JOURNAL OF DIFFERENTIAL EQUATIONS 99, 139–152 (1992)

The Global Feature of Unbounded Solutions to a Nonlinear Parabolic Equation

YUAN-WEI QI

Mathematical Institute, University of Oxford, Oxford, OX1 3LB, United Kingdom

Received April 10, 1990

1. INTRODUCTION

In this paper, we study nonlinear differential equation

$$u_t = u^\alpha(u_{xx} + u), \tag{1}$$

where $\alpha > 0$. The above equation consists of several interesting physical models, for instance, the case $0 < \alpha < 1$ corresponds to a special case of the well-known porous media equation (see [9]); the case $\alpha = 2$ arises from plasma physics, as demonstrated in [2] and [8].

It has been known (see [6, 3, 2]) for some time that the Dirichlet initial-boundary value problem defined in a finite domain or the Cauchy initial value problem defined in R^1 of the above equation may have solutions that blow-up in finite time. But the problem of finding the blow-up set and asymptotic behaviour when its solutions evolve to their blow-up time does not have a complete answer yet. In fact, only the case $0 < \alpha < 1$ has been studied in [4] but with a different formula from ours. In addition, the interesting problem concerning the similarity solutions (scaling invariant solutions) of the above equation is still open for $\alpha \geqslant 1$. Therefore the purpose of this paper is to study the blow-up set and asymptotic behaviour of the solution to (1) and the existence of related similarity solutions.

2. SIMILARITY SOLUTIONS TO EQUATION (1)

In the present section, we investigate the existence of similarity solutions to Eq. (1). We use these similarity solutions in Section 3 to study the blow-up of general solutions to the Cauchy problem

$$\begin{aligned} &u_t = u^\alpha(u_{xx} + u), \qquad x \in R^1, \qquad t > 0, \\ &u(x, 0) = u_0(x) \geqslant 0, \qquad x \in R^1, \qquad \operatorname{supp} u_0(x) < \infty, \end{aligned} \tag{2}$$

0022-0396/92 $5.00

where $\alpha > 0$, $u_0(x) \neq 0$ is a continuous function with compact support. We first mention an interesting property of the solutions to (2).

PROPOSITION 1. *Let u be a solution of* (2). *Suppose that* $u_0(x)$ *has compact support*

$$l(0) = \operatorname{supp} u_0(x) = (l_-(0), l_+(0)). \tag{3}$$

Then

$$l(t) = \operatorname{supp} u(x, t) = (l_-(t), l_+(t)) \tag{4}$$

is finite as long as u is bounded for all $(x, s) \in R^1 \times [0, t]$. *Furthermore,*

$$-\infty < l_-(t) \leqslant l_-(0) < l_+(0) \leqslant l_+(t) \tag{5}$$

and

$$l_+(t) - l_-(t) \leqslant l_+(0) - l_-(0) + 2L_s, \tag{6}$$

where $L_s = 2\pi/\alpha$.

For the proof see [9].

IIt is easy to verify by using the invariance property of equation (1) that the similarity solutions of (1) have the formula

$$u(x, t) = (T - t)^{-1/\alpha} w(x), \tag{7}$$

where the function $w(x)$ is a solution of the initial value problem

$$\begin{gathered} w^\alpha(w'' + w) - \frac{w}{\alpha} = 0, \\ w'(0) = 0, \qquad w(0) = \eta > 0. \end{gathered} \tag{8}$$

In the above equation "$''$" stands for the second derivative of w to x. We use w' and w'' to represent the first and second derivatives of w to x from here on. We note that the formula (7) is very similar to the technique of separation of variables for the linear equation

$$u_t = u_{xx} + u. \tag{9}$$

We show in the following that, for the problem (8), the case of $0 < \alpha < 2$ is more interesting than that of $\alpha > 2$. For this reason we treat these two cases separately. The existence result of $0 < \alpha < 2$ is stated in the following theorem.

