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Asymptotics of minimizers of variational problems involving curl functional

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In this paper we are concerned with singularly perturbed variational problems involving the curl functional, which arise in the mathematical theory of liquid crystals. The asymptotic behavior of the minimizers in the singular limiting process is discussed, which is closely related to the variational problems for curl functional under various constraints. © *2000 American Institute of Physics.*
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I. INTRODUCTION

In this paper we study variational problems which arise naturally in the mathematical theory of liquid crystals. The main focus is to study the asymptotic behavior of minimizers of the variational problems where the curl functional is dominant. Before stating our problems and main results in this paper, we shall explain our motivation first.

The phase transition phenomenon is an important topic in the mathematical theory of liquid crystals. Several mathematical models for phase transitions from nematic to smectic *A* based on order parameter theory were proposed by de Gennes,[1,2] also see McMillan.[3] In recent years various simplified mathematical models have been posed based on singular perturbation theory in calculus of variation, see for instance Refs. 4, 5, 6.

In the classical Oseen–Frank theory, nematic phase of liquid crystals can be described by a director field $\mathbf{n}:\Omega\to\mathbb{S}^2$, which is a minimizer of the following Oseen–Frank energy functional,[7]

$$\mathcal{W}_{\mathrm{OF}}(\mathbf{n})=\int_\Omega W_{\mathrm{OF}}(\mathbf{n},\nabla\mathbf{n})\,dx,$$

where $\Omega\subset\mathbb{R}^3$ is a smooth bounded domain occupied by the liquid crystal sample, and

$$W_{\mathrm{OF}}(\mathbf{n},\nabla\mathbf{n})=\frac{k_1}{2}|\mathrm{div}\,\mathbf{n}|^2+\frac{k_2}{2}|\mathbf{n}.\mathrm{curl}\,\mathbf{n}|^2+\frac{k_3}{2}|\mathbf{n}\wedge\mathrm{curl}\,\mathbf{n}|^2+\frac{k_2+k_4}{2}[\mathrm{tr}(\nabla\mathbf{n})^2-(\mathrm{div}\,\mathbf{n})^2].$$

Here the k_i are material constants, $k_1,k_2,k_3>0$. In this paper we shall only consider the Dirichlet boundary conditions. The last term $[\mathrm{tr}(\nabla\mathbf{n})^2-(\mathrm{div}\,\mathbf{n})^2]$ will be dropped, since it is a divergence term and can be reduced to a surface integral. So, in the following we shall assume

$$W_{\mathrm{OF}}(\mathbf{n},\nabla\mathbf{n})=\frac{k_1}{2}|\mathrm{div}\,\mathbf{n}|^2+\frac{k_2}{2}|\mathbf{n}\cdot\mathrm{curl}\,\mathbf{n}|^2+\frac{k_3}{2}|\mathbf{n}\wedge\mathrm{curl}\,\mathbf{n}|^2.$$

It is well-known that Oseen–Frank model has been used successfully to describe the point defects of nematic liquid crystals. But, to describe line defects one may need to use Ericksen's model,[8]

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$$\mathcal{W}_E(s,\mathbf{n})=\int_\Omega W_E(s,\mathbf{n})dx,$$

where

$$W_E(s,\mathbf{n})=\frac{s^2}{2}W_{\mathrm{OF}}(\mathbf{n},\nabla\mathbf{n})+\frac{k_5}{2}|\nabla s|^2+\frac{k_6}{2}|\nabla s\cdot\mathbf{n}|^2+\psi(s).$$

Here s is a scale function called the *degree of orientation*.

It is believed that, the Ericksen's model and its varieties may be useful to describe the transitions of liquid crystals from the nematic phase to smectic A phase. By dropping various not very important terms in the Ericksen's functional, one is led to the following simplified energy functional

$$\mathcal{W}(\mathbf{u})=\int_\Omega\left\{\frac{k_1}{2}|\nabla\mathbf{u}|^2+\frac{k_2}{2}|\mathrm{curl}\,\mathbf{u}|^2+\psi(|\mathbf{u}|)\right\}dx, \tag{1.1}$$

where $\mathbf{u}=s\mathbf{n}$ and k_2 is large.[6] The function ψ satisfies the following condition:

$$\psi \text{ is a positive } C^1 \text{ function and } \lim_{s\to\infty}\psi(s)=+\infty. \tag{1.2}$$

It is expected that, as $k_2\to\infty$, the asymptotic behavior of minimizers of (1.1) under suitable boundary conditions will provide a mathematical representation of the phase transition process of liquid crystals from nematic phase to smectic A phase. For this purpose, one may also use a slightly different model[5]

$$\mathcal{I}(\mathbf{u})=\int_\Omega\left\{\frac{k_1}{2}|\mathrm{div}\,\mathbf{u}|^2+\frac{k_2}{2}|\mathrm{curl}\,\mathbf{u}|^2+\psi(|\mathbf{u}|)\right\}dx. \tag{1.3}$$

The limiting behavior of minimizers of $\mathcal{W}(\mathbf{u})$ or $\mathcal{I}(\mathbf{u})$ as $k_2\to\infty$ is not at all clear. Several basic mathematical questions are open.

Let us first consider functional (1.1). We assume that Ω is a smooth bounded domain in $\mathbb{R}^n$, $n=2,3$. Given $\mathbf{u}_0\in H^{1/2}(\partial\Omega)\cap L^\infty(\partial\Omega)$, denote

$$H(\Omega,\mathbf{u}_0)=\{\mathbf{u}\in H^1(\Omega,\mathbb{R}^n):\mathbf{u}=\mathbf{u}_0 \text{ on } \partial\Omega\}.$$

Fix $k_1>0$ and denote by $\mathbf{u}(k_2)$ a minimizer of $\mathcal{W}$ in $H(\Omega,\mathbf{u}_0)$. Our general problem is

(Q1). *As $k_2\to+\infty$, how to derive sharp estimates of the minimal energy $\mathcal{W}(\mathbf{u}(k_2))$, and to study the asymptotic behavior of minimizers?*

Inspecting the functional $\mathcal{W}$, one may guess that, $\mathrm{curl}\,\mathbf{u}(k_2)$ should be approximately zero since k_2 is very large. So it is natural to ask

(Q1.1). *As $k_2\to+\infty$, will the total energy $\mathcal{W}(\mathbf{u}(k_2))$ remain bounded?*

If the answer to (Q1.1) is yes, then one further asks the following:

(Q1.2). *As $k_2\to+\infty$, does the corresponding minimizer $\mathbf{u}(k_2)$ converge (in some sense) to a limit which is a minimizer or stationary point of the following functional:*

$$\mathcal{J}_0(\mathbf{u})=\int_\Omega\left\{\frac{k_1}{2}|\nabla\mathbf{u}|^2+\psi(|\mathbf{u}|)\right\}dx, \tag{1.4}$$

with $\mathrm{curl}\,\mathbf{u}=0$?

These two questions, posed in Ref. 6, are closely related to the minimization problem of the curl functional,

$$\mathcal{F}(\mathbf{u})=\int_{\Omega}|\mathrm{curl}\,\mathbf{u}|^2dx.$$

Denote

$$\mathcal{R}(\mathbf{u}_0)=\inf_{\mathbf{u}\in H(\Omega,\mathbf{u}_0)}\mathcal{F}(\mathbf{u}).\tag{1.5}$$

It is surprising to us that $\mathcal{R}(\mathbf{u}_0)$ is achieved for every $\mathbf{u}_0$ given, as stated in the following:

Theorem 1: *Assume Ω is a bounded, smooth, simply connected domain in $\mathbb{R}^n$, $n=2,3$. For any $\mathbf{u}_0\in H^{1/2}(\partial\Omega)\cap L^{\infty}(\partial\Omega)$, $\mathcal{R}(\mathbf{u}_0)$ is achieved in $H(\Omega,\mathbf{u}_0)$. Moreover, the minimizers are unique modulo gradient fields. More precisely, fix a minimizer $\overline{\mathbf{u}}\in H(\Omega,\mathbf{u}_0)$, then the set $\Sigma(\mathbf{u}_0)$ of all the minimizers of the curl functional can be represented by*

$$\Sigma(\mathbf{u}_0)=\{\overline{\mathbf{u}}+\nabla\varphi:\varphi\in H^2(\Omega),\nabla\varphi=0\ \text{ on }\ \partial\Omega\}.\tag{1.6}$$

Remark 1.1: We note that every minimizer of $\mathcal{R}(u_0)$ satisfies the following Euler equation:

$$curl^2\,\mathbf{u}=0\ \text{ in }\ \Omega,\ \ \mathbf{u}=\mathbf{u}_0\ \text{ on }\ \partial\Omega.$$

Using Theorem 1 we can prove that, as $k_2\to\infty$, there is a sequence of minimizers $\mathbf{u}(k_2)$ which converges to a minimizer $\overline{\mathbf{u}}$ of the curl functional $\mathcal{F}$. Furthermore, $\overline{\mathbf{u}}$ minimizes $\mathcal{J}_0$, as defined in (1.4), among all the minimizers of the curl functional.

Theorem 2: *Assume Ω is a smooth, bounded, simply connected domain in $\mathbb{R}^n, n=2,3$. Assume ψ satisfies the condition (1.2) and $\mathbf{u}_0\in H^{1/2}(\partial\Omega)\cap L^{\infty}(\partial\Omega)$. Let $\mathbf{u}(k_2)$ be a minimizer of $\mathcal{W}$* in *$H(\Omega,u_0)$. Then, for any sequence $k_2\to+\infty$, there exists a subsequence such that the corresponding minimizers $\mathbf{u}(k_2)\to\overline{\mathbf{u}}$ strongly in $H^2(\Omega,\mathbb{R}^3)$, where $\overline{\mathbf{u}}$ satisfies*

$$\mathcal{F}(\overline{\mathbf{u}})=\mathcal{R}(\mathbf{u}_0),\ \ \mathcal{J}_0(\overline{\mathbf{u}})=\inf_{\mathbf{u}\in\Sigma(\mathbf{u}_0)}\mathcal{J}_0(\mathbf{u}).$$

Remark 1.2: It can be seen from the proof of Theorems 1 and 2 in Sec. II that the condition that Ω is simply connected is only used to guarantee that $\mathcal{R}(\mathbf{u}_0)$ is achieved.

As a consequence we find that, if $\mathcal{R}(\mathbf{u}_0)=0$, then we have positive answers to (Q1.1) and (Q1.2). Hence it is interesting to know the conditions under which $\mathcal{R}(\mathbf{u}_0)=0$ holds. Several results are given in Sec. II.

Similar results for the functional $\mathcal{I}(\mathbf{u})$ which is defined in (1.3) are also true, and can be proved by using Theorem 1.

In the following, we let $\psi(s)=\lambda(1-s^2)^2$ in (1.1) [or in (1.3), see Ref. 5],

$$\mathcal{W}(\mathbf{u},k_1,k_2,\lambda)=\int_{\Omega}\left\{\frac{k_1}{2}|\nabla\mathbf{u}|^2+\frac{k_2}{2}|\mathrm{curl}\,\mathbf{u}|^2+\lambda(1-|\mathbf{u}|^2)^2\right\}dx.$$

When we fix k_1 and λ, and let k_2 go to $+\infty$, the asymptotic behavior of minimizers has been given by Theorem 2. Now we ask

(Q2). *What is the limiting behavior of the minimizers of $\mathcal{W}(\cdot,k_1,k_2,\lambda)$ as we send both λ and k_2 to $+\infty$, with λ growing faster than k_2?*

For convenience we choose a proper scaling and consider the following functional:

$$\mathcal{E}_{\varepsilon}(\mathbf{u})=\int_{\Omega}\left\{\varepsilon|\nabla\mathbf{u}|^2+|\mathrm{curl}\,\mathbf{u}|^2+\frac{1}{2\varepsilon}(1-|\mathbf{u}|^2)^2\right\}dx,\tag{1.7}$$

where $\varepsilon\ll1$. Assume $|\mathbf{u}_0|=1$ a.e. on $\partial\Omega$ and denote

$$C_\varepsilon(\mathbf{u}_0)=\inf_{\mathbf{u}\in H(\Omega,\mathbf{u}_0)} \mathcal{E}_\varepsilon(\mathbf{u}).$$

The Euler equation for minimizers of $\mathcal{E}_\varepsilon$ in $H(\Omega,\mathbf{u}_0)$ is

$$\begin{cases} -\varepsilon\Delta\mathbf{u}+\operatorname{curl}^2\mathbf{u}=\dfrac{1}{\varepsilon}(1-|\mathbf{u}|^2)\mathbf{u} \quad \text{in} \quad \Omega, \\ \mathbf{u}=\mathbf{u}_0 \quad \text{on} \quad \partial\Omega. \end{cases} \tag{1.8}$$

We are interested in the asymptotic behavior of minimizers $\{\mathbf{u}_\varepsilon\}$ of $\mathcal{E}_\varepsilon$ as $\varepsilon\to 0$. Intuitively, there is a close relation between the asymptotic behavior of $\mathbf{u}_\varepsilon$, and the existence of minimizers of the curl functional among all the unit vector fields. To make this observation clear, we introduce several notations. Assume Ω is a bounded smooth domain in $\mathbb{R}^n$, $n=2,3$. Assume $|\mathbf{u}_0(x)|\equiv 1$. Denote

$$H(\Omega,\mathbb{S}^{n-1},\mathbf{u}_0)=\{\mathbf{u}\in W^{1,2}(\Omega,\mathbb{S}^{n-1}),\mathbf{u}|_{\partial\Omega}=\mathbf{u}_0\}.$$

If $H(\Omega,\mathbb{S}^{n-1},\mathbf{u}_0)\neq\emptyset$, we define

$$\mathcal{R}_h(\mathbf{u}_0)=\inf_{\mathbf{u}\in H(\Omega,\mathbb{S}^{n-1},\mathbf{u}_0)} \mathcal{F}(\mathbf{u}).$$

If a unit vector field $\mathbf{v}$ is a minimizer of $\mathcal{R}_h(\mathbf{u}_0)$ then $\mathbf{v}$ satisfies

$$\begin{cases} \operatorname{curl}^2\mathbf{v}=(\mathbf{v}\cdot\operatorname{curl}^2\mathbf{v})\mathbf{v} \quad \text{in} \quad \Omega, \\ \mathbf{v}=\mathbf{u}_0 \quad \text{on} \quad \partial\Omega. \end{cases} \tag{1.9}$$

Note that Eq. (1.9) implies that $\operatorname{curl}^2\mathbf{v}$ is parallel to $\mathbf{v}$ everywhere.

If $\mathcal{R}_h(\mathbf{u}_0)$ is achieved in $H(\Omega,\mathbb{S}^{n-1},\mathbf{u}_0)$, we denote by $\Sigma_h(\mathbf{u}_0)$ the set of all such minimizers of $\mathcal{R}_h(\mathbf{u}_0)$, that is,

$$\Sigma_h(\mathbf{u}_0)=\left\{\mathbf{u}\in H(\Omega,\mathbb{S}^{n-1},\mathbf{u}_0): \int_\Omega |\operatorname{curl}\mathbf{u}|^2dx=\mathcal{R}_h(\mathbf{u}_0)\right\}.$$

Then we define

$$a_h(\mathbf{u}_0)=\inf_{\mathbf{u}\in\Sigma_h(\mathbf{u}_0)}\int_\Omega|\nabla\mathbf{u}|^2dx. \tag{1.10}$$

Our next theorem indicates that, if $\mathcal{R}_h(\mathbf{u}_0)$ is achieved in $H(\Omega,\mathbf{u}_0)$, then the asymptotic behavior of $\{\mathbf{u}_\varepsilon\}$ is simple. In fact, $\mathbf{u}_\varepsilon$ converges to $\tilde{\mathbf{u}}$ as $\varepsilon\to 0$, and $\tilde{\mathbf{u}}$ is a minimizer of $\mathcal{R}_h(\mathbf{u}_0)$. If the minimizers of $\mathcal{R}_h(\mathbf{u}_0)$ are not unique, then the limit $\tilde{\mathbf{u}}$ has the least energy $\int_\Omega|\nabla\tilde{\mathbf{u}}|^2dx$ among all the minimizers of $\mathcal{R}_h(\mathbf{u}_0)$.

