference from interior estimates is that we cannot now take the average of $\nabla u$ in a ball. Choose a cut-off function $\chi(x)$, defined for all $x$ in $\Omega_R(x_0)$ with support contained strictly in $B_R(x_0)$, $\chi(x)=1$ in $\Omega_{R/2}(x_0)$, $0\leq\chi(x)\leq1$ and $|\nabla\chi|\leq\frac{C}{R}$, $|D^2\chi|\leq\frac{C}{R^2}$. Let $g(t)\in C^\infty(-\infty,+\infty)$, $0\leq g(t)\leq1$, $0\leq g'(t)\leq\frac{C}{R^4}$, $g(t)=1$ for $t\geq t_0-(\frac{R}{2})^4$ and $g(t)=0$ for $t<t_0-R^4$.

Multiplying the equation (1.1) by $g(t)\nabla\cdot(\chi^4\nabla u)$ and then integrating over $(t_0-R^4,t)\times B_R(x_0)$, we have

$$\int_{t_0-R^4}^t\int_{\Omega_R(x_0)}\frac{\partial u}{\partial t}g(t)\nabla\cdot[\chi^4\nabla u]dxdt$$
$$+\int_{t_0-R^4}^t\int_{\Omega_R(x_0)}g(t)\mathrm{div}[m(u)(k\nabla\Delta u-\nabla A(u))]\nabla\cdot[\chi^4\nabla u]dxdt=0$$

that is,

$$\int_{t_0-R^4}^t\frac12\frac{d}{dt}\int_{\Omega_R(x_0)}g(t)\chi^4|\nabla u|^2dxdt-\int_{t_0-R^4}^t g'(t)\int_{\Omega_R(x_0)}\chi^4|\nabla u|^2dxdt$$
$$+\int_{t_0-R^4}^t\int_{\Omega_R(x_0)}g(t)[m(u)(k\nabla\Delta u-\nabla A(u))]\nabla[\nabla\chi^4\nabla u+\chi^4\Delta u]dxdt=0.$$

It follows that

$$\frac12\int_{\Omega_R(x_0)}g(t)\chi^4|\nabla u|^2dx+\int_{t_0-R^4}^t\int_{\Omega_R(x_0)}kg(t)m(u)\chi^4|\nabla\Delta u|^2dxdt$$
$$=-\int_{t_0-R^4}^t\int_{\Omega_R(x_0)}kg(t)m(u)\nabla\Delta uD^2u\cdot4\chi^3\nabla\chi dxdt$$
$$-\int_{t_0-R^4}^t\int_{\Omega_R(x_0)}kg(t)m(u)\nabla\Delta uD^2\chi^4\nabla udxdt$$
$$+\int_{t_0-R^4}^t\int_{\Omega_R(x_0)}g(t)m(u)A'(u)\nabla u\cdot\nabla\Delta u\chi^4dxdt$$
$$+\int_{t_0-R^4}^t\int_{\Omega_R(x_0)}g(t)m(u)A'(u)\nabla uD^2u\cdot8\chi^3\nabla\chi dxdt$$

$$+\int_{t_0-R^4}^{t}\int_{\Omega_R(x_0)} g(t)m(u)A'(u)\nabla u D^2\chi^4\nabla u dxdt$$
$$+\int_{t_0-R^4}^{t} g'(t)\int_{\Omega_R(x_0)}\chi^4|\nabla u|^2 dxdt.$$