THEOREM 1. *Let $w(x, \eta)$ be a solution of* (8) *which has the initial value $w(0)=\eta$. Let $0<\alpha<2$. If $\eta \in (0, \eta_s)$ with $\eta_s=(2/\alpha(2-\alpha))^{1/\alpha}$, then $w(x, \eta)>0$ in $(0, \infty)$ and w oscillates around $w_0 \equiv (1/\alpha)^{1/\alpha}$ an infinite number of times. Consequently $w(\infty, \eta) \neq 0$. If $\eta > \eta_s$, then $w(x, \eta)$ vanishes at some point $x=x_\eta \in R^+$ and $w'(x_\eta, \eta) \neq 0$. The only non-trivial ($w \neq 0$, $w \neq w_0$) non-negative solution in $C^1(R^1)$ is $w(x, \eta_s)$ which terminates at some $x=x_{\eta_s} \in R^+$ and $w'(x_{\eta_s}, \eta)=0$.*

Proof. It is easy to prove by using classical theory that, for each $\eta>0$, the solution $w(x, \eta)$ of (8) exists for x close to zero and that w is twice continuously differentiable as long as w is positive. Set

$$F(x) \equiv \frac{(w')^2}{2}(x) + \frac{w^2(x)}{2} - \frac{w^{2-\alpha}(x)}{\alpha(2-\alpha)}. \tag{10}$$

By multiplying (8) by w' and integrating over $[0, x]$, we have

$$F(x)=F(0)=\frac{w^2(0)}{2} - \frac{w^{2-\alpha}(0)}{\alpha(2-\alpha)}. \tag{11}$$

Let us take $\eta > \eta_s$. In this case, we obtain from (11) that

$$F(x)=F(0)>0, \tag{12}$$

as long as w is non-negative on $[0, x]$. Furthermore, since $w(0)>\eta_s>w_0$, (8) yields that $w''(0)<0$, and so $w'(x)<0$ for all $0<x \ll 1$. Moreover, by writing (8) as the equivalent integral equation

$$w'(x)-w'(x_0)=\int_{x_0}^{x} \left(\frac{w^{1-\alpha}}{\alpha} - w\right) dx, \tag{13}$$

we can deduce easily that if $w'(x_0)=0$ for some $x_0 \in [0, \infty)$ and w is positive on $[0, \infty)$, then there will be an intersection of w with w_0 on (x_0, ∞). Therefore, any solution w of (8) must have at least one intersection with w_0. Suppose that w is positive on $(0, \infty)$. Then (12) implies that w is bounded above. Thus, there are only two possibilities:

(a) the solution intersects w_0 more than once;

(b) the solution goes to zero monotonically as $x \to \infty$.

If (a) holds, then we can see clearly that there exists a $x_0>0$, such that

$$w'(x_0)=0, \qquad w'(x)<0 \qquad \text{for all} \quad 0<x<x_0. \tag{14}$$

In addition, since $w''(x_0) \geq 0$,

$$w(x_0)<w_0<\eta_s. \tag{15}$$

In that case,

$$F(x_0) = \frac{w^2(x_0)}{2} - \frac{w^{2-\alpha}(x_0)}{\alpha(2-\alpha)} < 0. \tag{16}$$

But, this contradicts (12). Therefore (a) cannot hold. Similarly, (b) cannot hold, for otherwise this would contradict the fact that $F(x) = F(0) > 0$ on $(0, \infty)$, since (b) yields that $w'(x) \to 0$ as $x \to \infty$, which in turn implies $F(x) \to 0$ as $x \to \infty$. Thus, the solution cannot be positive on $(0, \infty)$, i.e., the solution must terminate at zero at some finite point x_η. The relation $F(x_\eta) = F(0) > 0$ then yields that $w'(x_\eta) \neq 0$. This proves the claim concerning solutions with initial values $\eta > \eta_s$.