Theorem 3: *Assume $\mathbf{u}_0$ is a unit vector field, and assume $\mathcal{R}_h(\mathbf{u}_0)$ is achieved in $H(\Omega,\mathbb{S}^{n-1},\mathbf{u}_0)$. Let u_ε be a minimizer of $\mathcal{E}_\varepsilon$. Then for any sequence $\varepsilon_n\to 0$, there exists a subsequence, which we still write as $\{\varepsilon_n\}$, such that the corresponding u_{ε_n} converges to $\tilde{\mathbf{u}}$ strongly in $W^{1,2}(\Omega,\mathbb{R}^n)$ and $\tilde{\mathbf{u}}$ is a minimizer of $\mathcal{R}_h(\mathbf{u}_0)$. Moreover, $\tilde{\mathbf{u}}$ minimizes the functional $\int_\Omega|\nabla\mathbf{u}|^2dx$ among all the minimizers of $\mathcal{R}_h(\mathbf{u}_0)$, that is,*

$$\int_\Omega|\nabla\tilde{\mathbf{u}}|^2dx=a_h(\mathbf{u}_0).$$

As an application of Theorem 3, we consider the variational problem of $\mathcal{E}_\varepsilon$ on the unit ball B in $\mathbb{R}^3$. Let $\mathbf{u}_0 = \pm x$ on ∂B. We will show that $\mathcal{R}_h(\pm x) = 0$ and the only minimizer in $H(B,\mathbb{S}^2, \pm x)$ is $\pm x/|x|$. As $\varepsilon \to 0$, the minimizer $\mathbf{u}_\varepsilon$ of $\mathcal{E}_\varepsilon$ will converge to $\pm x/|x|$ strongly in $W^{1,2}(B,\mathbb{R}^3)$. For more details, see Theorem 3.2.

Note that the vector fields $\pm x/|x|$ are spherically symmetric. It is interesting that the *only* spherically symmetric minimizers of the curl functional (under suitable boundary conditions) in $W^{1,2}(B,\mathbb{S}^2)$ are $\pm x/|x|$, see Theorem 3.1.

From Theorem 3 we see that, when $\mathcal{R}_h(\mathbf{u}_0)$ is achieved, then the situation is simple, and the limiting behavior of the minimizers of $\mathcal{E}_\varepsilon$ is clear. However, as we shall see later, in many practical cases, $H(\Omega,\mathbb{S}^{n-1},\mathbf{u}_0)$ is empty, or $\mathcal{R}_h(\mathbf{u}_0)$ is not achieved in $H(\Omega,\mathbb{S}^{n-1},\mathbf{u}_0)$. We believe that a natural class for the variational problem of the curl functional $\mathcal{F}$ is

$$\mathcal{L}(\Omega,\text{curl},\mathbf{u}_0) = \{\mathbf{u} \in L^\infty(\Omega) : \text{curl}\,\mathbf{u} \in L^2(\Omega) \text{ and } |\mathbf{u}| = 1 \text{ a.e. in } \Omega\}.$$

Set

$$\mathcal{R}_l(\mathbf{u}_0) = \inf_{\mathbf{u} \in \mathcal{L}(\Omega,\text{curl},\mathbf{u}_0)} \mathcal{F}(\mathbf{u}).$$

Note that

$$\max\{\mathcal{R}(\mathbf{u}_0), \mathcal{R}_l(\mathbf{u}_0)\} \leqslant \mathcal{R}_h(\mathbf{u}_0)$$

if $H(\Omega,\mathbb{S}^{n-1},\mathbf{u}_0) \neq \emptyset$. We expect that $\mathcal{R}_l(\mathbf{u}_0)$ is achieved in $\mathcal{L}(\Omega,\text{curl},\mathbf{u}_0)$, and in case $\mathcal{R}_h(\mathbf{u}_0)$ is not achieved, the minimizer of $\mathcal{E}_\varepsilon$ converges to a minimizer of $\mathcal{R}_l(\mathbf{u}_0)$.

Due to the complexity of these problems, in this paper we only discuss the special case where the domain is the unit disk in the plane. We assume that $\mathbf{u}_0$ makes a constant angle ϕ_0 with the normal vector of ∂D, i.e.,

$$\mathbf{u}_0 = e^{i[\theta + \phi_0]} \text{ on } \partial D. \tag{1.11}$$

Without loss of generality we assume $0 \leqslant \phi_0 \leqslant \pi/2$. Note that when $\phi_0 = 0$, $\mathbf{u}_0 = x/|x|$ on ∂D, which is in the outer normal direction; and when $\phi_0 = \pi/2$, $\mathbf{u}_0 = (-x_2, x_1)$ on ∂D, which is in the tangential direction. We shall prove that $\mathcal{R}_h(\mathbf{u}_0)$ is not achieved but $\mathcal{R}_l(\mathbf{u}_0)$ is achieved, and the minimizer $\mathbf{u}_\varepsilon$ of the functional $\mathcal{E}_\varepsilon$ converges weakly to the minimizer of $\mathcal{R}_l(\mathbf{u}_0)$.

Theorem 4: *Assume D is the unit disc in the plane and* $\mathbf{u}_0$ *is given in (1.11). Then we have the following results:*

(1) $\mathcal{R}_l(\mathbf{u}_0) = 4\pi \sin^2 \phi_0$, *and the unique minimizer of* $\mathcal{R}_l(\mathbf{u}_0)$ *is*

$$\mathbf{v} = e^{i[\theta + \arcsin(r \sin \phi_0)]}. \tag{1.12}$$

(2) *Let* $\mathbf{u}_\varepsilon$ *be the minimizer of* $\mathcal{E}_\varepsilon$. *When* $\mathbf{u}_0$ *is not tangential to the boundary* ∂D, *then* $\mathbf{u}_\varepsilon \to \mathbf{v}$ *weakly in* $H^1_{\text{loc}}(\overline{D}\setminus\{0\})$ *as* $\varepsilon \to 0$.

Remark 1.3: For the unit disk case, under the conditions of Theorem 4,

$$\mathcal{R}_h(\mathbf{u}_0) = \mathcal{R}_l(\mathbf{u}_0) = \mathcal{R}(\mathbf{u}_0).$$

Both $\mathcal{R}(\mathbf{u}_0)$ and $\mathcal{R}_l(\mathbf{u}_0)$ are achieved, but $\mathcal{R}_h(\mathbf{u}_0)$ is not achieved.

Two special cases are particularly interesting.

When $\phi_0 = \pi/2$, $\mathbf{u}_0$ is a tangential field on ∂D. We will see that for any planar domain Ω and any tangential field $\mathbf{u}_0$,

$$\mathcal{R}_l(\mathbf{u}_0) \geqslant 4\pi,$$

and the equality holds if and only if Ω is a disk, see Corollary 3.4 for more details.

When $\phi_0=0$, $\mathbf{v}(x)=x/|x|$. For any $\phi_0\neq 0$,$\mathbf{v}$ has a unique singular point at the origin with the same singularity as $x/|x|$.

Also note that, when $\phi_0=0$, $\mathcal{R}_l(\mathbf{u}_0)=0$ and $\mathbf{v}(x)=x/|x|$ is a *central field*. More precisely, if we write $\mathbf{v}=e^{i\psi}$, then ψ satisfies

$$\cos\psi\partial_1\psi+\sin\psi\partial_2\psi=0 \tag{1.13}$$

for $x\neq 0$, and the characteristic lines of Eq. (1.13) intersect with each other only at the origin $x=0$.

In general, given a vector field $\mathbf{u}_0$ defined on $\partial\Omega$, we say that $\mathbf{u}_0$ can be extended to become a central field, if there exists a (single-valued or multiple-valued) function ψ which satisfies (1.13) on $\Omega\backslash\{P\}$, such that $e^{i\psi}=\mathbf{u}_0$ on $\partial\Omega$, where P is a point in Ω. Therefore the characteristic lines of (1.13) starting from $(\partial\Omega,\psi|_{\partial\Omega})$ intersect with each other at one point only. It is easy to show that, if $\mathbf{u}_0$ can be extended to become a central field in Ω, then $\mathcal{R}_l(\mathbf{u}_0)=0$. In Sec. III we will discuss the characterization of curl-free unit planar vector fields from the view point of diffeomorphism.

The other interesting case where the variational problems for functionals $\mathcal{F}$ and $\mathcal{E}_\varepsilon$ can be thoroughly discussed is when Ω is a cylindrical domain. Since a related problem has been treated in Ref. 9, so we shall not present the results here. We mention that the variational problems in circular cylinders of various related functionals have been studied by Cladis and Kléman,[10] Mayer,[11] Bethuel, Brezis, Coleman, and Hélein,[12] Mizel, Poccato, and Virga.[13]

We should also mention that, the mathematical theory of liquid crystals has been studied by many authors, see for instance, Ambrosio,[14,15] Aviles and Giga,[4,16] Ambrosio and Virga,[17] Brezis,[18] Chou,[19] Hardt and Kinderlehrer,[5] Hardt, Kinderlehrer, and Lin,[20] Hardt and Lin,[21] Lin,[6,22,23] Calderer and Palffy-Muhoray,[24] Calderer, Liu, and Voss[27], and the references therein.

This paper is organized as follows: In Sec. II we discuss the variational problem for the functional $\mathcal{W}(u)$. Theorems 1 and 2 will be proven there. In Sec. III we discuss the variational problem for $\mathcal{E}_\varepsilon$ and $\mathcal{F}$ in general domains, and prove Theorem 3. We also study the characterization of curl-free unit planar vector fields. In Sec. IV we discuss the variational problems for $\mathcal{E}_\varepsilon$ and $\mathcal{F}$ in a disk in $\mathbb{R}^2$, and our special interest is on the case when $R_h(\mathbf{u}_0)$ is not achieved but $\mathcal{R}_l(\mathbf{u}_0)$ is achieved.

II. VARIATIONAL PROBLEMS FOR FUNCTIONAL $\mathcal{W}$ IN GENERAL DOMAINS

In this section we discuss the asymptotic behavior, as $k_2\to+\infty$, of the minimizers of the functional $\mathcal{W}$ given in (1.1). In particular, we shall prove Theorems 1 and 2. For convenience we assume Ω is a smooth bounded domain in $\mathbb{R}^3$. Our arguments also work for two dimensional domains. Denote $1/k_2=\varepsilon^2$ and rewrite $\mathcal{W}$ as

$$\mathcal{W}_\varepsilon(\mathbf{u})=\int_\Omega\left\{\frac{k_1}{2}|\nabla\mathbf{u}|^2+\frac{1}{2\varepsilon^2}|\mathrm{curl}\,\mathbf{u}|^2+\psi(|\mathbf{u}|)\right\}dx, \tag{2.1}$$

where k_1 and ε are positive constants with ε being very small, and $\psi(s)$ is a smooth function satisfying (1.2), see, Ref. 6, p. 808. Given $\mathbf{u}_0\in H^{1/2}(\partial\Omega)\cap L^\infty(\partial\Omega)$, we denote $H(\Omega,\mathbf{u}_0)=\{\mathbf{u}\in H^1(\Omega,\mathbb{R}^3):\mathbf{u}=\mathbf{u}_0 \text{ on } \partial\Omega\}$, and set

$$A_\varepsilon(\mathbf{u}_0)=\inf_{\mathbf{u}\in H(\Omega,\mathbf{u}_0)}\mathcal{W}_\varepsilon(\mathbf{u}).$$

We denote a minimizer of $\mathcal{W}_\varepsilon$ in $H(\Omega,\mathbf{u}_0)$ by $\mathbf{u}_\varepsilon$. As mentioned in the Introduction, we shall prove that, for a sequence ε_n goes to 0, there is a subsequence, which we still write as $\{\varepsilon_n\}$, such that the corresponding $\mathbf{u}_{\varepsilon_n}$ converges to a minimizer of the curl functional. Therefore we begin with discussions of the curl functional.

Denote by ν the unit outer normal to $\partial\Omega$. Define

$$H_\nu(\Omega;\mathrm{div})=\{\mathbf{u}\in H^1(\Omega,\mathbb{R}^3):\mathrm{div}\,u=0 \;\text{ in }\; \Omega,\mathbf{u}\cdot\nu=0 \;\text{ on }\; \partial\Omega\}.$$

The norm in $H_\nu(\Omega;\mathrm{div})$ is

$$\|\mathbf{u}\|=(\|\mathrm{curl}\,\mathbf{u}\|^2_{L^2(\Omega)}+\|\mathbf{u}\|^2_{L^2(\Omega)})^{1/2},$$

which is equivalent to the usual H^1 norm in this space.

Lemma 2.1: For all $\mathbf{w}\in H_\nu(\Omega;\mathrm{div})$, *it holds that*

$$\int_\Omega|\mathrm{curl}\,\mathbf{w}|^2dx=\int_\Omega|\nabla\mathbf{w}|^2dx-\int_{\partial\Omega}\Pi_\nu(\mathbf{w},\mathbf{w})dS, \tag{2.2}$$

where Π_ν *is the second fundamental form of* $\partial\Omega$.

Proof: In the following we denote by $\langle\cdot,\cdot\rangle$ *the inner product in* $\mathbb{R}^3$.

Step 1. First we assume that $\mathbf{w}\in C^2(\overline{\Omega})\cap H_\nu(\Omega;\mathrm{div})$.

Since,

$$|\mathrm{curl}\,\mathbf{w}|^2=\mathrm{div}(\mathbf{w}\wedge\mathrm{curl}\,\mathbf{w})-\mathbf{w}\Delta\mathbf{w}+\mathbf{w}\nabla(\mathrm{div}\,\mathbf{w})$$

and $\mathrm{div}\,\mathbf{w}=0$ in Ω, we have

$$\int_\Omega|\mathrm{curl}\,\mathbf{w}|^2dx=\int_\Omega|\nabla\mathbf{w}|^2dx+\int_{\partial\Omega}\left\{(\mathbf{w}\wedge\mathrm{curl}\,\mathbf{w})\cdot\nu-\mathbf{w}\cdot\frac{\partial\mathbf{w}}{\partial\nu}\right\}dS. \tag{2.3}$$

Now we prove that on the boundary $\partial\Omega$,

$$(\mathbf{w}\wedge\mathrm{curl}\,\mathbf{w})\cdot\nu-\mathbf{w}\cdot\frac{\partial\mathbf{w}}{\partial\nu}=-\Pi_\nu(\mathbf{w},\mathbf{w}). \tag{2.4}$$

Fix a point $P\in\partial\Omega$. Without loss of generality we may assume $P=O$, the origin. After rotating the coordinates we may assume that, at the point P, $\nu=(0,0,1)$. Since $\mathbf{w}\cdot\nu=0$ on $\partial\Omega$ we see that $\mathbf{w}(P)=(w_1(P),w_2(P),0)$. So at P we have

$$(\mathbf{w}\wedge\mathrm{curl}\,\mathbf{w})\cdot\nu=\mathbf{w}\cdot\frac{\partial\mathbf{w}}{\partial\nu}-w_1\partial_1w_3-w_2\partial_2w_3.$$

When x is near P and on the boundary $\partial\Omega$, the unit outer normal ν can be represented by

$$x_3=\phi(x')=-\frac{1}{2}\sum_{i=1}^{2}\alpha_ix_i^2+O(|x'|^3),$$

$$\nu=\frac{1}{\sqrt{1+|\nabla\phi|^2}}(-\partial_1\phi,-\partial_2\phi,1),$$

where $x'=(x_1,x_2)$, and α_1,α_2 are the principal curvatures of $\partial\Omega$ at P. Since $\mathbf{w}\cdot\nu=0$ on $\partial\Omega$ we have

$$w_3=w_1\partial_1\phi+w_2\partial_2\phi\quad\text{on}\quad\partial\Omega. \tag{2.5}$$

Since $\nu=(0,0,1)$ at the point P, we have $\nabla\phi(P)=0$. It follows from (2.5) that, at the point P,

$$\begin{aligned}w_1\partial_1w_3+w_2\partial_2w_3&=w_1^2\partial_{11}\phi(P)+2w_1w_2\partial_{12}\phi(P)+w_2^2\partial_{22}\phi(P)\\&=\langle[D^2\phi(P)]\mathbf{w},\mathbf{w}\rangle=-\Pi_\nu(\mathbf{w},\mathbf{w}).\end{aligned}$$

This verifies (2.4). Now (2.2) follows from (2.3) and (2.4).

Step 2. Next we assume $\mathbf{w}\in H_\nu(\Omega;\mathrm{div})$.

We can write $\mathbf{w}=w_1\tau_1+w_2\tau_2$, where (τ_1,τ_2) is the orthogonal tangent field of $\partial\Omega$, and $\Pi_\nu(\tau_i,\tau_j)=a_{ij}$. Then,

$$\Pi_\nu(\mathbf{w},\mathbf{w})=a_{ij}w_iw_j=a_{ij}\langle\mathbf{w},\tau_i\rangle\langle\mathbf{w},\tau_j\rangle.$$

Thus $|\Pi_\nu(\mathbf{w},\mathbf{w})|\leqslant C|\mathbf{w}|^2$ on $\partial\Omega$. It follows that the functional

$$\int_{\partial\Omega}\Pi_\nu(\mathbf{w},\mathbf{w})dS$$

is continuous in $H_\nu(\Omega;\mathrm{div})$ with respect to the H^1 norm. Since $C^2(\overline{\Omega})$ is a dense subset in $H_\nu(\Omega;\mathrm{div})$, we easily see that (2.2) is valid for all $\mathbf{w}\in H_\nu(\Omega;\mathrm{div})$. □

Corollary 2.2: Assume Ω *is a smooth, bounded, convex domain in* $\mathbb{R}^3$. *Then there exists a constant* $C(\Omega)>0$ *such that*

$$\int_\Omega|\nabla\mathbf{w}|^2dx\leqslant\int_\Omega|\mathrm{curl}\,\mathbf{w}|^2dx\leqslant C(\Omega)\left\{\int_\Omega|\nabla\mathbf{w}|^2dx+\int_\Omega|\mathbf{w}|^2dx\right\}\tag{2.6}$$

for all $\mathbf{w}\in H_\nu(\Omega;\mathrm{div})$.