Therefore

$$\begin{aligned}
&\frac{1}{2}\int_{\Omega_R(x_0)} g(t)\chi^4|\nabla u|^2dx+\int_{t_0-R^4}^{t}\int_{\Omega_R(x_0)} kg(t)m(u)\chi^4|\nabla\Delta u|^2dxdt\\
&\le\int_{t_0-R^4}^{t} g'(t)\int_{\Omega_R(x_0)}\chi^4|\nabla u|^2dxdt\\
&\quad+\frac{1}{8}\int_{t_0-R^4}^{t}\int_{\Omega_R(x_0)} kg(t)m(u)\chi^4|\nabla\Delta u|^2dxdt\\
&\quad+C\int_{t_0-R^4}^{t}\int_{\Omega_R(x_0)} kg(t)m(u)\chi^2|\nabla\chi|^2|D^2u|^2dxdt\\
&\quad+\frac{1}{8}\int_{t_0-R^4}^{t}\int_{\Omega_R(x_0)} kg(t)m(u)\chi^4|\nabla\Delta u|^2dxdt\\
&\quad+C\int_{t_0-R^4}^{t}\int_{\Omega_R(x_0)} kg(t)m(u)(4\chi|D^2\chi|+12|\nabla\chi|^2)^2|\nabla u|^2dxdt\\
&\quad+\frac{1}{8}\int_{t_0-R^4}^{t}\int_{\Omega_R(x_0)} kg(t)m(u)\chi^4|\nabla\Delta u|^2dxdt\\
&\quad+C\int_{t_0-R^4}^{t}\int_{\Omega_R(x_0)} g(t)m(u)|A'(u)|^2\chi^4|\nabla u|^2dxdt\\
&\quad+\frac{1}{8}\int_{t_0-R^4}^{t}\int_{\Omega_R(x_0)} g(t)\chi^4|D^2u|^2dxdt\\
&\quad+C\int_{t_0-R^4}^{t}\int_{\Omega_R(x_0)} g(t)m^2(u)|A'(u)|^2\chi^2|\nabla u|^2|\nabla\chi|^2dxdt\\
&\quad+C\int_{t_0-R^4}^{t}\int_{\Omega_R(x_0)} g(t)\chi^4(|D^2\chi|+|\nabla\chi|^2)^2|\nabla u|^2dxdt\\
&\quad+\int_{t_0-R^4}^{t}\int_{\Omega_R(x_0)} g(t)m^2(u)|A'(u)|^2|\nabla u|^2dxdt.
\end{aligned}$$

By virtue of (2.1)–(2.3), we have

$$\begin{aligned}
&\int_{\Omega_R(x_0)} g(t)\chi^4|\nabla u|^2dx+\int_{t_0-R^4}^{t}\int_{\Omega_R(x_0)} kg(t)m(u)\chi^4|\nabla\Delta u|^2dxdt\\
&\le\frac{C}{R^4}\int_{t_0-R^4}^{t}\int_{\Omega_R(x_0)}\chi^4|\nabla u|^2dxdt+C.
\end{aligned}$$

Therefore

$$\sup_{t_0-(R/2)^4<t<t_0+(R/2)^4}\int_{\Omega_{R/2}(x_0)}|\nabla u|^2dx\le\frac{C}{R^4}\int_{t_0-R^4}^{t}\int_{\Omega_R(x_0)}|\nabla u|^2dxdt+C \tag{2.7}$$

and

$$\iint_{\Omega_{R/2}(x_0)}|\nabla\Delta u|^2dxdt\le\frac{C}{R^4}\int_{t_0-R^4}^{t}\int_{\Omega_R(x_0)}|\nabla u|^2dxdt+C. \tag{2.8}$$

By the Sobolev–Poincaré inequality,

$$\iint_{\Omega_R}|\nabla u|^2dxdt=\int_{t_0-R^4}^{t}\left[\int_{\Omega_R}|\nabla u|^2dx\right]^{2/(n+2)+n/(n+2)}dt$$

$$\leq C \sup_{t\in I_R}\left(\int_{B_R}|\nabla u|^2dx\right)^{2/(n+2)}\int_{t_0-R^4}^{t}\left[\int_{\Omega_R}|\nabla u|^2dx\right]^{n/(n+2)}dt$$
$$\leq C \sup_{t\in I_R}\left(\int_{B_R}|\nabla u|^2dx\right)^{2/(n+2)}\int_{t_0-R^4}^{t}\int_{\Omega_R}|\nabla\Delta u|^{2n/(n+2)}dxdt$$
$$\leq \varepsilon R^4\sup_{t\in I_R}\int_{B_R}|\nabla u|^2dx + C(\varepsilon)R^{-8/n}\left(\int_{t_0-R^4}^{t}\int_{\Omega_R}|\nabla\Delta u|^{2n/(n+2)}dxdt\right)^{(n+2)/n}.$$