If $0 < \eta < \eta_s$, then $F(x) = F(0) < 0$ as long as w is non-negative on $[0, x]$. We observe further that if $w(x) = \eta_s$ at some point x, then $F(x) \geqslant 0$. So $0 \leqslant w < \eta_s$ on the same interval $[0, x]$ where w is non-negative. If $w(x) = 0$ at some point, then we would have $F(x) > 0$, which is impossible. Therefore, $w(x) > 0$ on $(0, \infty)$. We prove further that $w(x)$ oscillates around w_0 an infinite number of times. First, it is clear that if $w'(x_1) = 0$ at some $x_1 > 0$, then there exists an $x_2 > x_1$ such that $w'(x)$ has a fixed sign on $[x_1, x_2]$ and $w(x_2) = w_0$. Therefore if w has only a finite number of intersections with w_0 on $(0, \infty)$, there must hold $w' \neq 0$ for all $x \geqslant 1$. But, as w is bounded from both above and below on $(0, \infty)$, w'' has to have the opposite sign from w' for all x large. In this case, (2) would yield that $w \to w_0$ or $w \to 0$ as $x \to \infty$, which in turn implies that $w'(x) \to 0$ as $x \to \infty$. If $w \to 0$ as $x \to \infty$, then $F(0) < F(\infty) = 0$, which contradicts $F(x) = F(0)$ on $[0, \infty]$. On the other hand, if $w \to w_0$ as $x \to \infty$, then $F(\infty) < F(0)$, which again contradicts the condition $F(x) = F(0)$ on $[0, \infty]$. Hence, the hypothesis that w has only a finite number of intersections with w_0 is absurd. Thus, w oscillates around w_0 an infinite number of times if $w(0) = \eta < \eta_s$.

If $\eta = \eta_s$, then $w(x)$ terminates at zero at some finite point $x = x_{\eta_s}$ and from the condition $F(x) = F(0) = 0$, we have $w'(x_{\eta_s}) = 0$. This completes the proof of Theorem 1. Q.E.D.

For $\eta = \eta_s$, we can actually give the analytic expression of $w(x, \eta_s)$. Indeed, suppose

$$w(x) = \begin{cases} A(\cos^2 Bx)^{1/\alpha} & x \in [-\pi/2B, \pi/2B], \\ 0 & \text{otherwise} \end{cases}$$

is a solution of (2), where A and B are two positive constants. Then it is easy to calculate that for $x \in [-\pi/2B, \pi/2B]$,

$$\begin{aligned} w'(x) &= -\frac{AB}{\alpha}\sin 2Bx(\cos^2 Bx)^{(1-\alpha)/\alpha}, \\ w''(x) &= \frac{AB^2}{\alpha^2}(-4(\cos^2 Bx)^{1/\alpha} + (4-2\alpha)(\cos^2 Bx)^{(1-\alpha)/\alpha}). \end{aligned} \tag{17}$$

So

$$w^{\alpha}(w'' + w) - \frac{w}{\alpha} \tag{18}$$

$$= (\cos^2 Bx)^{(1+\alpha)/\alpha} A^{1+\alpha}\left(1 - \frac{4B^2}{\alpha^2}\right) + A(\cos^2 Bx)^{1/\alpha}\left(A^{\alpha}B^2\frac{4-2\alpha}{\alpha^2} - \frac{1}{\alpha}\right). \tag{19}$$

Thus, if we take

$$B = \frac{\alpha}{2}, \qquad A = \left(\frac{2}{\alpha(2-\alpha)}\right)^{1/\alpha}. \tag{20}$$

together with $0 < \alpha < 2$, the function

$$w(x) = \begin{cases} \left(\frac{2}{\alpha(2-\alpha)}\right)^{1/\alpha}\left(\cos^2 \frac{\alpha}{2}x\right)^{1/\alpha} & x \in [-\pi/\alpha, \pi/\alpha], \\ 0 & \text{otherwise,} \end{cases}$$

will be the solution of (8) which is C^1 continuous in R^1. It is easy to verify that with the constants A and B given in (20), $0 < \alpha < 2$,

$$w(x) = \begin{cases} A(\sin^2 Bx)^{1/\alpha} & x \in [0, \pi/B], \\ 0 & \text{otherwise,} \end{cases}$$

is also a solution of (8). But this time instead of having $w'(0) = 0$, we have $w(0) = 0$.