Proof: When Ω is convex, $\Pi_\nu(\mathbf{w},\mathbf{w})\leqslant 0$ for all $\mathbf{w}\in H_\nu(\Omega;\mathrm{div})$. Hence (2.2) implies

$$\int_\Omega|\nabla\mathbf{w}|^2dx\leqslant\int_\Omega|\mathrm{curl}\,\mathbf{w}|^2dx.$$

On the other hand, by the Sobolev embedding theorem,

$$\int_{\partial\Omega}\Pi_\nu(\mathbf{w},\mathbf{w})dS\leqslant C_1(\Omega)\int_{\partial\Omega}|\mathbf{w}|^2dS\leqslant C(\Omega)\|\mathbf{w}\|^2_{H^1(\Omega)}.$$

Thus (2.6) follows from (2.2). □

Now we are ready to prove Theorem 1.

Proof of Theorem 1:

Step 1. For any $\mathbf{u}\in H^1(\Omega,\mathbb{R}^3)$, we can decompose $\mathbf{u}$ such that $\mathbf{u}=\mathbf{v}+\nabla\varphi$, where $\mathbf{v}\in H^1_\nu(\Omega;\mathrm{div})$ and $\varphi\in H^2(\Omega)$ is a solution of the following Neumann problem:

$$\Delta\varphi=\mathrm{div}\,\mathbf{u}\ \text{ in }\ \Omega,\ \ \frac{\partial\varphi}{\partial\nu}=\mathbf{u}\cdot\nu\ \text{ on }\ \partial\Omega.\tag{2.7}$$

Note that φ is unique modulo an additive constant. We may assume

$$\int_\Omega\varphi dx=0.$$

It is easily seen that

$$\int_\Omega|\nabla\varphi-\mathbf{u}|^2dx\leqslant\int_\Omega|\nabla f-\mathbf{u}|^2dx\tag{2.8}$$

for all $f\in H^1(\Omega)$. By the choice of φ, it is obvious that $\mathbf{v}=\mathbf{u}-\nabla\varphi\in H_\nu(\Omega;\mathrm{div})$.

Step 2. Recall the definition $\mathcal{R}(\mathbf{u}_0)$ in (1.5). Let $\{\mathbf{u}_n\}\subset H(\Omega,u_0)$ be a minimizing sequence of $\mathcal{R}(\mathbf{u}_0)$. As in Step 1, we decompose $\mathbf{u}_n$ as

$$\mathbf{u}_n=\mathbf{v}_n+\nabla\varphi_n,$$

where $\mathbf{v}_n \in H_\nu(\Omega;\text{div})$ and φ_n is the solution of (2.7) with $\mathbf{u}$ replaced by $\mathbf{u}_n$. Then we have

$$\int_\Omega |\text{curl}\, \mathbf{v}_n|^2 dx = \int_\Omega |\text{curl}\, \mathbf{u}_n|^2 dx \to \mathcal{R}(\mathbf{u}_0) \quad \text{as} \quad n\to\infty.$$

Claim 1: $\|\nabla \mathbf{v}_n\|_{L^2(\Omega)}$ is bounded, i.e.,

$$\int_\Omega |\nabla \mathbf{v}_n|^2 dx \leqslant C, \quad n=1,2,\ldots. \tag{2.9}$$

Proof of Claim 1: Suppose (2.9) were false. Then we may assume

$$C_n = \|\nabla \mathbf{v}_n\|_{L^2(\Omega)} \to \infty \quad \text{as} \quad n\to\infty.$$

Set $\mathbf{w}_n = \mathbf{v}_n / C_n$. We have two cases to consider.

Case 1: $\int_\Omega |\mathbf{w}_n|^2 dx \geqslant a_0$ for $a_0 > 0$ and all $n \geqslant 1$.

In this case, we denote

$$\underline{\mathbf{w}}_n = \frac{1}{|\Omega|}\int_\Omega \mathbf{w}_n dx, \quad \tilde{\mathbf{w}}_n = \mathbf{w}_n - \underline{\mathbf{w}}_n.$$

Then

$$\int_\Omega |\nabla \tilde{\mathbf{w}}_n|^2 dx = 1 \quad \text{and} \quad \int_\Omega |\text{curl}\, \tilde{\mathbf{w}}_n|^2 dx \to 0 \quad \text{as} \quad n\to\infty.$$

By Poincaré inequality we have

$$\int_\Omega |\tilde{\mathbf{w}}_n|^2 dx \leqslant C(\Omega) \int_\Omega |\nabla \tilde{\mathbf{w}}_n|^2 dx = C(\Omega).$$

Passing to a subsequence we may assume $\tilde{\mathbf{w}}_n \to \tilde{\mathbf{w}}$ weakly in $H^1(\Omega,\mathbb{R}^3)$. Therefore $\text{curl}\, \tilde{\mathbf{w}}_n \to \text{curl}\, \tilde{\mathbf{w}}$ weakly in $L^2(\Omega,\mathbb{R}^3)$. On the other hand, $\text{curl}\, \tilde{\mathbf{w}}_n \to 0$ strongly in $L^2(\Omega,\mathbb{R}^3)$. Thus $\text{curl}\, \tilde{\mathbf{w}} = 0$. Since Ω is simply connected, there exists a function $\tilde{f} \in H^2(\Omega)$ such that $\tilde{\mathbf{w}} = \nabla \tilde{f}$. Now we see that

$$\mathbf{w}_n = \tilde{\mathbf{w}}_n + \underline{\mathbf{w}}_n = \nabla \tilde{f} + \nabla h_n + \mathbf{g}_n,$$

where $h_n = \underline{\mathbf{w}}_n \cdot x$, and $\mathbf{g}_n \to 0$ weakly in $H^1(\Omega,\mathbb{R}^3)$. Hence

$$\mathbf{u}_n = \nabla \varphi_n + \mathbf{v}_n = \nabla \varphi_n + \nabla (C_n \tilde{f} + C_n h_n) + C_n \mathbf{g}_n.$$

It follows from (2.8) that

$$\int_\Omega |\mathbf{v}_n|^2 dx = \int_\Omega |\mathbf{u}_n - \nabla \varphi_n|^2 dx \leqslant \int_\Omega |\mathbf{u}_n - \nabla(\varphi_n + C_n \tilde{f} + C_n h_n)|^2 dx = C_n \int_\Omega |\mathbf{g}_n|^2 dx.$$

Therefore,

$$0 < a_0 \leqslant \int_\Omega |\mathbf{w}_n|^2 dx \leqslant \int_\Omega |\mathbf{g}_n|^2 dx \to 0 \quad \text{as} \quad n\to\infty,$$

which is a contradiction.

Case 2: $\int_\Omega |\mathbf{w}_n|^2 dx \to 0$ as $n\to\infty$.

In this case we have $\mathbf{w}_n \to 0$ weakly in $H^1(\Omega,\mathbb{R}^3)$ and strongly in $L^2(\partial\Omega)$. Thus $\int_{\partial\Omega}\Pi_\nu(\mathbf{w}_n,\mathbf{w}_n)dS\to 0$. From (2.2) we have

$$1=\int_\Omega|\nabla\mathbf{w}_n|^2dx=\int_\Omega|\mathrm{curl}\,\mathbf{w}_n|^2dx+\int_{\partial\Omega}\Pi_\nu(\mathbf{w},\mathbf{w})dS\to 0,$$

again a contradiction.

Thus Claim 1 is true.

Step 3. Now we show that $\mathcal{R}(\mathbf{u}_0)$ is achieved.

Denote

$$\underline{\mathbf{v}}_n=\frac{1}{|\Omega|}\int_\Omega \mathbf{v}_n dx,\quad \tilde{\mathbf{v}}_n=\mathbf{v}_n-\underline{\mathbf{v}}_n\,.$$

From (2.9) and Poincaré inequality we have

$$\int_\Omega|\tilde{\mathbf{v}}_n|^2dx\leqslant C_1\,.$$

Therefore we have, after passing to a subsequence, $\tilde{\mathbf{v}}_n\to\tilde{\mathbf{v}}$ weakly in $H^1(\Omega,\mathbb{R}^3)$ and weakly in $H^{1/2}(\partial\Omega)$. Denote

$$f_n=\varphi_n+\underline{\mathbf{v}}_n\cdot x,\quad \tilde{f}_n=f_n-\frac{1}{|\partial\Omega|}\int_{\partial\Omega}f_n dS.$$

Then

$$\nabla\tilde{f}_n=\mathbf{u}_n-\tilde{\mathbf{v}}_n\to\mathbf{u}_0-\tilde{\mathbf{v}}|_{\partial\Omega}\ \text{ weakly in }\ H^{1/2}(\partial\Omega).$$

Now we consider the following minimization problem:

$$\lambda_n=\inf\left\{\int_\Omega|D^2\phi|^2dx:\nabla\phi=\nabla\tilde{f}_n\ \text{ on }\ \partial\Omega,\ \ \int_\Omega\phi dx=0\right\}.$$

By choosing $\phi=\tilde{f}_n$ as a test function, we see that $\lambda_n<\infty$. Since the functional involved is convex, we see that a minimizer exists, which is denoted by ϕ_n. Then $\nabla\phi_n=\nabla\tilde{f}_n$ on $\partial\Omega$ and

$$\int_\Omega\phi_n=0.$$

Since

$$\nabla\tilde{f}_n|_{\partial\Omega}=\mathbf{u}_0-\tilde{\mathbf{v}}_n|_{\partial\Omega}\to\mathbf{u}_0-\tilde{\mathbf{v}}|_{\partial\Omega}\,,$$

we have $\nabla\phi_n|_{\partial\Omega}\to\mathbf{u}_0-\tilde{\mathbf{v}}|_{\partial\Omega}$ weakly in $H^{1/2}(\partial\Omega)$. Thus $\{\lambda_n\}$ is bounded and $\{\phi_n\}$ is bounded in $H^2(\Omega)$.

Therefore, we may assume that $\phi_n\to\phi$ weakly in $H^2,\nabla\phi|_{\partial\Omega}=\mathbf{u}_0-\tilde{\mathbf{v}}|_{\partial\Omega}$. Set $\tilde{\mathbf{u}}=\tilde{\mathbf{v}}+\nabla\phi$. Then $\tilde{\mathbf{u}}|_{\partial\Omega}=\mathbf{u}_0$, $\mathrm{curl}\,\tilde{\mathbf{u}}=\mathrm{curl}\,\tilde{\mathbf{v}}$, and

$$\begin{aligned}\mathcal{R}(\mathbf{u}_0)\leqslant\int_\Omega|\mathrm{curl}\,\tilde{\mathbf{u}}|^2dx&=\int_\Omega|\mathrm{curl}\,\tilde{\mathbf{v}}|^2dx\\&\leqslant\liminf_{n\to\infty}\int_\Omega|\mathrm{curl}\,\tilde{\mathbf{v}}_n|^2dx\\&=\liminf_{n\to\infty}\int_\Omega|\mathrm{curl}\,\mathbf{v}_n|^2dx=\mathcal{R}(\mathbf{u}_0).\end{aligned}$$

That is, $\tilde{\mathbf{u}}\in H(\Omega,u_0)$ is a minimizer of $\mathcal{R}(\mathbf{u}_0)$.

Step 4. Assume $\mathbf{u}$, $\mathbf{v}\in H(\Omega,\mathbf{u}_0)$ are minimizers of $\mathcal{R}(\mathbf{u}_0)$. Then $\mathrm{curl}(\mathbf{u}-\mathbf{v})=0$ a.e. in Ω.

In fact, for $0<\alpha<1,\beta=1-\alpha$, set $\mathbf{w}=\alpha\mathbf{u}+\beta\mathbf{v}$. Then $\mathbf{w}\in H(\Omega,\mathbf{u}_0)$, and we have

$$\begin{aligned}\mathcal{R}(\mathbf{u}_0)\leqslant\int_\Omega|\mathrm{curl}\,\mathbf{w}|^2dx&=\alpha^2\int_\Omega|\mathrm{curl}\,\mathbf{u}|^2dx+2\alpha\beta\int_\Omega\mathrm{curl}\,\mathbf{u}\cdot\mathrm{curl}\,\mathbf{v}\,dx+\beta^2\int_\Omega|\mathrm{curl}\,\mathbf{v}|^2dx\\&\leqslant(\alpha^2+\beta^2)\mathcal{R}(\mathbf{u}_0)+2\alpha\beta\int_\Omega|\mathrm{curl}\,\mathbf{u}|\cdot|\mathrm{curl}\,\mathbf{v}|dx\\&\leqslant(\alpha^2+\beta^2)\mathcal{R}(\mathbf{u}_0)+\alpha\beta\left\{\int_\Omega|\mathrm{curl}\,\mathbf{u}|^2dx+\int_\Omega|\mathrm{curl}\,\mathbf{v}|^2dx\right\}\\&=(\alpha+\beta)^2\mathcal{R}(\mathbf{u}_0)=\mathcal{R}(\mathbf{u}_0).\end{aligned}$$

Thus $\mathrm{curl}\,\mathbf{u}=C\,\mathrm{curl}\,\mathbf{v}$ a.e. in Ω for some constant $C>0$. Then

$$\mathcal{R}(\mathbf{u}_0)=\int_\Omega|\mathrm{curl}\,\mathbf{u}|^2dx=C^2\int_\Omega|\mathrm{curl}\,\mathbf{v}|^2dx=C^2\mathcal{R}(\mathbf{u}_0).$$

Therefore $C=1$ and $\mathrm{curl}\,\mathbf{u}=\mathrm{curl}\,\mathbf{v}$ a.e. in Ω.

Step 5. Recall that Ω is simply connected. Now we show that the set $\Sigma(\mathbf{u}_0)$ of all the minimizers of $\mathcal{R}(\mathbf{u}_0)$ is given by (1.6).

If $\mathbf{u}$ and $\mathbf{v}$ are both minimizers of $\mathcal{R}(\mathbf{u}_0)$, then from Step 4, there is $\varphi\in H^2(\Omega)$ such that $\mathbf{v}=\mathbf{u}+\nabla\varphi$, and $\Delta\varphi=0$ on $\partial\Omega$.

Fix a minimizer $\overline{\mathbf{u}}$ of $\mathcal{R}(\mathbf{u}_0)$. Then we see (1.6) is true.

The proof of Theorem 1 is now complete. □

In the following we show that, among all the minimizers of the curl functional, there exists a vector field $\mathbf{u}$ which has the least value of $\mathcal{J}_0(\mathbf{u})$, where $\mathcal{J}_0$ is defined in (1.4).

Set

$$\begin{aligned}a(\mathbf{u}_0)&=\inf_{\mathbf{u}\in H(\Omega,\mathbf{u}_0)}\mathcal{J}_0(\mathbf{u}),\\b(\mathbf{u}_0)&=\inf_{\mathbf{u}\in\Sigma(\mathbf{u}_0)}\mathcal{J}_0(\mathbf{u}).\end{aligned}\tag{2.10}$$

Obviously $a(\mathbf{u}_0)\leqslant b(\mathbf{u}_0)$.

Proposition 2.3: Assume ψ satisfies the condition (1.2), and $\mathbf{u}_0\in H^{1/2}(\partial\Omega)\cap L^\infty(\partial\Omega)$. Then $b(\mathbf{u}_0)$ is achieved, that is, there exists a $\varphi\in H^2(\Omega)$, with $\nabla\varphi=0$ on $\partial\Omega$, such that $\mathcal{J}_0(\overline{\mathbf{u}}+\nabla\varphi)=b(\mathbf{u}_0)$.

Proof: For $\mathbf{u}=\overline{\mathbf{u}}+\nabla\varphi\in\Sigma(\mathbf{u}_0)$,

$$\mathcal{J}_0(\mathbf{u})=\mathcal{J}_0(\overline{\mathbf{u}}+\nabla\varphi)=\int_\Omega\left\{\frac{k_1}{2}|\nabla\overline{\mathbf{u}}+D^2\varphi|^2+\psi(|\overline{\mathbf{u}}+\nabla\varphi|)\right\}dx.$$

Assume $\overline{\mathbf{u}}+\nabla\varphi_n\in\Sigma(\mathbf{u}_0)$ is a minimizing sequence of $\mathcal{J}_0$ on $\Sigma(\mathbf{u}_0)$. Then $\overline{\mathbf{u}}+\nabla\varphi_n$ is bounded in $H^1(\Omega)$. Passing to a subsequence we have $\nabla\varphi_n\to\nabla\overline{\varphi}$ weakly in H^1 and strongly in $L^2(\partial\Omega)$. Thus $\nabla\overline{\varphi}=0$ on $\partial\Omega$.