Taking this into account and using (2.7) and (2.8), we have

$$\iint_{\Omega_{R/4}}(R^4|\nabla\Delta u|^2+|\nabla u|^2)dxdt$$
$$\leq \iint_{\Omega_{R/4}}R^4|\nabla\Delta u|^2dxdt+\iint_{\Omega_{R/4}}|\nabla u|^2dxdt$$
$$\leq C_1\iint_{\Omega_{R/2}}|\nabla u|^2dxdt + C_2$$
$$\leq \varepsilon R^4\sup_{t\in I_{R/2}}\int_{\Omega_R}|\nabla u|^2dx + C(\varepsilon)R^{-8/n}\left(\int_{t_0-(R/2)^4}^{t}\int_{\Omega_{R/2}}|\nabla\Delta u|^{2n/(n+2)}dxdt\right)^{(n+2)/n}$$
$$\leq \varepsilon\iint_{\Omega_R}|\nabla u|^2dxdt$$
$$+C(\varepsilon)R^{-8/n-4}\left(\int_{t_0-R^4}^{t}\int_{B_R}\left[R^4|\nabla\Delta u|^2+|\nabla u|^2\right]^{n/(n+2)}dxdt\right)^{(n+2)/n}$$
$$\leq \varepsilon\iint_{\Omega_R}|\nabla u|^2dxdt$$
$$+C(\varepsilon)R^{-8/n-4}\left(⨌_{\Omega_R}\left[R^4|\nabla\Delta u|^2+|\nabla u|^2\right]^{n/(n+2)}dxdt\cdot R^{4+n}\right)^{(n+2)/n}+C$$
$$\leq \varepsilon\iint_{\Omega_R}|\nabla u|^2dxdt + C(\varepsilon)R^{n+4}\left(⨌_{\Omega_R}\left[R^4|\nabla\Delta u|^2+|\nabla u|^2\right]^{n/(n+2)}dxdt\right)^{(n+2)/n}+C$$

and hence as in the proof of Lemma 2.1, for some $p>2$, there holds

$$\left(⨌_{\Omega_{R/4}}(R^4|\nabla\Delta u|^2+|\nabla u|^2)^{p/2}dxdt\right)^{1/p}\leq C_1\left(⨌_{\Omega_R}(R^4|\nabla\Delta u|^2+|\nabla u|^2)dxdt\right)^{1/2}+C_2.$$

The proof is complete.

## 3 Hölder Estimates

We establish the Hölder norm estimates based on the $L^p$-estimates obtained in the last section.

**Lemma 3.1** *Let $u$ be a smooth solution of the problem* (1.1)–(1.3). *Then there exists a constant $C$ depending only on the bounds of $A$, $m$ and the initial value $u_0$, such that for any* $(x_1,t_1),(x_2,t_2)\in\overline{Q_T}$ *and some* $0<\alpha<1$, $|u(x_1,t_1)-u(x_2,t_2)|\leq C(|t_1-t_2|^{\alpha/4}+|x_1-x_2|^{\alpha})$.

*Proof* For simplicity, we only show the interior Hölder norm estimates. The boundary estimates are essentially the same with apparent modification. Choose a cut-off function $\eta(x)$ with support in $B_\rho(x_0)$, such that $\eta(x)=1$ on $B_{\rho/2}(x_0)$, $0\leq\eta(x)\leq 1$. Multiplying the equation (1.1) by $\nabla\cdot(\chi^4\nabla u)$ and then integrating over $(0,t)\times B_\rho(x_0)$, we have

$$\int_0^t\frac{1}{2}\frac{d}{dt}\int_{B_\rho(x_0)}\chi^4|\nabla u|^2dxdt+\int_0^t\int_{B_\rho(x_0)}[m(u)(k\nabla\Delta u-\nabla A(u))]\nabla[\nabla\chi^4\nabla u+\chi^4\Delta u]dxdt=0.$$