Remark. If $\alpha < 0$, which corresponds to the fast-diffusion problem, we can construct some explicit similarity solutions of (2) which exist on $(0, \infty)$ and tend to infinity at $t \to \infty$. These similarity solutions are

$$u(x, t) = \begin{cases} (T+t)^{-1/\alpha} A(\cos^2 Bx)^{1/\alpha}, & x \in [-\pi/2B, \pi/2B], \\ 0 & \text{otherwise,} \end{cases}$$

with

$$B=-\frac{\alpha}{2}, \qquad A=\left(\frac{2}{-\alpha(2-\alpha)}\right)^{-1/\alpha}. \tag{21}$$

We shall discuss the similarity solutions of a more general class of fast-diffusion equation in a forthcoming paper. Next we study (8) for the case of $\alpha > 2$. The main result is the following proposition:

PROPOSITION 2. *Let $\alpha > 2$ and let $w(x, \eta)$ be a solution of* (8) *with the initial value $w(0) = \eta$. Then every solution is positive and bounded both from above and below away from zero on* $(0, \infty)$. *Furthermore, it oscillates around w_0 an infinite number of times.*

Proof. Let $F(x)$ be the function defined in (10). Then for $\alpha > 2$, the identity holds

$$F(x) = F(0) > 0, \qquad x > 0. \tag{22}$$

So w cannot tend to zero (as $w^{2-\alpha}$ will tend to infinity) or tend to infinity (as w^2 will tend to infinity). Therefore, w is positive for all $x > 0$ and bounded both from above and below. This means that the first assertion in the proposition holds. The second assertion follows from the same kind of reasoning as in Theorem 1. Q.E.D.

Note. Because of the above result for the case of $\alpha > 2$, there exist no similarity solutions of (2) which blow-up and which have finite support. So instead of seeking similarity solutions which blow-up and are of the form (2), we look for similarity solutions which exist for all time and decay to zero as $t \to \infty$, that is, the solutions u of (2) which take the form

$$u(x, t) = (T+t)^{-1/\alpha}\, w(x), \qquad T > 0. \tag{23}$$

It is easy to prove that, for $\alpha > 2$,

$$u(x, t) = \begin{cases} (T+t)^{-1/\alpha} A(\cos^2 Bx)^{1/\alpha} & x \in [-\pi/2B, \pi/2B], \\ 0 & \text{otherwise,} \end{cases}$$

where

$$B = \frac{\alpha}{2}, \qquad A = \left(\frac{2}{\alpha(\alpha-2)}\right)^{1/\alpha}, \tag{24}$$

is a solution of the initial boundary value problem

$$u_t = u^\alpha(u_{xx} + u), \qquad x \in \left(-\frac{\pi}{\alpha}, \frac{\pi}{\alpha}\right). \qquad t > 0, \tag{25}$$

$$u = 0, \qquad x = \pm\frac{\pi}{\alpha}, \qquad t > 0, \tag{26}$$

$$u(x, 0) = u_0(x) \geqslant 0, \qquad x \in \left[-\frac{\pi}{\alpha}, \frac{\pi}{\alpha}\right].$$

The solutions of the above initial boundary value problem can also be regarded as generalized solutions of the Cauchy problem (2), which have time-independent compact support $[-\pi/\alpha, \pi/\alpha]$, but they are no longer C^1 continuous in R^1.

Again as before, for $\alpha > 2$,

$$u(x, t) = \begin{cases} u(x, t) = (T + t)^{-1/\alpha} A(\sin^2 Bx)^{1/\alpha} & x \in [0, \pi/B], \\ 0 & \text{otherwise,} \end{cases}$$

is a solution of (2) which exists for all $t > 0$, where A and B are as given in (24).

We conclude this section by proving a result which describes the asymptotic profile of solutions of (8) as $\eta \to \infty$.

THEOREM 2. *Let $w(x, \eta)$ be a solution of (8). We assume that $\alpha > 0$. Then for any fixed $x \in [0, \pi/2)$ there holds*

$$w(x, \eta) \sim \eta \cos x \tag{27}$$

as $\eta \to \infty$.