Denote $\mathbf{u}_n=\overline{\mathbf{u}}+\nabla\varphi_n$, and $\tilde{\mathbf{u}}=\overline{\mathbf{u}}+\nabla\overline{\varphi}$. Then $\mathbf{u}_n\to\tilde{\mathbf{u}}$ weakly in $H^1(\Omega,\mathbb{R}^3)$ and strongly in $L^p(\Omega,\mathbb{R}^3)$ for all $1<p<6$, and

$$\int_\Omega|\nabla\tilde{\mathbf{u}}|^2dx=\int_\Omega|\nabla\overline{\mathbf{u}}+D^2\overline{\varphi}|^2dx\leqslant\liminf_{n\to\infty}\int_\Omega|\nabla\mathbf{u}_n|^2dx.$$

Since the function ψ satisfies the condition (1.2), we have

$$\int_\Omega\psi(|\tilde{\mathbf{u}}|)dx\leqslant\liminf_{n\to\infty}\int_\Omega\psi(|\mathbf{u}_n|)dx.$$

Therefore $\mathcal{J}_0(\tilde{\mathbf{u}})=\inf_{\mathbf{u}\in\Sigma(\mathbf{u}_0)}\mathcal{J}_0(\mathbf{u})$. □

Replacing $\overline{\mathbf{u}}$ by $\tilde{\mathbf{u}}$ if necessary, we always assume that the vector field $\overline{\mathbf{u}}$ in (1.6) satisfies

$$\int_\Omega|\mathrm{curl}\,\overline{\mathbf{u}}|^2dx=\mathcal{R}(\mathbf{u}_0),\quad \mathcal{J}_0(\overline{\mathbf{u}})=b(\mathbf{u}_0). \tag{2.11}$$

In the following we show the existence of minimizers of the functional $\mathcal{W}_\varepsilon$ given in (2.1) before we prove Theorem 2. Recall the definition of the minimum value $A_\varepsilon(\mathbf{u}_0)$ given at the beginning of this section.

Theorem 2.4: *Given* $\mathbf{u}_0\in H^{1/2}(\partial\Omega)\cap L^\infty(\partial\Omega)$, *there exists* $\mathbf{u}_\varepsilon\in H(\Omega,\mathbf{u}_0)$ *such that* $\mathcal{W}_\varepsilon(\mathbf{u}_\varepsilon)=A_\varepsilon(\mathbf{u}_0)$.

Proof: Assume $\{\mathbf{u}_n\}\subset H(\Omega,\mathbf{u}_0)$ such that $\mathcal{W}_\varepsilon(\mathbf{u}_n)\to A_\varepsilon(\mathbf{u}_0)$ as $n\to\infty$. After passing to a subsequence we may assume $\mathbf{u}_n\to\mathbf{u}_\varepsilon$ weakly in $H^1(\Omega,\mathbb{R}^3)$ and $\mathbf{u}_\varepsilon\in H(\Omega,u_0)$. Hence $\mathrm{curl}\,\mathbf{u}_n\to\mathrm{curl}\,\mathbf{u}_\varepsilon$ weakly in $L^2(\Omega,\mathbb{R}^3)$. By condition (1.2) we find $\mathcal{W}_\varepsilon(\mathbf{u}_\varepsilon)\leqslant\liminf_{n\to\infty}\mathcal{W}_\varepsilon(\mathbf{u}_n)$. Thus $\mathcal{W}_\varepsilon(\mathbf{u}_\varepsilon)=A_\varepsilon(\mathbf{u}_0)$. □

Remark 2.1: It is easy to see that, a minimizer $\mathbf{u}_\varepsilon$ of the functional $\mathcal{W}_\varepsilon$ satisfies the following equation in the weak sense:

$$\begin{cases}-k_1\Delta\mathbf{u}+\dfrac{1}{\varepsilon^2}\mathrm{curl}^2\,\mathbf{u}+\xi(|\mathbf{u}|)\mathbf{u}=0 & \text{in } \Omega,\\ \mathbf{u}=\mathbf{u}_0 & \text{on } \partial\Omega,\end{cases}$$

where $\xi(s)=\psi'(s)/s$.

Proof of Theorem 2: In the following we always denote by $\mathbf{u}_\varepsilon$ a minimizer of $\mathcal{W}_\varepsilon$ in $H(\Omega,\mathbf{u}_0)$. First, we recall the definitions of $a(\mathbf{u}_0)$ and $b(\mathbf{u}_0)$ given in (2.10), and $\mathcal{R}(\mathbf{u}_0)$ given in (1.5).

For every $\mathbf{u}\in H(\Omega,\mathbf{u}_0)$ we have

$$\mathcal{J}_0(\mathbf{u})\geqslant a(\mathbf{u}_0)\quad\text{and}\quad\int_\Omega|\mathrm{curl}\,\mathbf{u}|^2dx\geqslant\mathcal{R}(\mathbf{u}_0).$$

Hence,

$$\mathcal{W}_\varepsilon(\mathbf{u})=\mathcal{J}_0(\mathbf{u})+\frac{1}{2\varepsilon^2}\int_\Omega|\mathrm{curl}\,\mathbf{u}|^2dx\geqslant a(\mathbf{u}_0)+\frac{\mathcal{R}(\mathbf{u}_0)}{2\varepsilon^2}.$$

Let $\overline{\mathbf{u}}$ be a minimizer of the curl functional that satisfies (2.11). Then

$$\mathcal{W}_\varepsilon(\overline{\mathbf{u}})=\mathcal{J}_0(\overline{\mathbf{u}})+\frac{\mathcal{R}(\mathbf{u}_0)}{2\varepsilon^2}=b(\mathbf{u}_0)+\frac{\mathcal{R}(\mathbf{u}_0)}{2\varepsilon^2}.$$

Thus,

$$a(\mathbf{u}_0)+\frac{\mathcal{R}(\mathbf{u}_0)}{2\varepsilon^2}\leqslant A_\varepsilon(\mathbf{u}_0)=\mathcal{W}_\varepsilon(\mathbf{u}_\varepsilon)\leqslant b(\mathbf{u}_0)+\frac{\mathcal{R}(\mathbf{u}_0)}{2\varepsilon^2}.$$

So,

$$a(\mathbf{u}_0)\leqslant \mathcal{J}_0(\mathbf{u}_\varepsilon)=\mathcal{W}_\varepsilon(\mathbf{u}_\varepsilon)-\frac{1}{2\varepsilon^2}\int_\Omega|\mathrm{curl}\,\mathbf{u}_\varepsilon|^2dx$$

$$\leqslant b(\mathbf{u}_0)-\frac{1}{2\varepsilon^2}\left\{\int_\Omega|\mathrm{curl}\,\mathbf{u}_\varepsilon|^2dx-\mathcal{R}(\mathbf{u}_0)\right\}\leqslant b(\mathbf{u}_0)=\mathcal{J}_0(\overline{\mathbf{u}}).$$

Hence,

$$a(\mathbf{u}_0)\leqslant \mathcal{J}_0(\mathbf{u}_\varepsilon)\leqslant b(\mathbf{u}_0).$$

We also have

$$\int_\Omega|\mathrm{curl}\,\mathbf{u}_\varepsilon|^2dx=2\varepsilon[\mathcal{W}_\varepsilon(\mathbf{u}_\varepsilon)-\mathcal{J}_0(\mathbf{u}_\varepsilon)]\leqslant\mathcal{R}(\mathbf{u}_0)+2\varepsilon[b(\mathbf{u}_0)-\mathcal{J}_0(\mathbf{u}_\varepsilon)]$$

$$\leqslant\mathcal{R}(\mathbf{u}_0)+2\varepsilon[b(\mathbf{u}_0)-a(\mathbf{u}_0)].$$

Sending ε to 0 we find

$$\limsup_{\varepsilon\to 0}\int_\Omega|\mathrm{curl}\,\mathbf{u}_\varepsilon|^2dx\leqslant\mathcal{R}(\mathbf{u}_0). \tag{2.12}$$

Since $\psi\geqslant 0$, we find that $\{\mathbf{u}_\varepsilon\}$ is bounded in $H^1(\Omega,\mathbb{R}^3)$. Passing to a subsequence we have $\mathbf{u}_\varepsilon\to u^*$ weakly in $H^1(\Omega,\mathbb{R}^3)$. So $\mathrm{curl}\,\mathbf{u}_\varepsilon\to\mathrm{curl}\,\mathbf{u}^*$ weakly in $L^2(\Omega,\mathbb{R}^3)$ as $\varepsilon\to 0$. Using (2.12) we have $\int_\Omega|\mathrm{curl}\,\mathbf{u}^*|^2dx\leqslant\mathcal{R}(\mathbf{u}_0)$. On the other hand, since $\mathbf{u}^*\in H(\Omega,\mathbf{u}_0)$, we also have

$$\int_\Omega|\mathrm{curl}\,\mathbf{u}^*|^2dx\geqslant\mathcal{R}(\mathbf{u}_0).$$

Hence,

$$\int_\Omega|\mathrm{curl}\,\mathbf{u}^*|^2dx=\mathcal{R}(\mathbf{u}_0),\quad\text{and}\quad \mathbf{u}^*\in\Sigma(\mathbf{u}_0),$$

where $\Sigma(\mathbf{u}_0)$ was defined in (1.6).

By the condition (1.2) we have

$$\int_\Omega\psi(|\mathbf{u}^*|)dx\leqslant\liminf_{\varepsilon\to 0}\int_\Omega\psi(|\mathbf{u}_\varepsilon|)dx.$$

Therefore,

$$b(\mathbf{u}_0)\leqslant\mathcal{J}_0(\mathbf{u}^*)\leqslant\liminf_{\varepsilon\to 0}\mathcal{J}_0(\mathbf{u}_\varepsilon)\leqslant b(\mathbf{u}_0).$$

Now we see that $\mathbf{u}^*$ also satisfies (2.11). We may assume $\mathbf{u}^*=\overline{\mathbf{u}}$. So,

$$\mathbf{u}_\varepsilon\to\overline{\mathbf{u}}\ \text{ weakly in }\ H^1(\Omega,\mathbb{R}^3),\ \text{ and }\ \mathcal{J}_0(\mathbf{u}_\varepsilon)\to b(\mathbf{u}_0)=\mathcal{J}_0(\overline{\mathbf{u}}).$$

Next we compute

$$\frac{k_1}{2}\int_\Omega |\nabla \overline{\mathbf{u}}|^2 dx = \mathcal{J}_0(\overline{\mathbf{u}}) - \int_\Omega \psi(|\overline{\mathbf{u}}|)dx \geq \mathcal{J}_0(\overline{\mathbf{u}}) - \liminf_{\varepsilon\to 0}\int_\Omega \psi(|\mathbf{u}_\varepsilon|)dx$$

$$= \mathcal{J}_0(\overline{\mathbf{u}}) - \liminf_{\varepsilon\to 0}\left\{ \mathcal{J}_0(\mathbf{u}_\varepsilon) - \int_\Omega \frac{k_1}{2}|\nabla \mathbf{u}_\varepsilon|^2 dx\right\}$$

$$= \mathcal{J}_0(\overline{\mathbf{u}}) - \lim_{\varepsilon\to 0} \mathcal{J}_0(\mathbf{u}_\varepsilon) + \limsup_{\varepsilon\to 0}\frac{k_1}{2}\int_\Omega |\nabla \mathbf{u}_\varepsilon|^2 dx$$

$$= \limsup_{\varepsilon\to 0}\frac{k_1}{2}\int_\Omega |\nabla \mathbf{u}_\varepsilon|^2 dx.$$

Thus $\mathbf{u}_\varepsilon \to \overline{\mathbf{u}}$ strongly in $H^1(\Omega,\mathbb{R}^3)$. This completes the proof of Theorem 2. □

An immediate consequence of Theorem 2 is the following:

Corollary 2.5: Assume Ω is a bounded, smooth, simply connected domain in $\mathbb{R}^n, n=2,3$. Assume ψ satisfies (1.2), and $\mathbf{u}_0 \in H^{1/2}(\partial\Omega)\cap L^\infty(\partial\Omega)$. Let $\mathbf{u}_\varepsilon$ be a minimizer of the functional $\mathcal{W}_\varepsilon$. Then the following conclusions are true:

(1) *As $\varepsilon\to 0$, the total energy $\mathcal{W}_\varepsilon(\mathbf{u}_\varepsilon)$ remains bounded if and only if $\mathcal{R}(\mathbf{u}_0)=0$.*
(2) *If $\mathcal{R}(\mathbf{u}_0)=0$, then for any convergent sequence of $\{\mathbf{u}_\varepsilon\}$ with $\varepsilon\to 0$, there is a function $\overline{\varphi} \in H^2(\Omega)$ with $\nabla\overline{\varphi}=\mathbf{u}_0$ on $\partial\Omega$, such that $\mathbf{u}_\varepsilon$ converges to $\nabla\overline{\varphi}$ strongly in $H^1(\Omega,\mathbb{R}^n)$ as $\varepsilon \to 0$, and*

$$\mathcal{J}_0(\nabla\overline{\varphi}) = \inf\{\mathcal{J}_0(\nabla\phi) : \phi\in H^2(\Omega), \nabla\phi|_{\partial\Omega}=\mathbf{u}_0\}.$$

Remark 2.2: Corollary 2.5 implies that the answer to the questions (Q1) and (Q2) is ''yes'' if and only if $\mathcal{R}(\mathbf{u}_0)=0$.

Corollary 2.5 also indicates that, it is interesting to find the exact conditions under which $\mathcal{R}(\mathbf{u}_0)=0$. The rest of this section is devoted to this problem. We will give a necessary and sufficient condition for $\mathcal{R}(\mathbf{u}_0)=0$ in the two-dimensional case. We shall see that in the three-dimensional case more geometric and topology conditions will be involved.

First we consider planar domains.

Proposition 2.6: Assume Ω is a smooth bounded domain in $\mathbb{R}^2$ and $\mathbf{u}_0 \in H^{1/2}(\partial\Omega)\cap L^\infty(\partial\Omega)$. Then $\mathcal{R}(\mathbf{u}_0)=0$ if and only if the following condition holds:

$$\int_{\partial\Omega} \mathbf{u}_0\cdot \tau ds = 0, \tag{2.13}$$

where τ is the unit tangential vector field of $\partial\Omega$.

As a direct consequence of Proposition 2.6, we have the following:

Corollary 2.7: Assume Ω is a smooth bounded domain in $\mathbb{R}^2$ and $\mathbf{u}_0 \in H^{1/2}(\partial\Omega)\cap L^\infty(\partial\Omega)$. If $\mathbf{u}_0$ is parallel to the normal direction of $\partial\Omega$ everywhere, then $\mathcal{R}(\mathbf{u}_0)=0$.

Proposition 2.6 follows from the following Lemma 2.8 and an approximation process, the details will be omitted.

Assume Ω is a C^k domain in $\mathbb{R}^n$ and $\mathbf{g}\in C^k(\partial\Omega, R^n)$. We say that $\mathbf{g}$ has a C^k curl-free extension if there exists a function $\phi\in C^{k+1}(\overline{\Omega})$ such that $\nabla\phi=\mathbf{g}$ on $\partial\Omega$.

Lemma 2.8: Assume Ω is a smooth (say C^{k+1}) bounded domain in $\mathbb{R}^2$, and $\mathbf{g}\in C^k(\partial\Omega,\mathbb{R}^2)$ for some integer $k>0$. Then $\mathbf{g}$ has a C^k curl-free extension if and only if (2.13) holds.

Proof: Denote by ν the unit outer normal of $\partial\Omega$ and τ the unit tangential vector. Denote $\mathbf{n} = -\nu$. Assume $\partial\Omega$ consists a finite number of smooth simple close curves $\Gamma_j, 1\leq j\leq m$. Each curve can be represented as $z=z_j(s), 0\leq s\leq L_j$, where s is the arc-length parameter, and $z_j(L_j)$

$=z_j(0)$. We choose the positive direction of Γ_j such that the orientation of $(\tau,\mathbf{n})$ is the same as that of x_1x_2-plane, and s is increasing along the positive direction of Γ_j. Then $\partial/\partial\tau=\partial/\partial s$ on $\partial\Omega$.

Let $d(x)=\mathrm{dist}(x,\partial\Omega)$ be the distance function from x to $\partial\Omega$, and for $\mu>0$ we set $\Omega(\mu)=\{x\in\overline{\Omega}:d(x)<\mu\}$. Then there is a constant $\mu_0>0$ such that $d\in C^k(\overline{\Omega(\mu_0)})$, and for every $x\in\Omega(\mu_0)$ there exists a unique point $z=z(x)\in\partial\Omega$ such that

$$x=z-d(x)\nu(z),\quad \nabla d(x)=-\nu(z).$$

Fix a positive constant $\mu<\mu_0$. Then

$$\Omega(\mu)=\bigcup_{j=1}^{m}D_j,\quad\text{where}\quad D_j=\{z(s)-t\nu(s):0\leqslant s\leqslant L_j,0\leqslant t<\mu\}.$$

First we assume that there exists $\phi\in C^{k+1}(\overline{\Omega})$ such that $\nabla\phi=\mathbf{g}$. Then

$$\int_{\Gamma_j}\mathbf{g}\cdot\tau ds=\int_{\Gamma_j}\nabla\phi\cdot\tau ds=\int_0^{L_j}\nabla\phi(z_j(s))\cdot z_j'(s)ds=\phi(z_j(L_j))-\phi(z_j(0))=0.$$

Hence (2.13) holds true.