Using Hölder inequality and Poincaré inequality, we have

$$\frac{1}{2}\int_{B_\rho(x_0)}\chi^4|\nabla u|^2dx+\int_0^t\int_{B_\rho(x_0)}km(u)\chi^4|\nabla\Delta u|^2dxdt$$

$$
\begin{aligned}
&\le \frac{1}{8}\int_0^t\int_{B_\rho(x_0)} km(u)\chi^4|\nabla\Delta u|^2dxdt + C\int_0^t\int_{B_\rho(x_0)} km(u)\chi^2|\nabla\chi|^2|D^2u|^2dxdt\\
&\quad + \frac{1}{8}\int_0^t\int_{B_\rho(x_0)} km(u)\chi^4|\nabla\Delta u|^2dxdt\\
&\quad + C\int_0^t\int_{B_\rho(x_0)} km(u)(4\chi|D^2\chi| + 12|\nabla\chi|^2)^2|\nabla u|^2dxdt\\
&\quad + \frac{1}{8}\int_0^t\int_{B_\rho(x_0)} km(u)\chi^4|\nabla\Delta u|^2dxdt + C\int_0^t\int_{B_\rho(x_0)} m(u)\chi^4|\nabla u|^2dxdt\\
&\quad + \frac{1}{8}\int_0^t\int_{B_\rho(x_0)} \chi^4|D^2u|^2dxdt + C\int_0^t\int_{B_\rho(x_0)} m^2(u)\chi^2|\nabla u|^2|\nabla\chi|^2dxdt\\
&\quad + C\int_0^t\int_{B_\rho(x_0)} \chi^4(|D^2\chi| + |\nabla\chi|^2)^2|\nabla u|^2dxdt + \int_0^t\int_{B_\rho(x_0)} m^2(u)|\nabla u|^2dxdt.
\end{aligned}
$$

Therefore

$$
\begin{aligned}
&\int_{B_\rho(x_0)} \chi^4|\nabla u(t)|^2dx - \int_{B_\rho(x_0)} \chi^4|\nabla u_0|^2dx + \int_0^t\int_{B_\rho(x_0)} km(u)\chi^4|\nabla\Delta u|^2\\
&\quad \le C_1\int_0^t\int_{B_\rho(x_0)} |\nabla\Delta u|^2dxdt + \frac{C_2}{\rho^4}\int_0^t\int_{B_\rho(x_0)} |\nabla u|^2dx.
\end{aligned}
$$

Then by the $L^p$-estimates established in Section 2, we have

$$
\begin{aligned}
&\int_{B_\rho(x_0)} \chi^4|\nabla u(t)|^2dx + \int_0^t\int_{B_\rho(x_0)} km(u)\chi^4|\nabla\Delta u|^2\\
&\le \int_{B_\rho(x_0)} \chi^4|\nabla u_0|^2dx + C_1\int_0^t\int_{B_\rho(x_0)} |\nabla\Delta u|^2dxdt + \frac{C_2}{\rho^4}\int_0^t\int_{B_\rho(x_0)} |\nabla u|^2dx\\
&\le C\rho^n + C\Big(\iint_{S_\rho} (\rho^4|\nabla\Delta u|^2 + |\nabla u|^2)^{p/2}dxdt\Big)^{2/p}\Big(\int_0^t\int_{B_\rho(x_0)} dxdt\Big)^{(p-2)/p}\\
&\le C_1\rho^n + C_2\rho^{n(p-2)/p}\\
&\le C\rho^{n(p-2)/p}.
\end{aligned}
$$

Due to the arbitrariness of $t$ and $\rho$, we obtain

$$
\sup_{0<t<T}\int_{B_{\rho/2}} |\nabla u(t)|^2dx \le C\rho^{n(p-2)/p}.
$$

The conclusion follows from the Morrey Theorem for the integral description of Hölder continuous functions, namely, for $\alpha = \min\{1, n(p-2)/2p\}$, $|u(x_1,t_1) - u(x_2,t_2)| \le C(|t_1-t_2|^{\alpha/4} + |x_1-x_2|^\alpha)$. The proof is complete.

## 4 Proof of the Main Result

In this section, we prove the theorem that there exists a classical solution of the problem (1.1)–(1.3), under our assumptions on $m$ and $A$. The proof is quite similar to the corresponding part in [20]. However, for the convenience of readers, here we list the lemmas from [20], and show the main idea of the existence proof by two propositions.