Proof. Let

$$v(x, \eta) = \frac{w(x, \eta)}{\eta}. \tag{28}$$

Then v satisfies the differential equation

$$v'' + v - \frac{\eta^{-\alpha} v^{1-\alpha}}{\alpha} = 0,$$
$$v'(0) = 0, \qquad v(0) = 1. \tag{29}$$

It is easy to verify that $z \equiv \cos x$ is the solution of the differential equation

$$\begin{aligned} z'' + z = 0, \\ z'(0) = 0, \qquad z(0) = 1, \end{aligned} \tag{30}$$

and $\phi = \varepsilon + (1-\varepsilon)\cos x$ is a solution of the differential equation

$$\begin{aligned} \phi'' + \phi = \varepsilon, \\ \phi'(0) = 0, \qquad \phi(0) = 1. \end{aligned} \tag{31}$$

Therefore, by using the Sturmian Comparison Theorem to compare v with z, we can deduce that

$$v(x, \eta) \geqslant \cos x, \qquad x \in [0, \pi/2). \tag{32}$$

Thus,

$$0 \leqslant v'' + v \leqslant \frac{\eta^{-\alpha} M}{\alpha}, \qquad x \in [0, x_1], \tag{33}$$

where $0 < x_1 < \pi/2$, $M = \max(\cos^{1-\alpha} x_1, 1)$. Next, by using the Sturmian Comparison Theorem again to compare v with ϕ, we get

$$v \leqslant \frac{\eta^{-\alpha} M}{\alpha} + \left(1 - \frac{\eta^{-\alpha} M}{\alpha}\right) \cos x, \qquad x \in [0, x_1]. \tag{34}$$

Therefore,

$$0 \leqslant v - \cos x \leqslant \frac{\eta^{-\alpha} M}{\alpha} (1 - \cos x), \qquad x \in [0, x_1]. \tag{35}$$

Thus,

$$w(x, \eta) \sim \eta \cos x, \qquad x \in \left[0, \frac{\pi}{2}\right), \tag{36}$$

as $\eta \to \infty$. Q.E.D.

3. Applications to General Blow-up

As the first application of our similarity solutions we give a sufficient condition for the blow-up of solutions of (2). More precisely, we have the following theorem:

THEOREM 3. *Let* $0<\alpha<2$ *and let* $u_0(x)\geqslant 0$ *be a function which satisfies* $\operatorname{supp} u_0(x)\supset(a,b)$. *If* $b-a>2\pi/\alpha$, *then the solution* $u(x,t)$ *of* (2) *with initial value* $u(x,0)=u_0(x)$ *will blow-up in finite time.*

An obvious consequence of the above theorem is the following result:

COROLLARY 1. *Let* $0<\alpha<2$. *Let* $u(x,t)$ *be a solution of the Cauchy problem* (2) *with a non-identically zero initial value* $u_0(x)\geqslant 0$. *If there exists* $t>0$, *such that* $\operatorname{supp} u(x,t)\supset(a,b)$ *and* $b-a>2\pi/\alpha$, *then the solution* $u(x,t)$ *will blow-up in finite time.*

The proof of Theorem 3 depends on the maximum principle which we state as a lemma in what follows.

LEMMA 1. *Let* $u(x,t)$ *be a generalized solution (see* [6] *for the definition) of the Cauchy problem* (2) *in* $Q_T=R^1\times(0,T)$. *Suppose the function* $z\colon D_T\to R^1_+$ *is such that* $z\in C^{2,1}(D_T)\cap C(\overline{D_T})$, *where*

$$D_T=\{x\in R^1 \mid \|x\|<\xi(t)\times[0,T)\}, \tag{37}$$

$\xi(t)\in C([0,T))$ *is some non-negative function,* $z=0$ *everywhere in* $Q_T\backslash D_T$, *and satisfies the inequality* $z_t-z^\alpha(z_{xx}+z)\leqslant 0$ $(z_t-z^\alpha(z_{xx}+z)\geqslant 0)$ *in* D_T. *In addition let the derivative* z_x *be continuous in* Q_T, *and let* $z(x,0)\leqslant u_0(x)$ $(z(x,0)\geqslant u_0(x))$ *everywhere in* R^1. *Then* $u\geqslant z$ $(u\leqslant z)$ *everywhere in* Q_T.

For the proof of the above lemma see [6].

Proof of Theorem 3. This is immediate on using the maximum principle stated in the above lemma to compare the solutions of (2) with the similarity solution we constructed in Section 2. Q.E.D.