In the following we assume (2.13) holds. We shall construct a function $\phi\in C^{k+1}(\overline{\Omega})$ such that $\nabla\phi=\mathbf{g}$ on $\partial\Omega$.

Let $\mathbf{g}_1$ be any C^k extension of $\mathbf{g}$, that is, $\mathbf{g}_1\in C^k(\overline{\Omega},\mathbb{R}^2)$ and $\mathbf{g}_1=\mathbf{g}$ on $\partial\Omega$.

Let ϕ_1 be the solution of the following equation:

$$\Delta\phi_1=\mathrm{div}\,\mathbf{g}_1\quad\text{in}\quad\Omega,\quad\frac{\partial\phi_1}{\partial\nu}=\mathbf{g}_1\cdot\nu\quad\text{on}\quad\partial\Omega,$$

and $\int_\Omega\phi_1dx=0$. Then $\phi_1\in C^{k+1}(\overline{\Omega})$.

Set $\mathbf{g}_2=\mathbf{g}_1-\nabla\phi_1$. Then

$$\mathbf{g}_2\in C^k(\overline{\Omega},\mathbb{R}^2),\quad\mathrm{div}\,\mathbf{g}_2=0\quad\text{in}\quad\Omega,\quad\mathbf{g}_2\cdot\nu=0\quad\text{on}\quad\partial\Omega.$$

Hence $\mathbf{g}_2=(\mathbf{g}_2\cdot\tau)\tau$ on $\partial\Omega$.

On each curve $\Gamma_j:z=z_j(s),0\leqslant s\leqslant L_j$, we define a function $\xi_j\in C^{k+1}[0,L_j]$ satisfying following conditions:

$$\begin{cases}\xi_j'(t)=g_2\cdot\tau=g_2(z_j(s))\cdot z_j'(s)\quad\text{for}\quad 0<s<L_j,\\ \xi_j(0)=0.\end{cases}$$

Note that ξ_j is unique, and

$$\frac{d^i\xi_j}{ds^i}(L_j)=\frac{d^i\xi}{ds^i}(0)\quad\text{for}\quad 1\leqslant i\leqslant k+1.$$

Since $\mathbf{g}_2=\mathbf{g}-\nabla\phi_1$ on $\partial\Omega$, using the condition (2.13) we find $\xi_j(L_j)=\xi_j(0)=0$.

Now we define a function ξ on $\partial\Omega$ by

$$\xi(z)=\xi_j(s)\quad\text{if}\quad z=z_j(s)\in\Gamma_j.$$

Then $\xi\in C^{k+1}(\partial\Omega)$, and $\partial\xi/\partial\tau=\mathbf{g}_2\cdot\tau$ on $\partial\Omega$.

Fix a constant $\mu,0<\mu<\mu_0$. Choose a function $\eta\in C^{k+1}[0,\mu]$ such that

$$\eta(0)=1,\quad\eta'(0)=0,\quad\eta(\mu)=\eta'(\mu)=0,\quad 0<\eta(t)<1\quad\text{for}\quad 0<t<\mu.$$

Define a function ϕ_2 on $\Omega(\mu)$ in the following way. For $x\in D_j$, we can write $x=z-t\nu(z)$, where $z=z_j(s)\in\Gamma_j, 0\leqslant s\leqslant L_j$ and $0\leqslant t<\mu$. We set

$$\phi(x)=\eta(t)\xi(z)=\eta(t)\xi_j(s).$$

We easily see that $\phi_2\in C^{k+1}(\Omega(\mu))$. Computation shows that, on Γ_j we have

$$\frac{\partial\phi_2}{\partial\tau}=\mathbf{g}_2\cdot\tau,\quad \frac{\partial\phi_2}{\partial\nu}=-\xi_j(s)\eta'(0)=0.$$

Now we define a function ϕ by $\phi=\phi_1+\phi_2$. Then $\phi\in C^{k+1}(\Omega(\mu))$. On $\partial\Omega$ we have

$$\frac{\partial\phi}{\partial\tau}=\frac{\partial\phi_1}{\partial\tau}+\mathbf{g}_2\cdot\tau=\mathbf{g}\cdot\tau,\quad \frac{\partial\phi}{\partial\nu}=\frac{\partial\phi_1}{\partial\nu}=\mathbf{g}\cdot\nu.$$

So $\nabla\phi=\mathbf{g}$ on $\partial\Omega$. □

When $n=3$ the situation is more complicated. Assume Ω is a bounded smooth (say, C^{k+1}) domain in $\mathbb{R}^3$ and $\mathbf{g}\in C^k(\partial\Omega,\mathbb{R}^3)$. Denote by $\mathbf{g}_\tau$ the tangential component of $\mathbf{g}$, i.e., $\mathbf{g}_\tau=\mathbf{g}-(\mathbf{g}\cdot\nu)\nu$. Denote by $\nabla_\tau\psi$ the tangential component of $\nabla\psi$.

Lemma 2.9: Assume Ω is a bounded C^k domain in $\mathbb{R}^3$ and $\mathbf{g}\in C^k(\partial\Omega,\mathbb{R}^3)$. Then $\mathbf{g}$ has a C^k curl-free extension if and only if

$$\mathbf{g}_\tau=\nabla_\tau\psi \quad \text{on} \quad \partial\Omega \tag{2.14}$$

for some $\psi\in C^{k+1}(\partial\Omega)$.

Proof: Obviously condition (2.14) is necessary.

Now assume $\mathbf{g}_\tau=\nabla_\tau\psi$ for some $\psi\in C^{k+1}(\partial\Omega)$. We extend ψ to a C^{k+1} function on $\overline{\Omega}$. Then on $\partial\Omega$ we have

$$\mathbf{g}-\nabla\psi=\left(\mathbf{g}\cdot\nu-\frac{\partial\psi}{\partial\nu}\right)\nu.$$

Choose a function $\zeta\in C^{k+1}(\overline{\Omega})$ such that $\zeta=0$ on $\partial\Omega$ and

$$\frac{\partial\zeta}{\partial\nu}=\mathbf{g}\cdot\nu-\frac{\partial\psi}{\partial\nu}.$$

For example we may choose ζ to be the unique solution of the following equation:

$$\begin{cases}\Delta^2\zeta=0 \quad \text{in} \quad \Omega,\\ \zeta=0 \quad \text{and} \quad \dfrac{\partial\zeta}{\partial\nu}=\mathbf{g}\cdot\nu-\dfrac{\partial\psi}{\partial\nu} \quad \text{on} \quad \partial\Omega.\end{cases}$$

Then $\nabla(\psi+\zeta)=\mathbf{g}$ on $\partial\Omega$. □

Remark 2.3: Let (y_1,y_2) be the isothermal coordinates on the surface $\partial\Omega$. Then the tangential field $\mathbf{g}_\tau$ can the represented as $\mathbf{g}_\tau=g_1(\partial/\partial y_1)+g_2(\partial/\partial y_2)$. Denote

$$\text{curl}_\tau(\mathbf{g}_\tau)=\frac{\partial g_2}{\partial y_1}-\frac{\partial g_1}{\partial y_2}.$$

It is easy to see that, if $\mathbf{g}$ has a C^k curl-free extension, then

$$\text{curl}_\tau(\mathbf{g}_\tau)=0. \tag{2.15}$$

Define a 1-form on the $\partial\Omega$ by

$$\mathbf{g}^* = g_1 dy_1 + g_2 dy_2 .$$

Obviously $\mathrm{curl}_\tau(\mathbf{g}_\tau) = 0$ if and only if $d\mathbf{g}^* = 0$, i.e., if and only if $\mathbf{g}^*$ is a closed 1-form. On the other hand, (2.14) means that $\mathbf{g}^* = d\psi$ for some function ψ, i.e., $\mathbf{g}^*$ is an exact 1-form. Recall that if the first cohomology group $H^1(\partial\Omega)$ is zero, then every close form is exact. Hence we get the following:

Proposition 2.10: Assume Ω is a bounded C^k domain in $\mathbb{R}^3$ and $g \in C^k(\partial\Omega, \mathbb{R}^3)$. Then $\mathbf{g}$ has a C^k curl-free extension if and only if $\mathbf{g}^*$ is an exact 1-form. Especially if the first cohomology group $H^1(\partial\Omega) = 0$, then $\mathbf{g}$ has a C^k curl-free extension if and only if (2.15) holds.

It is well known that if $\partial\Omega$ is diffeomorphic to the sphere $\mathbb{S}^2$ then $H^1(\partial\Omega) = 0$. So we get the following conclusion:

Proposition 2.11: Assume $\overline{\Omega}$ is C^k diffeomorphic to a ball in $\mathbb{R}^3$ and $\mathbf{g} \in C^{1+\alpha}(\partial\Omega, \mathbb{R}^3)$. Then $\mathcal{R}(\mathbf{g}) = 0$ if and only if (2.15) holds.

Proof: Since $\overline{\Omega}$ is C^k diffeomorphic to a ball, so Ω is simply connected and $H^1(\partial\Omega) = 0$.

From Proposition 2.10 we see that, (2.15) implies $\mathbf{g}$ has a C^k curl-free extension. Hence $\mathcal{R}(\mathbf{g}) = 0$.

Now assume $\mathcal{R}(\mathbf{g}) = 0$. From Theorem 1 we know that $\mathcal{R}(\mathbf{g})$ is achieved by a vector field $\mathbf{u} \in W^{1,2}(\Omega, \mathbb{R}^3)$. Since $\mathcal{R}(\mathbf{g}) = 0$, there exists a function $\psi \in W^{2,2}(\Omega)$ such that $\mathrm{curl}\, \mathbf{u} = 0$ a.e. and $\mathbf{u} = \mathbf{g}$ on $\partial\Omega$. Since Ω is simply connected, there is a function $\psi \in W^{1,2}(\Omega)$ such that $\nabla\psi = \mathbf{u}$ in Ω and $\nabla\psi = \mathbf{g}$ on $\partial\Omega$. So $\nabla_\tau\psi = \mathbf{g}_\tau$ on $\partial\Omega$. Hence

$$\Delta_\tau\psi = \mathrm{div}_\tau\, \mathbf{g}_\tau \quad \text{on} \quad \partial\Omega,$$

where Δ_τ and div_τ are the Laplacian and divergence operators on $\partial\Omega$. Note that $\mathbf{g} \in C^{1+\alpha}(\partial\Omega, \mathbb{R}^3)$. Hence $\mathrm{div}_\tau\, \mathbf{g}_\tau \in C^\alpha(\partial\Omega)$. Using the elliptic estimates on $\partial\Omega$ we find that $\psi \in C^{2+\alpha}(\partial\Omega)$. Therefore,

$$\mathrm{curl}_\tau(\mathbf{g}_\tau) = \mathrm{curl}_\tau(\nabla_\tau\psi) = 0 \quad \text{on} \quad \partial\Omega.$$

This verifies (2.15). □

III. VARIATIONAL PROBLEMS FOR FUNCTIONAL $\mathcal{E}_\varepsilon$ AND $\mathcal{F}$ IN GENERAL DOMAINS

In this section we discuss the variational problem for the functional $\mathcal{E}_\varepsilon$ defined in (1.7). We begin this section with the proof of Theorem 3 stated in Sec. I.

Proof of Theorem 3: Denote by $\mathbf{u}_\varepsilon$ a minimizer of $\mathcal{E}_\varepsilon$. Obviously

$$\mathcal{E}_\varepsilon(\mathbf{u}_\varepsilon) \geqslant \int_\Omega \left\{ \varepsilon|\nabla\mathbf{u}_\varepsilon|^2 + \frac{1}{2\varepsilon}(1 - |\mathbf{u}_\varepsilon|^2)^2 \right\} dx + \mathcal{R}_h(\mathbf{u}_0). \tag{3.1}$$

By the assumption $\mathcal{R}_h(\mathbf{u}_0)$ is achieved. So $\Sigma_h(\mathbf{u}_0)$, the set of minimizers of $\mathcal{R}_h(\mathbf{u}_0)$, is not empty. Without loss of generality we assume $\mathbf{u}_0$ is not a constant vector. Then $a_h(\mathbf{u}_0) > 0$, where $a_h(\mathbf{u}_0)$ is defined in (1.10). So we can choose $\mathbf{v}_\varepsilon \in \Sigma_h(\mathbf{u}_0)$ such that

$$\int_\Omega |\nabla\mathbf{v}_\varepsilon|^2 dx = [1 + o(1)] a_h(\mathbf{u}_0) \quad \text{as} \quad \varepsilon \to 0.$$

Note that $|\mathbf{v}_\varepsilon| = 1$. Then

$$\mathcal{E}_\varepsilon(\mathbf{u}_\varepsilon) \leqslant \int_\Omega \{\varepsilon|\nabla\mathbf{v}_\varepsilon|^2 + |\mathrm{curl}\, \mathbf{v}_\varepsilon|^2\} dx = \varepsilon[1 + o(1)] a_h(\mathbf{u}_0) + \mathcal{R}_h(\mathbf{u}_0). \tag{3.2}$$

Combining (3.1) and (3.2) we get

$$\int_\Omega \left\{ |\nabla \mathbf{u}_\varepsilon|^2 + \frac{1}{2\varepsilon^2}(1-|\mathbf{u}_\varepsilon|^2)^2 \right\} dx \leqslant [1+o(1)]a_h(\mathbf{u}_0).$$

So

$$\begin{gathered}\int_\Omega |\nabla \mathbf{u}_\varepsilon|^2 dx = [1+o(1)]a_h(\mathbf{u}_0),\\ \int_\Omega (1-|\mathbf{u}_\varepsilon|^2)^2 dx = O(\varepsilon^2).\end{gathered} \tag{3.3}$$

Passing to a subsequence we may assume $\mathbf{u}_\varepsilon \to \tilde{\mathbf{u}}$ weakly in $W^{1,2}(\Omega,\mathbb{R}^n)$, $|\tilde{\mathbf{u}}(x)|=1$ a.e. in Ω. Hence $\tilde{\mathbf{u}} \in H(\Omega,\mathbb{S}^{n-1},\mathbf{u}_0)$ and

$$\int_\Omega |\nabla \tilde{\mathbf{u}}|^2 dx \leqslant \liminf_{\varepsilon\to 0} \int_\Omega |\nabla \mathbf{u}_\varepsilon|^2 dx \leqslant a_h(\mathbf{u}_0). \tag{3.4}$$

Using (3.2), (3.3) we compute

$$\begin{aligned}\int_\Omega |\mathrm{curl}\, \mathbf{u}_\varepsilon|^2 dx = \mathcal{E}_\varepsilon(\mathbf{u}_\varepsilon) - \int_\Omega \left\{ \varepsilon |\nabla \mathbf{u}_\varepsilon|^2 + \frac{1}{2\varepsilon}(1-|\mathbf{u}_\varepsilon|^2)^2 \right\} dx \leqslant \{\mathcal{R}_h(\mathbf{u}_0) + \varepsilon[1+o(1)]a_h(\mathbf{u}_0)\}\\ - \left\{ \int_\Omega \left[\varepsilon |\mathbf{u}_\varepsilon|^2 + \frac{1}{2\varepsilon}(1-|\mathbf{u}_\varepsilon|^2)^2 \right] dx \right\} = \mathcal{R}_h(\mathbf{u}_0) + O(\varepsilon),\end{aligned}$$

so

$$\int_\Omega |\mathrm{curl}\, \tilde{\mathbf{u}}|^2 dx \leqslant \liminf_{\varepsilon\to 0} \int_\Omega |\mathrm{curl}\, \mathbf{u}_\varepsilon|^2 dx \leqslant \mathcal{R}_h(\mathbf{u}_0).$$

Hence

$$\int_\Omega |\mathrm{curl}\, \tilde{\mathbf{u}}|^2 dx = \mathcal{R}_h(\mathbf{u}_0),$$

i.e., $\tilde{\mathbf{u}} \in \Sigma_h(\mathbf{u}_0)$.

Using (3.4) we find

$$\int_\Omega |\nabla \tilde{\mathbf{u}}|^2 dx = a_h(\mathbf{u}_0).$$

This and (3.3) together imply that

$$\lim_{\varepsilon\to 0} \int_\Omega |\nabla \mathbf{u}_\varepsilon|^2 dx = \int_\Omega |\nabla \tilde{\mathbf{u}}|^2 dx.$$

Hence $\mathbf{u}_\varepsilon \to \tilde{\mathbf{u}}$ strongly in $W^{1,2}(\Omega,\mathbb{R}^n)$. □

As an application of Theorem 3, we consider the variational problems for $\mathcal{E}_\varepsilon$ on the unit ball B in $\mathbb{R}^3$ under a spherically symmetric boundary condition on ∂B.

