



# Traveling waves solutions to general isothermal diffusion systems



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## ABSTRACT

This article studies propagating traveling waves in a class of reaction–diffusion systems in one dimensional space which model isothermal diffusion, bio-reactor and auto-catalytic chemical reactions. In addition, the reaction terms are non-KPP, which denies a linear theory for determining minimum speed. By using novel techniques, it is shown that the minimum speed depends, for a given reaction term, on the size of ratio of diffusion coefficients  $D$  in a subtle way; with the case  $D > 1$  demonstrating very different properties from that of  $D < 1$ .

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## 1. Introduction

In this paper we study reaction–diffusion systems of the form

$$\begin{cases} u_t = u_{xx} - H(u, v), \\ v_t = Dv_{xx} + H(u, v), \end{cases} \quad (1.1)$$

where  $H$  is a non-negative  $C^1$  function defined on  $[0, \infty) \times [0, \infty)$ ,  $D > 0$  is a positive number. We assume  $H(u, v)$  satisfies the following conditions.

(A1)  $H(u, v) > 0$  on  $(0, 1) \times (0, 1)$ ,  $H(0, 1) = H(1, 0) = 0$ .

The particular feature we are interested in is the existence and non-existence of traveling waves connecting the two equilibrium points  $(u, v) = (0, 1)$  and  $(u, v) = (1, 0)$ . Let  $u(x, t) = u(z)$ ,  $v(x, t) = v(z)$ , with

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$z = x - ct$ ,  $c > 0$ , we arrive at the traveling wave problem:  $(u, v)$  is a positive solution of

$$(TW) \begin{cases} u''(z) + cu'(z) - H(u, v) = 0, & -\infty < z < \infty, \\ Dv''(z) + cv'(z) + H(u, v) = 0, & -\infty < z < \infty, \\ \lim_{z \rightarrow -\infty} (u(z), v(z)) = (0, 1), & \lim_{z \rightarrow \infty} (u(z), v(z)) = (1, 0), \\ \lim_{z \rightarrow \pm\infty} u'(z) = \lim_{z \rightarrow \pm\infty} v'(z) = 0, \end{cases} \quad (1.2)$$

where the positive constant  $c$  is the wave speed.

Many important phenomena in applications such as isothermal diffusion, population dynamics, bio-reactors and chemical reactions can be modeled by a system of the form as in (1.1). Since we are only interested in developing a mathematical theory on traveling waves, we refer the interested reader to [1–10] for the modeling aspects.

The main purpose of this work is to demonstrate that a number of special cases being studied in the literature can be united into a general case with minimum assumption on the nonlinearity  $H$ , thus revealing that results established for special cases are valid for more general systems. Moreover, we study some new systems which are totally different from what appears in the literature.

It is clear by adding the two equations in (TW) that  $u''(z) + Dv''(z) + c(u'(z) + v'(z)) = 0$ , and by integration on  $(-\infty, z)$ ,

$$u'(z) + Dv'(z) + c(u(z) + v(z) - 1) = 0. \quad (1.3)$$

In particular, if  $D = 1$ , we obtain that  $u(z) + v(z) \equiv 1$  and the problem reduces to a single equation

$$v''(z) + cv'(z) + F(v) = 0, \quad (1.4)$$

where  $F(v) = H(1 - v, v) > 0$  on  $(0, 1)$ ,  $F(0) = F(1) = 0$ , and  $F'(0) \geq 0 \geq F'(1)$ . By classical theory, see [11,12,10], there is a minimum speed  $c_1 > 0$  such that traveling wave exists for any  $c \geq c_1$ .

One important aspect of our approach is to link the general case of  $D > 0$  to the special case of  $D = 1$  using comparison argument, where previous studies of (TW) with general  $H$  fail to explore.