In the following we prove that the explicit similarity solutions which were constructed in Section 2 correctly describe the asymptotic behaviour of the evolution of the unbounded solutions to the Cauchy problem (2). This is one of the major motivations for studying similarity solutions. Our approach is a generalization of that of [4], where the case $0<\alpha<1$ has been considered under the normal formulation of the porous media equations. Nevertheless, as there are some important details which have been neglected in [4] and more importantly, as the formulation of [4] does not extend to the case $1\leqslant\alpha<2$, we think it is worthwhile presenting a self-contained proof here and to include the necessary modifications. In the following we always assume that the initial function satisfies

$$\begin{gathered} u_0(-x)=u_0(x), \qquad x\in R^1, \\ u_0(x) \text{ decreasing for } x>0 \qquad \text{and} \qquad \sup u_0(x)=u_0(0). \end{gathered} \tag{38}$$

Our main result in this section is the following theorem:

THEOREM 4. *Let $u(x, t)$ be a solution of (2) which blows-up in finite time $T<\infty$. If $u_0(x) \in C^1(R^1)$ satisfies (38) and $0<\alpha<2$, then the following holds*:

$$(T-t)^{1/\alpha} u(x, t) \to w(x), \qquad \text{as} \quad t \to T^- \tag{39}$$

uniformly in R^1, where w is the similarity solution given in Section 2.

Following Giga and Kohn [7], we introduce the "new time" variable

$$s = -\log(T-t); \qquad [0, T) \to (0, \infty) \tag{40}$$

and define

$$g(x, s) = (T-t)^{1/\alpha} u(x, t). \tag{41}$$

Then $g(x, s)$ is a solution of the Cauchy problem

$$\begin{aligned} &g_s = g^\alpha(g_{xx} + g) - \frac{g}{\alpha}, \qquad s>0, \qquad x \in R^1, \\ &g(x, 0) = g_0(x) \equiv T^{1/\alpha} u_0(x), \qquad x \in R^1. \end{aligned} \tag{42}$$

It is easy to see that the similarity solution

$$w(x) = \begin{cases} \left(\dfrac{2}{\alpha(2-\alpha)}\right)^{1/\alpha} \left(\cos^2 \dfrac{\alpha}{2} x\right)^{1/\alpha} & x \in [-\pi/\alpha, \pi/\alpha], \\ 0 & \text{otherwise}, \end{cases}$$

is a "generalized" (for definition see [4]) stationary solution of (42). So the asymptotic convergence of (39) is reduced to the problem of proving that the solutions of (42) tends to the stationary solution as $s \to \infty$. Recall that the problem (2) has the heat localization property. Therefore, supp $u(x, t)$ and so supp $g(x, s)$ is uniformly bounded for all $0<s<\infty$. Thus, if we can establish bounds in $L^\infty(R^1)$ for $g(x, s)$, we may treat $g(x, s)$ as a solution of a boundary value problem in a finite domain. In this case, the study of the convergence of $g(x, s)$ to $w(x)$ in the Cauchy problem (42) can be regarded as equivalent to the study of the convergence of a boundary value problem. The convergence of boundary value problem is a direct consequence of some general results [1]. So the most important feature of the proposed "boundary value problem" approach is the establishment of bounds on $g(x, s)$. An upper bound will ensure that $g(x, s)$ is globally bounded, and a lower bound is needed to make sure that the limit function $g(x, \infty)$ is non-trivial. The real difficulty in our approach is that $g = w(x)$ is an unstable stationary solution of (42), which is satisfied by the solution $g(x, s)$. We show this later in this section. By the same token, in proving (39), we must distinguish in the initial function space $\{g(x, 0)\}$ a stability set which guarantee that $g(x, s)$ is in the stable manifold of $w(x)$.

LEMMA 2. *Let $u(x, t)$ be a solution of* (2) *with an initial value $u_0(x)$ satisfying* (38). *Let u have continuous first derivative to x. Suppose u blows-up in finite time $T < \infty$. Then there exists $t > 0$ such that*

$$\text{mes}(\text{supp}\, u(x, t)) > \pi. \tag{43}$$

Proof. Suppose (43) is not true, then for all $0 < t < T$

$$\text{supp}\, u(x, t) \subset \left[-\frac{\pi}{2}, \frac{\pi}{2} \right], \tag{44}$$

since $u(x, t)$ is even as a consequence of (38) (symmetry of u_0). The function

$$h(x) = \begin{cases} M \cos x & x \in [-\pi/2, \pi/2], \\ 0 & \text{otherwise}, \end{cases}$$

can be regarded as a generalized solution of (2). Therefore, a direct comparison of u with $h(x)$ with $M > 0$ being a large positive number, will yield that $u(0, T^-)$ is bounded which contradicts the fact that u blows-up at $t = T$. Thus, (43) must hold. Q.E.D.