In the spherical coordinates (ρ,θ,φ), a unit vector field $\mathbf{v}$ defined on B can be written as

$$\mathbf{v} = \cos\xi \cos\psi \mathbf{e}_\rho + \cos\xi \sin\psi \mathbf{e}_\theta + \sin\xi \mathbf{e}_\varphi. \tag{3.5}$$

If $\mathbf{v}$ is in the form of (3.5) with ξ and ψ depending only on ρ, then we call $\mathbf{v}$ a unit vector field with spherical symmetry. For such vector fields we assume ξ and ψ satisfy the following boundary conditions on the sphere,

$$\xi(1)=\xi_0, \quad \psi(1)=\psi_0.$$

Denote

$$\mathbf{u}_0=\cos\xi_0\cos\psi_0\mathbf{e}_\rho+\cos\xi_0\sin\psi_0\mathbf{e}_\theta+\sin\xi_0\mathbf{e}_\varphi. \tag{3.6}$$

Then we can show the following:

Theorem 3.1: *Let B be the unit ball in $\mathbb{R}^3$ and $\mathbf{u}_0$ be given in* (3.6). *Then $\mathcal{R}_h(\mathbf{u}_0)$ is achieved by a spherical minimizer $\mathbf{v}$ in the form of* (3.5) *if and only if $\mathbf{u}_0=\pm x$. In this case the only minimizers are $\mathbf{v}=\pm x/|x|$.*

Proof: Obviously, if $\mathbf{u}_0=\pm x$ on ∂B, then $\mathcal{R}_h(\mathbf{u}_0)=0$ and $\mathbf{v}=\pm x/|x|$ is a minimizer.

In the following we assume $\mathcal{R}_h(\mathbf{u}_0)$ is achieved by $\mathbf{v}$, which is a unit vector field with spherical symmetry. In the spherical coordinates (ρ, θ, φ), $\mathbf{v}$ can be written in the form of (3.5), where $\xi=\xi(\rho)$ and $\psi=\psi(\rho)$. In the proof of Theorem we denote $'=d/d\rho$. We compute

$$\operatorname{curl}\mathbf{v}=-\frac{\cot\varphi}{\rho}\cos\xi\sin\psi\mathbf{e}_\rho-\frac{(\rho\sin\xi)'}{\rho}\mathbf{e}_\theta+\frac{(\rho\cos\xi\sin\psi)'}{\rho}\mathbf{e}_\varphi.$$

So

$$\int_B|\operatorname{curl}\mathbf{v}|^2dx=2\pi\int_0^1\cos^2\xi\sin^2\psi d\rho\int_0^\pi\frac{\cos^2\varphi}{\sin\varphi}d\varphi$$
$$+4\pi\int_0^1\{|(\rho\sin\psi)'|^2+|(\rho\cos\xi\sin\psi)'|^2\}d\rho.$$

Hence, $|\operatorname{curl}\mathbf{v}|\in L^2(B)$ if and only if

$$\cos\xi\sin\psi\equiv 0. \tag{3.7}$$

Denote by $\mathcal{L}_{\text{sym}}(B,\text{curl},\mathbf{u}_0)$ the subset of $\mathcal{L}(B,\text{curl},\mathbf{u}_0)$ consisting of spherically symmetric vector fields in the form of (3.5). Then each vector field in this set satisfies the condition (3.7). The curl functional in $\mathcal{L}_{\text{sym}}(B,\text{curl},\mathbf{u}_0)$ is reduced to

$$\mathcal{F}(\mathbf{v})=4\pi\int_0^1|(\rho\sin\xi)'|^2d\rho.$$

Assume $\mathbf{v}\in\mathcal{L}_{\text{sym}}(B,\text{curl},\mathbf{u}_0)$ is a minimizer. Then it satisfies the following Euler equation:

$$\cos\xi(\rho\sin\xi)''=0. \tag{3.8}$$

Solving (3.8) we find

$$v=\cos\xi_0\mathbf{e}_\rho+\sin\xi_0\mathbf{e}_\varphi.$$

For this vector field $\mathbf{v}$ we have $\operatorname{div}\mathbf{v}=(2\cos\xi_0+\sin\xi_0\cot\varphi)/\rho$. Hence $\mathbf{v}\in W^{1,2}(B,\mathbb{S}^2)$ if and only if $\sin\xi_0=0$. Therefore, if $\mathcal{R}_h(\mathbf{u}_0)$ is achieved by a spherically symmetric unit vector field $\mathbf{v}$, we must have $\mathbf{u}_0=\pm\mathbf{e}_\rho=\pm x$ on ∂B, so $\mathcal{R}_h(\mathbf{u}_0)=0$, and $\mathbf{v}=\pm x/|x|$. □

Using Theorem 3 and Theorem 3.1 we can prove the following:

Theorem 3.2: *Let B be the unit ball in $\mathbb{R}^3$ and $\mathbf{u}_0=\pm x$ on ∂B. Let $\mathbf{u}_\varepsilon$ be the minimizer of $\mathcal{E}_\varepsilon$ in $H(B,\pm x)$. Then as $\varepsilon\to 0,\mathbf{u}_\varepsilon$ converges to $\pm x/|x|$ strongly in $W^{1,2}(B,\mathbb{R}^3)$.*

Proof: For simplicity we only consider the case where $\mathbf{u}_0 = x$.

Denote $\mathbf{v} = x/|x|$. Obviously $\mathbf{v} \in H(B, \mathbb{S}^2, x)$ and $\text{curl}\,\mathbf{v} = 0$ away from the origin. Hence $\mathcal{R}_h(x) = 0$ and $\mathbf{v}$ is a minimizer.

Now we show that $\mathbf{v} = x/|x|$ is the unique minimizer of $\mathcal{R}_h(x)$. Suppose there is a unit vector field $\mathbf{w} \in H(B, \mathbb{S}^2, x)$ such that curl $\mathbf{w} = 0$ in B. Then there is a function $\phi \in W^{2,2}(B)$ such that $\mathbf{w} = \nabla \phi$. Hence

$$|\nabla \phi| = 1 \quad \text{in} \quad B, \quad \nabla \phi = x \quad \text{on} \quad \partial B. \tag{3.9}$$

The solution of (3.9) is determined by the characteristic equations,

$$\dot{x} = p, \quad \dot{\phi} = 1, \quad \dot{p} = 0.$$

Using the boundary condition in (3.9) we find that each characteristic line is a ray from the center, and along each ray $\nabla \phi = \mathbf{u}_0(x/|x|) = x/|x|$. So $\mathbf{w} = \mathbf{v} = x/|x|$.

Using Theorem 3 we see that $\mathbf{u}_\varepsilon \to x/|x|$ strongly in $W^{1,2}(B, \mathbb{R}^3, x)$ as $\varepsilon \to 0$. □

Remark 3.1: Under the conditions of Theorem 3.2 one can show that $\mathbf{u}_\varepsilon \to \pm x/|x|$ uniformly in any compact subdomain away from 0.

From Theorem 3 we see that, if $H(\Omega, \mathbb{S}^{n-1}, \mathbf{u}_0) \neq \varnothing$ and if $\mathcal{R}_h(\mathbf{u}_0)$ is achieved, then the limiting behavior of the minimizers $\{\mathbf{u}_\varepsilon\}$ of $\mathcal{E}_\varepsilon$ is clear. However, as mentioned in Sec. I, in many practical cases, $H(\Omega, \mathbb{S}^{n-1}, \mathbf{u}_0) = \varnothing$. For example, if Ω is a bounded smooth domain in the plane and if the degree $\deg(\mathbf{u}_0)$ of $\mathbf{u}_0$ is not zero, then $H(\Omega, \mathbb{S}^1, \mathbf{u}_0) = \varnothing$. Note that if $\mathbf{u}_0 \equiv \nu$ or if $\mathbf{u}_0 \equiv \tau$, where ν is the unit outer normal of $\partial\Omega$ and τ is the unit tangential field of $\partial\Omega$, then $\deg(\mathbf{u}_0) = 1$. Therefore it is reasonable to consider the variational problem on a large set $\mathcal{L}(\Omega, \text{curl}, \mathbf{u}_0)$ and discuss the achievability of $\mathcal{R}_l(\mathbf{u}_0)$ on $\mathcal{L}(\Omega, \text{curl}, \mathbf{u}_0)$. For the definitions of $\mathcal{L}(\Omega, \text{curl}, \mathbf{u}_0)$ and $\mathcal{R}_l(\mathbf{u}_0)$ see Sec. I.

First we give a lower bound of $\mathcal{R}_l(\mathbf{u}_0)$. Denote by $|E|$ the measure of a set E.

Lemma 3.3: Assume Ω *is a bounded smooth domain in* $\mathbb{R}^2$. *For any* $\mathbf{u}_0 \in H^{1/2}(\partial\Omega, \mathbb{S}^1)$ *we have*

$$\mathcal{R}_l(\mathbf{u}_0) \geqslant \frac{1}{|\Omega|} \left\{ \int_{\partial\Omega} \mathbf{u}_0 \cdot \tau ds \right\}^2.$$

Especially if $\mathbf{u}_0$ *is a unit tangential field on* $\partial\Omega$, *then*

$$\mathcal{R}_l(\mathbf{u}_0) \geqslant \frac{|\partial\Omega|^2}{|\Omega|}.$$

Proof: Since

$$\int_\Omega \text{curl}\,\mathbf{u}\,dx = \int_{\partial\Omega} \mathbf{u} \cdot \tau ds,$$

using Hölder inequality we obtain the lower bound. □

Lemma 3.3 has an interesting consequence.

Corollary 3.4: For any bounded smooth domain Ω *in* $\mathbb{R}^2$ *and for any smooth unit tangential field* $\mathbf{u}_0$ *on* $\partial\Omega$, *it holds that*

$$\mathcal{R}_l(\mathbf{u}_0) \geqslant \frac{|\partial\Omega|^2}{|\Omega|} \geqslant 4\pi.$$

The equality holds if and only if Ω *is a disk.*

Proof: The first statement is the consequence of Lemma 3.3 and the isoperimetric inequality. In the proof of Theorem 4 we shall show that, when Ω is a disk and $\mathbf{u}_0$ is a unit tangential field, $\mathcal{R}_l(\mathbf{u}_0) = 4\pi$ and is achieved, see Sec. IV. Therefore, the second conclusion is true. □

Even though Lemma 3.3 gives a lower bound of $\mathcal{R}(\mathbf{u}_0)$ for any tangential vector field $\mathbf{u}_0$, we cannot prove whether $\mathcal{R}_l(\mathbf{u}_0)$ is achievable in general case at moment. We cannot even show it is achievable when $\mathcal{R}_l(\mathbf{u}_0)=0$. On the other hand, we can show that, if $\mathbf{u}_0$ is parallel to the normal vector field on $\partial\Omega$, or more general, if $\mathbf{u}_0$ can be extended to become a central field (for the definition see Sec. I), then $\mathcal{R}_l(\mathbf{u}_0)=0$ and is achieved. See Corollary 2.7 for the related statement for $\mathcal{R}(\mathbf{u}_0)$.

In the following we shall discuss this issue further from the diffeomorphism point of view. For simplicity we only consider the two-dimensional case. We begin our discussion with a special case. Assume $\mathbf{u}$ is a unit vector field which can be written as $\mathbf{u}=e^{i\phi}$ for some function ϕ. For such $\mathbf{u}$, if $\mathbf{u}$ is a minimizer of $\mathcal{R}_l(\mathbf{u}_0)$ then it satisfies the Euler equation (1.9), hence $e^{i\phi}\wedge curl^2\, e^{i\phi}=0$ in Ω. Especially if $\mathcal{R}_l(\mathbf{u}_0)=0$ and is achieved by such a vector field $\mathbf{u}$, then $\operatorname{curl} e^{i\phi}=0$. Computation shows that both these two conditions can be formulated with a differential operator D_ϕ, which is defined as follows:

$$D_\phi=\cos\phi\,\partial_1+\sin\phi\,\partial_2 .$$

In fact

$$\operatorname{curl} e^{i\phi}=D_\phi\phi,\quad e^{i\phi}\wedge\operatorname{curl}^2 e^{i\phi}=D_\phi^2\phi.$$

Let x be the original coordinates. Given $e^{i\phi(x)}$, we look for a diffeomorphism $x=F(y)$, where $F:G\to\Omega$ is a map such that

$$\frac{\partial F}{\partial y_1}=e^{i\phi}.$$

If such a map F exists, then in the new coordinates y we have

$$\frac{\partial x_1}{\partial y_1}=\cos\phi,\quad \frac{\partial x_2}{\partial y_1}=\sin\phi,$$

$$\frac{\partial}{\partial y_1}=D_\phi,\quad \frac{\partial^2}{\partial y_1^2}=D_\phi^2.$$

Hence

$$\operatorname{curl} e^{i\phi}=\frac{\partial\phi}{\partial y_1},\quad e^{i\phi}\wedge\operatorname{curl}^2 e^{i\phi}=\frac{\partial^2\phi}{\partial y_1^2}.$$

Therefore,

(i) $\operatorname{curl} e^{i\phi}=0$ in $\Omega\Leftrightarrow D_\phi\phi=0$ in $\Omega\Leftrightarrow\partial\phi/\partial y_1=0$ in G;
(ii) $e^{i\phi}\wedge\operatorname{curl}^2 e^{i\phi}=0$ in $\Omega\Leftrightarrow D_\phi^2\phi=0$ in $\Omega\Leftrightarrow\partial^2\phi/\partial y_1^2=0$ in G.

In order to get $\operatorname{curl} e^{i\phi}=0$, we need $\partial F/\partial y_1=e^{i\phi}$ and $\partial\phi/\partial y_1=0$. It suggests the conditions for $\operatorname{curl}\mathbf{u}=0$: *There are functions* $\zeta(y_2)$, *and* $V(y_2)$ *such that*

$$\mathbf{u}=e^{i\zeta(y_2)},\quad F(y)=y_1e^{i\zeta(y_2)}+V(y_2).$$

Here $\zeta(y_2)$ and $V(y_2)$ are determined by the boundary condition of $\mathbf{u}$. Essentially, $\mathbf{u}$ depends only on one variable.

In order to get $e^{i\phi}\wedge\operatorname{curl}^2 e^{i\phi}=0$, we need $\partial F/\partial y_1=e^{i\phi}$ and $\partial^2\phi/\partial y_1^2=0$. It suggests the condition for $e^{i\phi}\wedge\operatorname{curl}^2 e^{i\phi}=0$: *There are functions* $\zeta(y_2)$, $\xi(y_2)$, *and* $V(y_2)$ *such that*

$$\mathbf{u}=\exp\{i[2y_1\zeta(y_2)+\xi(y_2)]\},\quad \operatorname{curl}\mathbf{u}=2\zeta(y_2),$$

and

$$F(y)=\begin{cases}\zeta(y_2)^{-1}\sin(y_1\zeta(y_2))\exp\{i[y_1\zeta(y_2)+\xi(y_2)]\}+V(y_2) & \text{if } \zeta(y_2)\neq 0,\\ y_1\exp[i\xi(y_2)]+V(y_2) & \text{if } \zeta(y_2)=0.\end{cases}$$

Here $\xi(y_2)$ and $V(y_2)$ are determined by the boundary condition of $\mathbf{u}$. Hence, curl $\mathbf{u}$ only depends on one variable.

Locally we can always write a unit vector field $\mathbf{u}$ as $\mathbf{u}=e^{i\phi}$, but this may not hold in the entire domain Ω. Nevertheless, the above intuitive discussion is still helpful, which yields the conclusions for general case. The conditions for curl $\mathbf{u}=0$ for the general case will be presented in Proposition 3.5, and the conditions for $\text{curl}^2\,\mathbf{u}$ to be parallel to $\mathbf{u}$ will be given in Proposition 3.6. In the following, for a smooth map F, we denote by DF the Frechet differential of F, and by det (DF) the determinant of DF.

Proposition 3.5: Let Ω be a bounded smooth domains in $\mathbb{R}^2$.

(1) *Assume there is a diffeomorphism $F:G\to\Omega$ in the following form:*

$$F(y)=y_1U(y_2)+V(y_2), \tag{3.10}$$

where $U(y_2)$ is a unit vector field. Set

$$\mathbf{u}(x)=U(F^{-1}(x)),\quad x\in\Omega.$$

Then curl $\mathbf{u}=0$.

(2) *On the other hand, assume $\mathbf{u}$ is a unit vector field and curl $\mathbf{u}=0$ in Ω. Assume there is a smooth change of variables $x=F(y)$ such that*

$$\frac{\partial F(y)}{\partial y_1}=\mathbf{u}(F(y))\ \text{ and }\ \det(\text{DF})\neq 0,\quad x\in\Omega. \tag{3.11}$$

Then in the new variables y we have $\partial\mathbf{u}/\partial y_1=0$.