**Proposition 4.1** *If $u$ is Hölder continuous in the interior of $Q_T$, then $u$ is classical in the interior of $Q_T$.*

*Proof* We first change the equation (1.1) into the form

$$
\frac{\partial u}{\partial t} + \nabla[a(t,x)\nabla\Delta u] = \nabla\vec{F}, \tag{4.1}
$$

where $a(t,x) = km(u(t,x))$, $\vec{F} = m(u(t,x))\nabla A(u(t,x))$. We may think of $a(t,x)$ and $\vec{F}(t,x)$ as known functions and consider the reduced linear equation (4.1). Since $u$ is locally Hölder continuous, we see that $a(t,x)$ is locally Hölder continuous too. Without loss of generality, we may assume that $a(t,x)$ and $\vec{F}(t,X)$ are sufficiently smooth; otherwise we replace them by their approximation functions. According to the boundary value condition (1.2), $\vec{F}(t,X)\cdot\vec{n}=0$, $(t,x)\in(0,T)\times\partial\Omega$. The crucial step is to establish the estimates on the Hölder norm of $\nabla u$. Let $(t_0,x_0)\in(0,T)\times\Omega$ be fixed and define

$$\varphi(\rho)=\iint_{S_\rho}(|\nabla u-(\nabla u)_\rho|^2+\rho^4|\nabla\Delta u|^2)\,dtdx,\quad (\rho>0),$$

where $S_\rho=(t_0-\rho^4,t_0+\rho^4)\times B_\rho(x_0)$, $(\nabla u)_\rho=\frac{1}{|S_\rho|}\iint_{S_\rho}\nabla u\,dtdx$ and $B_\rho(x_0)$ is the ball centred at $x_0$ with radius $\rho$.

Let $u$ be the solution of the problem (4.1), (1.2), (1.3). We split $u$ on $S_R$ into $u=u_1+u_2$, where $u_1$ is the solution of the problem

$$\frac{\partial u_1}{\partial t}+a(t_0,x_0)\Delta^2u_1=0,\quad (t,x)\in S_R \tag{4.2}$$

$$\frac{\partial u_1}{\partial n}=\frac{\partial u}{\partial n},\quad \frac{\partial\Delta u_1}{\partial n}=\frac{\partial\Delta u}{\partial n},\ \ (t,x)\in(t_0-R^4,t_0+R^4)\times\partial B_R(x_0) \tag{4.3}$$

$$u_1=u,t=t_0-R^4,\ \ x\in B_R(x_0), \tag{4.4}$$

and $u_2$ is the solution of the problem

$$\frac{\partial u_2}{\partial t}+a(t_0,x_0)\Delta^2u_2=\nabla[(a(t_0,x_0)-a(t,x))\nabla\Delta u]+\nabla\vec{F},\ \ (t,x)\in S_R, \tag{4.5}$$

$$\frac{\partial u_2}{\partial n}=0,\quad \frac{\partial\Delta u_2}{\partial n}=0,\ \ (t,x)\in(t_0-R^4,t_0+R^4)\times\partial B_R(x_0), \tag{4.6}$$

$$u_2=0,t=t_0-R^4,\ \ x\in B_R(x_0). \tag{4.7}$$

By classical linear theory, the above decomposition is uniquely determined by $u$.

We need several lemmas on $u_1$ and $u_2$.

**Lemma 4.2** ([18]) *Assume that*

$$|a(t,x)-a(t_0,x_0)|\le a_\sigma(|t-t_0|^{\sigma/4}+|x-x_0|^\sigma),t\in(t_0-R^4,t_0+R^4),\quad x\in B_R(x_0).$$

*Then*

$$\sup_{(t_0-R^4,t_0+R^4)}\int_{B_R(x_0)}|\nabla u_2(t,x)|^2\,dx+\iint_{S_R}(\nabla\Delta u_2)^2\,dtdx$$
$$\le CR^{2\sigma}\iint_{S_R}(\nabla\Delta u)^2\,dtdx+C\sup_{S_R}|\vec{F}|^2R^6.$$

**Lemma 4.3** ([20]) *For any $(t_1,x_1)$, $(t_2,x_2)\in S_\rho$,*

$$\frac{|\nabla u_1(t_1,x_1)-\nabla u_1(t_2,x_2)|^2}{|t_1-t_2|^{1/4}+|x_1-x_2|}$$
$$\le C\sup_{(t_0-\rho^4,t_0+\rho^4)}\int_{B_\rho(x_0)}(\rho^{-3}|\nabla u_1(t,x)-(\nabla u_1)_\rho|^2+\rho|\nabla\Delta u_1(t,x)|^2)\,dx$$
$$+C\iint_{S_\rho}(\rho^{-3}|\nabla\Delta u_1|^2+\rho|\nabla\Delta^2u_1|^2)\,dtdx.$$