LEMMA 3. *Let all the conditions of Theorem 4 be satisfied. Then for $t \in [0, T)$, the following inequality holds*

$$\sup_{x \in R^1} u(x, t) > \alpha^{-1/\alpha}(T - t)^{-1/\alpha}. \tag{45}$$

On the other hand, there is an $\eta > \eta_s$ such that

$$\sup_{x \in R^1} u(x, t) < \eta(T - t)^{-1/\alpha}. \tag{46}$$

Proof. In this proof we assume that the solution $u(x, t)$ of (2) is classical at all points where it is positive and $u_x(x, t)$ is a continuous function of (x, t) in $R^1 \times (0, T)$. To prove (45), we define

$$U(t) = \sup_{x \in R^1} u(x, t). \tag{47}$$

Then we get from (2)

$$U'(t) \leqslant U^{\alpha+1}, \qquad 0<t<T. \tag{48}$$

An integration of the above inequality over $[t, T]$ implies

$$\frac{1}{U^{\alpha}(t)} \leqslant (T-t), \qquad 0<t<T, \tag{49}$$

and so,

$$U(t) > \alpha^{-1/\alpha}(T-t)^{-1/\alpha}, \qquad 0<t<T. \tag{50}$$

To prove (46) holds, we observe that the length of $\operatorname{supp} u(x, t)$ must be greater than π for all $t_0<t<T$, where $0 \leqslant t_0 < T$. So without loss of generality, we assume that $\operatorname{mes}(\operatorname{supp} u_0) > \pi$. We choose $\eta \geqslant \eta_s$ to be so large that

$$\eta T^{-1/\alpha} > u_0(0) = \sup u_0(x) \tag{51}$$

and, for the corresponding solution $w(x, \eta)$ to (8), $\operatorname{supp} w(x, \eta)$ is a proper subset of $\operatorname{supp} u_0$. Moreover, we can, if necessary, increase η so that the functions $T^{-1/\alpha} w(x, \eta)$ and $u_0(x)$ "intersect" not more than twice. At the same time, we may regard $w(x, \eta)$ as an even function in R^1 which coincides, whenever it is positive, with the solution of (8) which has the initial value η and it equals zero for $|x| > x_\eta$. In that case, (46) follows by comparing $u(x, t)$ with the unbounded solution

$$v(x, t) = (T-t)^{-1/\alpha} w(x, \eta) \tag{52}$$

of (2), which has the same blow-up time as $u(x, t)$, and by using the "intersection" decreasing property proved in [5] for solutions having the same blow-up time. Q.E.D.

Next we show that the similarity solution we constructed in Section 2 is unstable.

PROPOSITION 3. *The similarity solution $w(x, \eta_s)$ in Section 2 is not a stable stationary solution to the Cauchy problem* (42).

Proof. Let $g(x, 0) = \delta w(x, \eta_s)$ and $\delta \in (0, 1)$. Then the Cauchy problem (42) has a global solution

$$g(x, s) = (1 + C\exp(s))^{-1/\alpha} w(x, \eta), \tag{53}$$

where $C = (1-\delta^{1/\alpha})/\delta^{1/\alpha} > 0$ and

$$g(x, s) \to 0 \qquad \text{as} \quad s \to \infty \text{ in } R^1. \tag{54}$$

Conversely, if $\delta > 1$, and $g(x, 0) = \delta w(x, \eta_s)$, then the solution of Cauchy problem (42) is unbounded,

$$g(x, s) \to \infty \qquad \text{as} \quad s \to S_0(\delta) \tag{55}$$

for all $|x| < L_s/2$, where $S_0(\delta) = \log[\delta^{1/\alpha}/(\delta^{1/\alpha} - 1)]$. Therefore the stationary solution $g = w(x, \eta_s)$ is unstable. Q.E.D.