Proof: Step 1. Assume there is a diffeomorphism in the form of (3.10).
Denote $x=F(y)$ and $U=(U_1,U_2)$. From (3.10) we have

$$\frac{\partial x_1}{\partial y_1}=U_1,\quad \frac{\partial x_2}{\partial y_1}=U_2.$$

So,

$$\frac{\partial y_2}{\partial x_1}=-\frac{U_2}{\det(\text{DF})},\ \frac{\partial y_2}{\partial x_2}=\frac{U_1}{\det(\text{DF})}. \tag{3.12}$$

Since U is a unit vector field, $U\cdot\partial_2U=0$. Hence

$$\text{curl}\,u(x)=\frac{\partial u_2}{\partial x_1}-\frac{\partial u_1}{\partial x_2}=U_2'\frac{\partial y_2}{\partial x_1}-U_1'\frac{\partial y_2}{\partial x_2}=-\frac{1}{\det(\text{DF})}(U_1U_1'+U_2U_2')=0.$$

Step 2. Now assume curl $\mathbf{u}=0$ in Ω, and assume there is a change of variables $x=F(y)$ satisfying

$$\frac{\partial F(y)}{\partial y_1}=\mathbf{u}(x),\ \text{ and }\ \det(\text{DF})\neq 0\ \text{ in }\ \Omega.$$

Then (3.12) holds with U_1 and U_2 being replaced by u_1 and u_2. Since $\mathbf{u}$ is a unit vector field, so $\mathbf{u}\cdot\partial_1\mathbf{u}=\mathbf{u}\cdot\partial_2\mathbf{u}=0$. Hence,

$$0=\operatorname{curl}\mathbf{u}=\frac{\partial u_2}{\partial x_1}-\frac{\partial u_1}{\partial x_2}=\frac{\partial u_2}{\partial y_1}\frac{\partial y_1}{\partial x_1}-\frac{u_2}{\det(\mathrm{DF})}\frac{\partial u_2}{\partial y_2}-\frac{\partial u_1}{\partial y_1}\frac{\partial y_1}{\partial x_2}-\frac{u_1}{\det(\mathrm{DF})}\frac{\partial u_1}{\partial y_2}$$
$$=\frac{\partial u_2}{\partial y_1}\frac{\partial y_1}{\partial x_1}-\frac{\partial u_1}{\partial y_1}\frac{\partial y_1}{\partial x_2}. \tag{3.13}$$

Now we show $\partial u_1/\partial y_1=\partial u_2/\partial y_1=0$. Otherwise, we may assume $\partial u_2/\partial y_1\neq 0$. Then from (3.12) (with U_1 and U_2 replaced by u_1 and u_2) and (3.13) we have,

$$\det((\mathrm{DF})^{-1})=\frac{\partial y_1}{\partial x_1}\frac{\partial y_2}{\partial x_2}-\frac{\partial y_1}{\partial x_2}\frac{\partial y_2}{\partial x_1}=\frac{1}{\det(\mathrm{DF})}\left[u_1\frac{\partial y_1}{\partial x_1}+u_2\frac{\partial y_1}{\partial x_2}\right]$$
$$=\frac{1}{\det(\mathrm{DF})}\frac{\partial y_1}{\partial x_2}\left(\frac{\partial u_2}{\partial y_1}\right)^{-1}\left[u_1\frac{\partial u_1}{\partial y_1}+u_2\frac{\partial u_2}{\partial y_1}\right]=0,$$

a contradiction. □

Remark 3.2: Proposition 3.5 implies that a curl-free unit vector field in a planar domain depends on one variable only. We shall call a map in the form of (3.10) a *central* map.

Note that a map in the form of (3.10) is not a conformal map in general. In fact, a map F in the form of (3.10) is a conformal map if and only if F is an orthogonal linear transform, that is,

$$F(y)=y_1(\cos\phi_0,\sin\phi_0)+y_2(-\sin\phi_0,\cos\phi_0)+\mathbf{v}_0,$$

where ϕ_0 is a constant number and $\mathbf{v}_0$ is a constant vector.

Proposition 3.6: Let Ω be a smooth bounded domain in $\mathbb{R}^2$ and $\mathbf{u}$ be a unit vector field in Ω. Assume that there is a smooth change of variables $x=F(y)$, where the map $F:G\to\Omega$ satisfies (3.11) in Ω. Then $\mathbf{u}$ satisfies the equation in (1.9) if and only if curl $\mathbf{u}$ depends only on y_2.

Proof: Denote $H=\operatorname{curl}\mathbf{u}$. Then $\operatorname{curl}^2\mathbf{u}=(\partial_2 H,-\partial_1 H)$ and $\mathbf{u}\wedge\operatorname{curl}^2\mathbf{u}=\mathbf{u}\cdot\nabla_x H$.

Since $\mathbf{u}$ is a unit vector field, as in (3.13) we have

$$H=\frac{\partial u_2}{\partial y_1}\frac{\partial y_1}{\partial x_1}-\frac{\partial u_1}{\partial y_1}\frac{\partial y_1}{\partial x_2}.$$

Since $\partial x_1/\partial y_1=u_1$ and $\partial x_2/\partial y_1=u_2$, we have $\partial y_2/\partial x_1=-\lambda u_2$ and $\partial y_2/\partial x_2=\lambda u_1$, where $\lambda=1/\det(\mathrm{DF})$. So we find

$$\mathbf{u}\cdot\nabla_x H=u_1\left[\frac{\partial u_2}{\partial y_1}\frac{\partial^2 y_1}{\partial x_1^2}-\frac{\partial u_1}{\partial y_1}\frac{\partial^2 y_1}{\partial x_1\partial x_2}\right]+u_2\left[\frac{\partial u_2}{\partial y_1}\frac{\partial^2 y_1}{\partial x_1\partial x_2}-\frac{\partial u_1}{\partial y_1}\frac{\partial^2 y_1}{\partial x_2^2}\right]+u_1\left[\frac{\partial^2 u_2}{\partial y_1^2}\left(\frac{\partial y_1}{\partial x_1}\right)^2\right.$$
$$\left.-\frac{\partial^2 u_1}{\partial y_1^2}\frac{\partial y_1}{\partial x_1}\frac{\partial y_1}{\partial x_2}\right]+u_2\left[\frac{\partial^2 u_2}{\partial y_1^2}\frac{\partial y_1}{\partial x_1}\frac{\partial y_1}{\partial x_2}-\frac{\partial^2 u_1}{\partial y_1^2}\left(\frac{\partial y_1}{\partial x_2}\right)^2\right].$$

Since $u_1\partial y_1/\partial x_1+u_2\partial y_1/\partial x_2=1$, the sum of the last two terms in the right is

$$\frac{\partial^2 u_2}{\partial y_1^2}\frac{\partial y_1}{\partial x_1}-\frac{\partial^2 u_1}{\partial y_1^2}\frac{\partial y_1}{\partial x_2}.$$

Hence,

$$\mathbf{u}\cdot\nabla_x H=\frac{\partial^2 u_2}{\partial y_1^2}\frac{\partial y_1}{\partial x_1}+\frac{\partial u_2}{\partial y_1}\left[u_1\frac{\partial^2 y_1}{\partial x_1^2}+u_2\frac{\partial^2 y_1}{\partial x_1\partial x_2}\right]-\frac{\partial^2 u_1}{\partial y_1^2}\frac{\partial y_1}{\partial x_2}-\frac{\partial u_1}{\partial y_1}\left[u_1\frac{\partial^2 y_1}{\partial x_1\partial x_2}+u_2\frac{\partial^2 y_1}{\partial x_2^2}\right]$$
$$=\frac{\partial^2 u_2}{\partial y_1^2}\frac{\partial y_1}{\partial x_1}+\frac{\partial u_2}{\partial y_1}\frac{\partial}{\partial y_1}\left(\frac{\partial y_1}{\partial x_1}\right)-\frac{\partial^2 u_1}{\partial y_1^2}\frac{\partial y^2 y_1}{\partial x_2}-\frac{\partial u_1}{\partial y_1}\frac{\partial}{\partial y_1}\left(\frac{\partial y_1}{\partial x_2}\right)$$
$$=\frac{\partial}{\partial y_1}\left[\frac{\partial u_2}{\partial y_1}\frac{\partial y_1}{\partial x_1}-\frac{\partial u_1}{\partial y_1}\frac{\partial y_1}{\partial x_2}\right]=\frac{\partial H}{\partial y_1}.$$

Consequently,

$$\mathbf{u}\wedge\text{curl}^2\,\mathbf{u}=0\Leftrightarrow\mathbf{u}\cdot\nabla_x H=0\Leftrightarrow\partial H/\partial y_1=0.$$

The lemma is proved. □

IV. VARIATIONAL PROBLEMS FOR $\mathcal{E}_\varepsilon$ and $\mathcal{F}$ IN A DISK

In this section we discuss the variational problems for the functional $\mathcal{E}_\varepsilon$ on a disk D in the plane, and prove Theorem 4. In this section we denote

$$\mathbf{e}_r=\frac{x}{r}=e^{i\theta},\quad \mathbf{e}_\theta=\frac{1}{r}(-x_2,x_1)=e^{i(\theta+\pi/2)}.$$

Throughout this section we assume that $\mathbf{u}_0$ is a unit vector field on ∂D which makes a constant angle ϕ_0 with the outer normal. Without loss of generality we assume $0\leqslant\phi_0\leqslant\pi/2$. In the polar coordinates we can write $\mathbf{u}_0=e^{i[\theta+\phi_0]}$, see (1.11).

Lemma 4.1: For $\mathbf{u}_0$ given above we have

$$\mathcal{R}_l(\mathbf{u}_0)=4\pi\sin^2\phi_0,$$

and the only minimizer of the curl functional $\mathcal{F}$ in $\mathcal{L}(\Omega,\text{curl},\mathbf{u}_0)$ is the vector field $\mathbf{v}$ given in (1.12).

Proof: Step 1. We first look for a minimizer of the curl functional $\mathcal{F}$ among all unit vector fields in the form $\mathbf{v}=e^{i[\theta+\phi(r)]}$.

For such $\mathbf{v}$ we compute

$$\mathcal{F}(\mathbf{v})=\int_D|\text{curl}\,\mathbf{v}|^2dx=2\pi\int_0^1\left[\frac{\sin\phi}{r}+\phi'\cos\phi\right]^2 r\,dr$$

and the Euler equation is

$$\cos\phi\left\{(r\phi''+\phi')\cos\phi-\left(r\phi'^2+\frac{1}{r}\right)\sin\phi\right\}=0.$$

Note that the solution of this Euler equation must satisfy $\cos\phi\not\equiv0$. In fact, if $\cos\phi\equiv0$, then $\phi=\pm\pi/2$ and $v=\pm\mathbf{e}_\theta$. This solution must be dropped because it does not lie in $\mathcal{L}(D,\text{curl},S^2)$. Now since $\cos\phi\not\equiv0$, $\cos\phi$ can have only isolated zero points. Hence the second factor in the above equation must be zero, which can also be written as

$$r\frac{d}{dr}\left(r\frac{d}{dr}\sin\phi\right)=\sin\phi.$$

The solutions of this equation which satisfy $\phi(1)=\phi_0$ are

$$\sin\phi(r)=(\sin\phi_0)\cosh(-\log r)+C\sinh(-\log r).$$

The only solution which remains bounded as $r \to 0$ is

$$\sin \phi(r) = r \sin \phi_0 .$$

Hence $\mathbf{v}$ can be written as (1.12), or,

$$\mathbf{v} = e^{i[\theta + \phi(r)]} = \sqrt{1 - r^2 \sin^2 \phi_0}\,\mathbf{e}_r + r \sin \phi_0 \mathbf{e}_\theta . \tag{4.1}$$

Next we claim that the vector field $\mathbf{v}$ given in (4.1) achieves $\mathcal{R}_l(\mathbf{u}_0)$. In other words, $\mathbf{v}$ is a minimizer of the curl functional $\mathcal{F}$ among all vector fields in $\mathcal{L}(D,\text{curl},\mathbf{u}_0)$. In fact, from (4.1) we see that $\text{curl}\,\mathbf{v} = 2 \sin \phi_0$ and $\mathcal{F}(\mathbf{v}) = 4\pi \sin^2 \phi_0$. From Lemma (3.3) we see that $\mathcal{F}(\mathbf{v}) = \mathcal{R}_l(\mathbf{u}_0)$.
Step 2. Now we show that, the vector field $\mathbf{v}$ given in (4.1) is the only minimizer of $\mathcal{F}$ in the class $\mathcal{L}(D,\text{curl},\mathbf{u}_0)$.

To prove this conclusion, suppose $\mathbf{w} \in \mathcal{L}(\Omega,\text{curl},\mathbf{u}_0)$ is a minimizer of $\mathcal{F}$, and we shall show $\mathbf{w} = \mathbf{v}$. Since $\mathbf{w}$ is a minimizer,

$$\int_D |\text{curl}\,\mathbf{w}|^2 dx = 4\pi \sin^2 \phi_0 .$$

Hence,

$$\begin{aligned}\int_D |\text{curl}\,\mathbf{w} - \text{curl}\,\mathbf{v}|^2 dx &= \int_D |\text{curl}\,\mathbf{w}|^2 dx + \int_D |\text{curl}\,\mathbf{v}|^2 dx - 2\int_D (\text{curl}\,\mathbf{v})(\text{curl}\,\mathbf{w})\, dx \\ &= 8\pi \sin^2 \phi_0 - 4 \sin \phi_0 \int_D \text{curl}\,\mathbf{w}\, dx \\ &= 8\pi \sin^2 \phi_0 - 4 \sin \phi_0 \int_{\partial D} \mathbf{w} \cdot \tau\, ds = 0.\end{aligned}$$

So

$$\text{curl}\,\mathbf{w} = 2 \sin \phi_0 \quad \text{a.e. in } D. \tag{4.2}$$

Note that $|\mathbf{w}| = 1$ a.e. in D. Locally (at least near each point on ∂D) we can write

$$\mathbf{w} = e^{i\omega}, \text{ with } \omega = \theta + \phi_0 \text{ on } \partial D.$$

Using (4.2) we find that ω is a solution of Cauchy problem of the following partial differential equation of first order, with ∂D as the initial curve:

$$\cos \omega \partial_1 \omega + \sin \omega \partial_2 \omega = 2 \sin \phi_0. \tag{4.3}$$

When $\phi_0 \neq \pi/2$, ∂D is noncharacteristic for (4.3) and the solution is unique near ∂D. By continuous extension we find that the solution is actually unique on D. Hence $\mathbf{w} = \mathbf{v}$.

So we only need to prove $\mathbf{w} = \mathbf{v}$ when $\phi_0 = \pi/2$. However, the proof presented below works for all ϕ_0.

For an arbitrary point $e^{i\theta_0} \in \partial D$, we denote the characteristic line of (4.3) starting from this point by $\mathcal{C}(\theta_0)$. To show $\mathbf{w} = \mathbf{v}$, we prove the following:

Claim 1: For every point $x \in D$, there is a characteristic line $\mathcal{C}(\theta_0)$ passing through x; and $\mathbf{w} = \mathbf{v}$ along each characteristic curve $\mathcal{C}(\theta_0)$.

Proof of Claim 1: Note that, when $\phi_0 = 0$, $\phi \equiv 0$. In this case every characteristic line is the radial ray. So it is obvious that $\mathbf{w} = x/|x| = \mathbf{v}$. Thus, in the following we assume $0 < \phi_0 \leq \pi/2$. The equations for the characteristic lines of (4.3) are

$$\dot{x} = e^{i\omega}, \quad \dot{\omega} = 2 \sin \phi_0 .$$

Solving this differential system we find that

$$\omega=\theta_0+\phi_0+2t\sin\phi_0,$$

and the characteristic line $\mathcal{C}(\theta_0)$ can be represented by

$$x=\frac{\sin(\phi_0+t\sin\phi_0)}{\sin\phi_0}\exp[i(\theta_0+t\sin\phi_0)].$$

Therefore, in the polar coordinates, the characteristic line $\mathcal{C}(\theta_0)$ is represented as

$$r=\frac{\sin(\phi_0+t\sin\phi_0)}{\sin\phi_0},\quad \theta=\theta_0+t\sin\phi_0. \tag{4.4}$$

Hence the characteristic line lies in D for $-\phi_0/\sin\phi_0<t<0$, and reaches the center of the disk when $t=-\phi_0/\sin\phi_0$.

Now we see that D is covered by the family of characteristic lines. In fact, for each $x=re^{i\theta}\in D$, we can find a characteristic line $\mathcal{C}(\theta_0)$ passing through x. To see if this is true, for each $x=re^{i\theta}\in D$, we solve θ_0 and t from (4.4),

$$\theta_0=\phi_0+\theta-\arcsin(r\sin\phi_0),\quad t=\frac{\arcsin(r\sin\phi_0)-\phi_0}{\sin\phi_0},$$

Note that (θ_0,t) satisfy the condition $-\phi_0/\sin\phi_0\leqslant t<0$.