**Lemma 4.4** ([20]) (Caccioppoli type inequality)

$$\sup_{(t_0-(R/2)^4,t_0+(R/2)^4)}\int_{B_{R/2}(x_0)}|\nabla u_1(t,x)-(\nabla u_1)_R|^2\,dx+\iint_{S_{R/2}}|\nabla\Delta u_1|^2\,dtdx$$
$$\le\frac{C}{R^4}\iint_{S_R}|\nabla u_1(t,x)-(\nabla u_1)_R|^2\,dtdx$$

$$\sup_{(t_0-(R/2)^4,t_0+(R/2)^4)}\int_{B_{R/2}(x_0)}|\Delta u_1|^2\,dx+\iint_{S_{R/2}}|\Delta^2u_1|^2\,dtdx$$
$$\le\frac{C}{R^4}\iint_{S_R}|\Delta u_1|^2\,dtdx\le\frac{C}{R^6}\iint_{S_{2R}}|\nabla u_1(t,x)-(\nabla u_1)_R|^2\,dtdx$$

$$\sup_{(t_0-(R/2)^4,t_0+(R/2)^4)}\int_{B_{R/2}(x_0)}|\nabla\Delta u_1|^2\,dx+\iint_{S_{R/2}}|\nabla\Delta^2 u_1|^2\,dtdx\le\frac{C}{R^4}\iint_{S_R}|\nabla\Delta u_1|^2\,dtdx.$$

**Lemma 4.5** ([20]) *Assume that*

$$|a(t,x)-a(t_0,x_0)|\le a_\sigma(|t-t_0|^{\sigma/4}+|x-x_0|^\sigma),\quad t\in(t_0-R^4,t_0+R^4),\quad x\in B_R(x_0).$$

*Then for any* $\rho\in(0,R)$,

$$\frac{1}{\rho^7}\iint_{S_\rho}(|\nabla u_1-(\nabla u_1)_\rho|^2+\rho^4|\nabla\Delta u_1|^2)\,dtdx\le\frac{C}{R^7}\iint_{S_R}(|\nabla u_1-(\nabla u_1)_R|^2+R^4|\nabla\Delta u_1|^2)\,dtdx.$$

**Lemma 4.6** ([20]) *For* $\lambda\in(6,7)$, $\varphi(\rho)\le C_\lambda(1+\sup_{S_{R_0}}|\vec{F}|)\rho^\lambda$, $\rho\le R_0=\min(\mathrm{dist}(x_0,\partial\Omega),t_0^{1/4})$, *where* $C_\lambda$ *depends on* $\lambda$, $R_0$ *and the known quantities.*

*Proof of Proposition* 4.1 Similarly to the discussion about the Campanato spaces in [21], we first conclude from Lemma 4.6 that

$$\frac{|\nabla u(t_1,x_1)-\nabla u(t_2,x_2)|}{|t_1-t_2|^{(\lambda-6)/8}+|x_1-x_2|^{(\lambda-6)/2}}\le C\Big(1+\sup_{S_{R_0}}\vec{F}\Big)\le C\Big(1+\sup_{S_{R_0}}|\nabla u|\Big).$$

By the interpolation inequality, we thus obtain

$$|\nabla u(t_1,x_1)-\nabla u(t_2,x_2)|\le C(|t_1-t_2|^{(\lambda-6)/8}+|x_1-x_2|^{(\lambda-6)/2}).$$

The conclusion follows immediately from the classical theory, since we can transform the equation (1.1) into the form

$$\frac{\partial u}{\partial t}+a_1(t,x)\Delta^2u+\vec{B_1}(t,x)\nabla\Delta u+a_2(t,x)\Delta u+\vec{B_2}(t,x)\nabla u=0,$$

where the Hölder norms on $a_1(t,x)=km(u(t,x))$, $\vec{B_1}(t,x)=km'(u(t,x))\nabla u(t,x)$, $a_2(t,x)=-m(u(t,x))A'(u(t,x))$, $\vec{B_2}(t,x)=-\nabla(m(u(t,x))A(u(t,x)))$ have been obtained from the preceding lemmas. The proof is complete.