We note, however, under the condition of Theorem 4, i.e., if the initial value $u_0(x)$ is taken so that $u(x, t)$ and $u_A(x, t) = (T-t)^{-1/\alpha} w(x, \eta_s)$ have the same blow-up time, then the Cauchy problem (42) has a global solution which converges to the stationary solution $g(x, \eta_s) \equiv w(x, \eta_s)$. Hence, the stability set of the unstable stationary solution $g(x, \eta_s)$ is

$$\Phi = \{ g_0(x) \geqslant 0 \mid g_0(x) = (T)^{1/\alpha} u_0(x),\ u_0(x) \in C^1(R^1) \text{ satisfies (38) and } T = T(u_0) < \infty \text{ is the blow-up time of the solution } u(x, t) \}. \tag{56}$$

Proof of Theorem 4. The convergence

$$g(x, s) \to w(x, \eta_s), \qquad \text{as} \quad s \to \infty, \tag{57}$$

follows from the heat localization property, the upper bound (46) and the lower bound (45). Indeed, if we denote supp $u_0(x)$ by $[-l_0, l_0]$, $l_0 > 0$, then we have

$$\text{supp } g(x, s) \subset [-l_0 - L_s, l_0 + L_s], \tag{58}$$

$$\sup_{x \in R^1} g(x, s) > \alpha^{-1/\alpha}, \qquad s > 0, \tag{59}$$

$$g(x, s) < \eta, \qquad s > 0, \qquad x \in R^1. \tag{60}$$

Let $\Omega \in R^1$ be an arbitrary bounded region such that $[-l_0 - L_s, l_0 + L_s] \subset \Omega$. Therefore, (58) implies that

$$g(x, s) = 0, \qquad s > 0, \qquad x \in \partial\Omega. \tag{61}$$

Hence the Cauchy problem (42) is equivalent to the Dirichlet boundary value problem in $\Omega \times R^1$ with conditions (59) and (60) which ensure its global solvability. Thus, the assertion of Theorem 4 follows from the result of [1] and we omit the proof here. Q.E.D.

REFERENCES

1. V. S. BELONOSOV AND T. I. ZELENYAK, "Nonlocal Problems in the Theory of Quasi-linear Parabolic Equations," Izd. Novosib. Univ., Novosibirsk, 1976.
2. A. FRIEDMAN AND J. B. MCLEOD, Blow-up of solutions of nonlinear degenerate parabolic equations, *Arch. Rational Mech. Anal.* **96** (1987), 55–80.
3. V. A. GALAKTIONOV, Boundary-value problem for the nonlinear parabolic equation $u_t = \Delta u^{\sigma+1} + u^{\beta}$, *Differential Equations* **21** (1983), 1544–1559.
4. V. A. GALAKTIONOV, Asymptotic behaviour of unbounded solutions of the nonlinear parabolic equation $u_t = (u^{\sigma} u_x)_x + u^{\sigma+1}$, *Differential Equations* **21**, No. 1 (1985).
5. V. A. GALAKTIONOV, Proof of the localization of unbounded solutions of the nonlinear parabolic equation $u_t = (u^{\sigma} u_x)_x + u^{\beta}$, *Differential Equations* **21** (1985), 15–23.
6. V. A. GALAKTIONOV, S. P. KURDYUMOV, A. P. MIKHAILOV, AND A. A. SAMARSKII, Unbounded solutions of the Cauchy problem for the parabolic equation $u_t = \nabla(u^{\sigma} \nabla u) + u^{\beta}$, *Soviet Phys. Dokl.* **25** (1980), 458–459.
7. Y. GIGA AND R. V. KOHN, Asymptotically self-similar blow-up of semilinear heat equations, *Comm. Pure Appl. Math.* **38** (1985), 297–319.
8. B. C. LOW, Resistive diffusion of force-free magnetic fields in a plasma medium, *Astrophys. J.* **181** (1973), 209–226.
9. Y.-W. QI, "The Blow-Up of Quasilinear Parabolic Equations," Ph.D. Thesis, University of Oxford, 1989.