Next we show that along each characteristic line it holds that $\mathbf{w}=\mathbf{v}$. Write $\omega=\theta+\psi$. Then along $\mathcal{C}(\theta_0)$ we have, from (4.4),

$$\theta+\psi=\omega=\theta_0+\phi_0+2t\sin\phi_0=\theta+\phi_0+t\sin\phi_0.$$

Using (4.1) again

$$\psi=\phi_0+t\sin\phi_0=\arcsin(r\sin\phi_0)=\phi(r).$$

So $\mathbf{w}=e^{i(\theta+\psi)}=e^{i[\theta+\phi(r)]}=\mathbf{v}$. Claim 1 is proved. □

Therefore we have shown that $\mathbf{v}$ is the only minimizer. Step 2 is complete. □

Lemma 4.2: For $\mathbf{u}_0$ given above, let $\mathbf{u}_\varepsilon$ be a minimizer of $\mathcal{E}_\varepsilon$ among all the vector fields in $H(D,\mathbf{u}_0)$. For all $\varepsilon>0$ small we have the following energy estimates:

(1) *When* $0\leqslant\phi_0<\pi/2$,

$$\mathcal{E}_\varepsilon(\mathbf{u}_\varepsilon)=4\pi\sin^2\phi_0+2\pi\varepsilon\log\frac{1}{\varepsilon}+O(\varepsilon).$$

(2) *When* $\phi_0=\pi/2$,

$$4\pi+2\pi\varepsilon\log\frac{1}{\varepsilon}+O(\varepsilon)\leqslant\mathcal{E}_\varepsilon(\mathbf{u}_\varepsilon)\leqslant4\pi+3\pi\varepsilon\log\frac{1}{\varepsilon}+O(\varepsilon).$$

Proof: Step 1. We first derive a lower bound,

$$\mathcal{E}_\varepsilon(\mathbf{u}_\varepsilon)\geqslant4\pi\sin^2\phi_0+2\pi\varepsilon\log\frac{1}{\varepsilon}+O(\varepsilon).$$

Using Hölder inequality we have

$$\int_D|\mathrm{curl}\,\mathbf{u}_\varepsilon|^2dx\geqslant\frac{1}{|D|}\left\{\int_{\partial D}\mathbf{u}_0\cdot\tau ds\right\}^2=4\pi\sin^2\phi_0. \tag{4.5}$$

Also note that $\deg(\mathbf{u}_\varepsilon)=\deg(\mathbf{u}_0)=1$, it is true that

$$\int_D\left\{\varepsilon|\nabla\mathbf{u}_\varepsilon|^2+\frac{1}{2\varepsilon}(1-|\mathbf{u}_\varepsilon|^2)^2\right\}dx\geqslant 2\pi\varepsilon\log\frac{1}{\varepsilon}+O(\varepsilon), \tag{4.6}$$

see Ref. 25, p. 82, Lemma 4. Hence the energy lower bound is true.

Step 2. We derive a energy upper bound for $\phi_0=\pi/2$.

Let

$$0<\delta\ll 1,\ \ 0<\rho\ll 1,\ \ \theta_0=\frac{\pi}{2(1+\rho)}.$$

We choose a test field $\mathbf{u}$ in the form

$$\mathbf{u}=f(r)e^{i[\theta+\phi(r)]}, \tag{4.7}$$

where

$$f(r)=\begin{cases}\dfrac{r}{\delta} & \text{if } 0<r\leqslant\delta,\\ 1 & \text{if } \delta<r\leqslant 1,\end{cases}$$

$$\phi(r)=\begin{cases}\dfrac{\pi}{2\theta_0}\arcsin r & \text{if } 0\leqslant r\leqslant\sin\theta_0,\\ \dfrac{\pi}{2} & \text{if}\sin\theta_0<r\leqslant 1.\end{cases}$$

Note that $\phi(r)=(1+\rho)\arcsin r$ for $0\leqslant r\leqslant\sin\theta_0$. For $\mathbf{u}$ in the form of (4.7),

$$|\nabla\mathbf{u}|^2=f'^2+f^2\phi'^2+\frac{f^2}{r^2},$$

$$\operatorname{curl}\mathbf{u}=\frac{(rf\sin\phi)'}{r}.$$

We compute

$$\begin{aligned}\mathcal{E}_\varepsilon(\mathbf{u})&=\int_D\left\{\varepsilon|\nabla\mathbf{u}|^2+|\operatorname{curl}\mathbf{u}|^2+\frac{1}{2\varepsilon}(1-|\mathbf{u}|^2)^2\right\}dx\\ &=2\pi\int_0^1\left\{\varepsilon\left[rf'^2+rf^2\phi'^2+\frac{f^2}{r}\right]+\frac{1}{r}|(rf\sin\phi)'|^2+\frac{r}{2\varepsilon}(1-f^2)^2\right\}dr\\ &=4\pi+2\pi\varepsilon\left[\log\frac{1}{\delta}+\log\frac{1}{\rho}\right]+2\pi\varepsilon\left(1-\log\frac{\pi}{2}\right)+4\pi\varepsilon\rho\log\frac{1}{\rho}+\frac{\delta^2}{12\varepsilon}+O(\varepsilon\rho+\rho^2+\delta^2).\end{aligned}$$

Choose $\delta=\sqrt{12\pi\varepsilon},\ \rho=\varepsilon^{1/2}$ we get

$$\mathcal{E}_\varepsilon(\mathbf{u}_\varepsilon)\leqslant\mathcal{E}_\varepsilon(\mathbf{u})=4\pi+3\pi\varepsilon\log\frac{1}{\varepsilon}+O(\varepsilon).$$

Step 3. We derive a energy upper bound for $0\leqslant\phi_0<\pi/2$.

Choose $\mathbf{u}$ in the form of (4.7), where f is the same as above, and

$$\phi = \arcsin(r \sin \phi_0).$$

We have

$$\mathcal{E}_\varepsilon(\mathbf{u}) = 4\pi \sin^2 \phi_0 + 2\pi\varepsilon \log\frac{1}{\delta} + 2\pi\varepsilon(1 - 2\log\cos\phi_0) + \frac{\delta^2}{12\varepsilon} + O(\delta^2).$$

Choose $\delta = \sqrt{12\pi}\varepsilon$ we get

$$\mathcal{E}_\varepsilon(\mathbf{u}_\varepsilon) \leqslant \mathcal{E}_\varepsilon(\mathbf{u}) = 4\pi \sin^2 \phi_0 + 2\pi\varepsilon \log\frac{1}{\varepsilon} + O(\varepsilon).$$

Now the lemma is proved. □

Corollary 4.3: For $\mathbf{u}_0$ *given above, let* $\mathbf{u}_\varepsilon \in H(D, \mathbf{u}_0)$ *be a minimizer of* $\mathcal{E}_\varepsilon$. *Then we have the following results:*

(1)

$$\int_D |\mathrm{curl}\, \mathbf{u}_\varepsilon|^2 dx = 4\pi \sin^2 \phi_0 + O\left(\varepsilon \log\frac{1}{\varepsilon}\right).$$

(2) *When* $0 \leqslant \phi_0 < \pi/2$ *we have*

$$\int_D \left\{ |\nabla \mathbf{u}_\varepsilon|^2 + \frac{1}{2\varepsilon^2}(1 - |\mathbf{u}_\varepsilon|^2)^2 \right\} dx = 2\pi \log\frac{1}{\varepsilon} + O(1),$$

and when $\phi_0 = \pi/2$ *we have*

$$\int_D \left\{ |\nabla \mathbf{u}_\varepsilon|^2 + \frac{1}{2\varepsilon^2}(1 - |\mathbf{u}_\varepsilon|^2)^2 \right\} dx \leqslant 3\pi \log\frac{1}{\varepsilon} + O(1).$$

Proof: Denote

$$K(\phi_0) = \begin{cases} 2\pi & \text{if } 0 \leqslant \phi_0 < \dfrac{\pi}{2}, \\ 3\pi & \text{if } \phi_0 = \dfrac{\pi}{2}. \end{cases}$$

Since

$$4\pi \sin^2 \phi_0 \leqslant \int_D |\mathrm{curl}\, u|^2 dx \leqslant \mathcal{E}_\varepsilon(\mathbf{u}_\varepsilon) \leqslant 4\pi \sin^2 \phi_0 + K(\phi_0)\varepsilon \log\frac{1}{\varepsilon} + O(\varepsilon),$$

we obtain the first estimate.

Since

$$\int_D \left\{ |\nabla \mathbf{u}_\varepsilon|^2 + \frac{1}{2\varepsilon^2}(1 - |\mathbf{u}_\varepsilon|^2)^2 \right\} dx = \frac{1}{\varepsilon}\left\{ \mathcal{E}_\varepsilon(\mathbf{u}_\varepsilon) - \int_D |\mathrm{curl}\, \mathbf{u}_\varepsilon|^2 dx \right\} = \frac{1}{\varepsilon}\{\mathcal{E}_\varepsilon(\mathbf{u}_\varepsilon) - 4\pi \sin^2 \phi_0\}$$
$$- \frac{1}{\varepsilon}\left\{ \int_D |\mathrm{curl}\, \mathbf{u}_\varepsilon|^2 dx - 4\pi \sin^2 \phi_0 \right\},$$

when $0 \leqslant \phi_0 < \pi/2$, using (4.5), (4.6) and Lemma 4.2 we obtain the second estimate. Whereas $\phi_0 = \pi/2$ we use (4.5) and Lemma 4.2 to get the third estimate. □

Proof of Theorem 4: Assume $0 \leqslant \phi_0 < \pi/2$.

Before starting the proof of Theorem 4, we recall the definition $\mathcal{S}(c_0, K, \varepsilon, \mathbf{u}_0, \Omega)$ given in Ref. 25, p. 83, where $\mathbf{u}_0 : \partial\Omega \to \mathbb{S}^1$ and $\deg(\mathbf{u}_0) = d > 0$. A map $\mathbf{u}: \Omega \to \mathbb{R}^2$ is said to belong to the class $\mathcal{S}(c_0, K, \varepsilon, \mathbf{u}_0, \Omega)$, if $\mathbf{u} \in H(\Omega, \mathbf{u}_0)$, and

(i) $\int_D \{|\nabla \mathbf{u}_\varepsilon|^2 + 1/(2\varepsilon^2)(1 - |\mathbf{u}_\varepsilon|^2)^2\} dx \leqslant 2\pi d \log(1/\varepsilon) + K$;
(ii) For $x_0 \in \Omega$ with $|\mathbf{u}(x_0)| \leqslant 1/2$, then $|\mathbf{u}(x)| \leqslant 3/4$ whenever $x \in \Omega$ and $|x - x_0| \leqslant c_0 \varepsilon$.

Recall that $\deg(\mathbf{u}_0) = 1$. We first prove that, when $0 \leqslant \phi_0 < \pi/2$, the minimizer $\mathbf{u}_\varepsilon \in \mathcal{S}(c_0, K, \varepsilon, \mathbf{u}_0, D)$ for some constants c_0 and K. In fact, from Corollary 4.3(2) we see that (i) is true. Using the elliptic estimates for the Euler equation (1.8) we see that

$$|\nabla \mathbf{u}_\varepsilon(x)| \leqslant \frac{C}{\varepsilon} \quad \text{on} \quad \Omega,$$

where $C > 0$ is independent of ε. (ii) follows from this result.

Having proved $\mathbf{u}_\varepsilon \in \mathcal{S}(c_0, K, \varepsilon, \mathbf{u}_0, D)$, using the Compactness Theorem in Ref. 25, p. 85, for any sequence $\varepsilon_n \to 0$, there is a subsequence ε_{n_j} and a point $b \in D$ such that, for $\mathbf{u}_j = \mathbf{u}_{\varepsilon_{n_j}}$,

$$\mathbf{u}_j \to \mathbf{u} \quad \text{weakly in} \quad H^1_{\mathrm{loc}}(\bar{D} \setminus \{b\}) \quad \text{as} \quad j \to \infty,$$

$$\mathbf{u} = \frac{x-b}{|x-b|} e^{ih}, \quad h \in H^1_{\mathrm{loc}}(D \setminus \{b\}),$$

$$\mathrm{dist}(b, \partial D) \geqslant \delta_*(c_0, K, \mathbf{u}_0, D) > 0.$$

Since $\{\mathrm{curl}\, \mathbf{u}_j\}$ is uniformly bounded in $L^2(D)$, there is a function $H \in L^2(D)$ such that $\mathrm{curl}\, \mathbf{u}_j \to H$ weakly in $L^2(D)$ as $j \to \infty$. On the other hand, since $\mathbf{u}_j \to \mathbf{u}$ weakly in $H^1_{\mathrm{loc}}(D \setminus \{b\})$ we have $\mathrm{curl}\, \mathbf{u}_j \to \mathrm{curl}\, \mathbf{u}$ weakly in $L^2_{\mathrm{loc}}(D \setminus \{b\})$. Therefore $H = \mathrm{curl}\, \mathbf{u}$ and

$$\mathrm{curl}\, \mathbf{u}_j \to \mathrm{curl}\, \mathbf{u} \quad \text{weakly in} \quad L^2(D) \quad \text{as} \quad j \to \infty.$$

So

$$\int_D |\mathrm{curl}\, \mathbf{u}|^2 dx \leqslant \liminf_{j \to \infty} \int_D |\mathrm{curl}\, \mathbf{u}_j|^2 dx = 4\pi \sin^2 \phi_0,$$

which implied $\mathbf{u} \in \mathcal{L}(D, \mathrm{curl}, \mathbf{u}_0)$, and hence

$$\int_D |\mathrm{curl}\, \mathbf{u}|^2 dx \geqslant \inf_{\mathbf{v} \in \mathcal{L}(D, \mathrm{curl}, \mathbf{u}_0)} \mathcal{F}(\mathbf{v}) = 4\pi \sin^2 \phi_0.$$

Thus $\mathbf{u}$ is the minimizer of the curl functional $\mathcal{F}$. By the uniqueness of the minimizers we get

$$\mathbf{u} = e^{i[\theta + \phi(r)]}, \quad \phi(r) = \arcsin(r \sin \phi_0).$$

Hence $b = 0$ and $h = \phi(r) = \arcsin(r \sin \phi_0)$.

The above argument implies that, for any sequence $\varepsilon_n \to 0$, there is a subsequence ε_{n_j} such that

$$\mathbf{u}_{\varepsilon_{nj}} \to e^{i[\theta + \arcsin(r \sin \phi_0)]} \quad \text{weakly in} \quad H^1_{\mathrm{loc}}(\bar{D} \setminus \{0\}).$$

Therefore the entire sequence $\{\mathbf{u}_\varepsilon\}$ must converge, i.e.,

$$\mathbf{u}_\varepsilon \to e^{i[\theta + \arcsin(r \sin \phi_0)]} \quad \text{weakly in} \quad H^1_{\mathrm{loc}}(\bar{D} \setminus \{0\}).$$

Now the proof of Theorem 4 is complete. □

Remark 4.1: Under the conditions of Theorem 4, we can prove that, when $0 \leqslant \phi_0 < \pi/2$, i.e., when $\mathbf{u}_0$ is not tangential to the boundary ∂D, then

$$\mathbf{u}_\varepsilon \rightarrow e^{i[\theta + \arcsin(r \sin \phi_0)]} \text{ in } C^k_{\text{loc}}(\overline{D} \setminus \{0\}). \tag{4.8}$$

In fact, from the Structure Theorem in Ref. 25, p. 83, there is a disk B_j centered at the point b and with radius $(\varepsilon_{n_j})^\alpha$, α is independent of ε, such that

$$\deg\left(\frac{\mathbf{u}_j}{|\mathbf{u}_j|}, \partial B_j\right) = 1,$$

and $|\mathbf{u}_j(x)| > 1/2$ for $x \in \overline{D} \setminus B_j$. Fix a constant δ_0 such that $0 < \delta_0 < \delta_*$. Then the ball $B(b, \delta_0)$ with center b and radius δ_0 is contained in D, and

$$\deg\left(\frac{\mathbf{u}_j}{|\mathbf{u}_j|}, \partial B(b, \delta_0)\right) = 1.$$

So

$$\deg\left(\frac{\mathbf{u}_j}{|\mathbf{u}_j|}, \partial[D \setminus B(b, \delta_0)]\right) = 0.$$

Using Lemma 4 in Ref. 25, p. 82, we also have

$$\int_{B(b,\delta_0)} \left\{ |\nabla \mathbf{u}_j|^2 + \frac{1}{2\varepsilon^2_{n_j}} (1 - |\mathbf{u}_j|^2)^2 \right\} dx \geqslant 2\pi \log \frac{1}{\varepsilon_{n_j}} + O(1).$$

From this and Corollary 4.3 we get

$$\int_{D \setminus B(b,\delta_0)} \left\{ |\nabla \mathbf{u}_j|^2 + \frac{1}{2\varepsilon^2_{n_j}} (1 - |\mathbf{u}_j|^2)^2 \right\} dx \leqslant C \tag{4.9}$$

for some $C > 0$ independent of ε. From (4.9), we can use the methods in Ref. 26 to prove (4.8).

Remark 4.2: In Theorem 4 we get the convergence of $\mathbf{u}_\varepsilon$ when $\mathbf{u}_0$ is not tangential to the boundary ∂D. We believe that the same conclusion remains true when $\mathbf{u}_0$ is tangential to ∂D.

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