**Proposition 4.7** *If* $u$ *is Hölder continuous in* $\overline{Q_T}$, *then* $u$ *is classical in* $\overline{Q_T}$.

*Proof* Let $(t_0,x_0)\in(0,T)\times\partial\Omega$ be fixed, and assume that in some neighbourhood of $x_0$, $\partial\Omega$ is explicitly expressed by a function $y=\varphi(x)$. We split $u$ into $u_1+u_2$ in $(t_0-R^4,t_0+R^4)\times\Omega_R(x_0)$ with $\Omega_R(x_0)=B_R(x_0)\cap\Omega$, where

$$\begin{aligned}&\frac{\partial u_1}{\partial t}+a(t_0,x_0)\Delta^2u_1=0,\quad \text{in}\quad S_R,\\&\frac{\partial u_1}{\partial n}=\frac{\partial u}{\partial n},\quad\frac{\partial\Delta u_1}{\partial n}=\frac{\partial\Delta u}{\partial n},\quad(t,x)\in(t_0-R^4,t_0+R^4)\times\partial B_R(x_0),\\&u_1=u,t=t_0-R^4,\ \ x\in\Omega_R(x_0),\end{aligned}$$

and

$$\begin{aligned}&\frac{\partial u_2}{\partial t}+a(t_0,x_0)\Delta^2u_2=\nabla[(a(t_0,x_0)-a(t,x))\nabla\Delta u]+\nabla\vec{F},\quad\text{in}\quad S_R,\\&\frac{\partial u_2}{\partial n}=0,\quad\frac{\partial\Delta u_2}{\partial n}=0,\quad(t,x)\in(t_0-R^4,t_0+R^4)\times\partial\Omega_R(x_0),\\&u_2=0,t=t_0-R^4,\ \ x\in\Omega_R(x_0).\end{aligned}$$

Define the normal and tangential derivatives as $\partial_n=\varphi'(x)\frac{\partial}{\partial x_1}-\frac{\partial}{\partial x_2}$, $\partial_\tau=\frac{\partial}{\partial x_1}+\varphi'(x)\frac{\partial}{\partial x_2}$.

Now, we modify the function $\varphi(\rho)$ as $\varphi(\rho)=\iint_{S_\rho}(|\partial_nu|^2+|\partial_\tau u-(\partial_\tau u)_\rho|^2+\rho^4|\nabla\Delta u|^2)\,dtdx$. Similarly to the proof of Proposition 1, we conclude that

$$\begin{aligned}&\frac{|\nabla u_1(t_1,x_1)-\nabla u_1(t_2,x_2)|^2}{|t_1-t_2|^{1/4}+|x_1-x_2|}\\&\le C\sup_{(t_0-\rho^4,t_0+\rho^4)}\int_{\Omega_\rho(x_0)}(\rho^{-3}|\partial_nu_1|^2+\rho^{-3}|\partial_\tau u_1-(\partial_\tau u_1)_\rho|^2+\rho|\nabla\Delta u_1|^2)\,dx\\&\quad+C\iint_{S_\rho}(\rho^{-3}|\nabla\Delta u_1|^2+\rho|\nabla\Delta^2u_1|^2)\,dtdx\end{aligned}$$

and

$$\sup_{(t_0-(R/2)^4,t_0+(R/2)^4)} \int_{\Omega_{R/2}(x_0)} |\partial_n u_1|^2 + |\partial_\tau u_1 - (\partial_\tau u_1)_{1/2}|^2 \,dx + \iint_{S_{R/2}} |\nabla \Delta u_1|^2 \,dtdx$$
$$\leq \frac{C}{R^4} \iint_{S_R} |\partial_n u_1|^2 + |\partial_\tau u_1 - (\partial_\tau u_1)_R|^2 \,dtdx$$
$$+ \frac{C}{R^6} \iint_{S_R} |u - u_R|^2 \,dtdx + \frac{C}{R^4} \iint_{S_R} |\nabla u_2|^2 \,dtdx.$$

The remaining part of the proof is similar to that of Proposition 4.1, and we omit the details.

*Proof of Theorem* It is a direct consequence of Lemmas 4.2–4.6 and Propositions 4.1 and 4.7.

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