The present paper concentrates on two classes of nonlinearity  $H$  with two very different behavior about  $(0, 1)$ : (i)  $H_u(0, 1) > 0$  but  $H_v(0, 1) = 0$  and (ii)  $H(u, v) \approx u^\alpha v^\beta$  with  $\alpha, \beta > 1$ . The case of  $H_u(0, 1) > 0$  and  $H_v(0, 1) = 0$  put previous works in the literature on the case of  $H(u, v) = uv^n$  with  $n > 1$  and  $H(u, v) = uv^n/(1 + u)$  (see [2,13]) into a more general framework. The case where both  $H_u(0, 1) = 0$  and  $H_v(0, 1) = 0$  is completely different and unknown. Our study of the case of  $H(u, v) \approx u^\alpha v^\beta$  with  $\alpha, \beta > 1$  reveals some very interesting phenomena.

**Theorem 1.** *Let  $D > 1$ . Suppose  $H(u, v)$  is a  $C^2$  function which satisfies the following conditions:*

- (i) *Assumption (A1), and  $H_u(0, 1) > 0$  and  $H_v(0, 1) = 0$ ;*
- (ii)  *$H(u, v) \leq H_u(0, 1)u$ ,  $H(u, v)$  is an increasing function of  $u$  on  $(0, D)$ , for any  $v \in (0, 1)$  fixed, and  $H(\alpha u, v) \leq \lambda_0 \alpha H(u, v)$  on  $(0, 1) \times (0, 1)$  for any  $0 < \alpha < D$ , where  $\lambda_0 > 0$  is a fixed number;*
- (iii)  *$X(c)^T M X(c) < 0$  for any  $c > 0$ , where  $M$  is the Hessian matrix of  $H$  at  $(0, 1)$  and*

$$X(c) = \left[ \left( \frac{2}{c + \sqrt{c^2 + 4H_u(0, 1)}} \right)^2, -H_u(0, 1) \right].$$

*Then, there exists a unique traveling wave (up to translation) for any*

$$c \geq c_1 \frac{\sqrt{D\lambda_0}}{\sqrt{1 - (1 - \frac{1}{D})^{\frac{L-1}{L+1}}}},$$

where

$$L = \sqrt{1 + \frac{4H_u(0,1)}{\lambda_0 c_1^2}}.$$

On the other hand, there exists no traveling wave when

$$c < \sqrt{D}c_1.$$

**Remark 1.** It is clear from [Theorem 1](#) that the minimum speed is of order  $\sqrt{D}$  when  $D \gg 1$ , and the result, which is qualitatively similar to the case of  $H(u, v) = uv$ , holds for very wide class of functions  $H(u, v)$ . The key assumption is  $H(u, v) \leq H_u(0, 1)u$  which corresponds directly to  $F'(v) \leq F'(0)v$  for the single equation [\(1.4\)](#), a condition that characterizes the KPP-type of nonlinearity of  $F$ . Nevertheless, our result follows from extensive analysis rather than directly from the linearized system at the equilibrium points. Furthermore, as the following result shows, the case of  $D < 1$  has very different property and more detailed information on  $H$  is called upon.

**Theorem 2.** Let  $0 < D < 1$ . Suppose  $H(u, v)$  is a  $C^1$  function which satisfies the following conditions:

- (i) Assumption (A1), and  $H_u(0, 1) > 0$  and  $H_v(0, 1) = 0$ ;
- (ii)  $H(u, v)$  is an increasing function of  $u$  on  $(0, D)$ , for any  $v \in (0, 1)$  fixed;
- (iii)  $\lambda_2 uv^n \leq H(u, v) \leq \lambda_3 uv^n$  on  $(0, 1) \times (0, 1)$  with  $n > 1$ ,  $0 < \lambda_2 \leq \lambda_3$ .

Then, there exists a unique traveling wave for any

$$c \geq \frac{4D\lambda_3}{\sqrt{4\lambda_3 D + \lambda_2}}$$

if  $n \geq 2$  and

$$c \geq \max \left( \frac{4D\sqrt{\lambda_3}(n-1)}{\sqrt{1+4D(n-1)}}, \frac{8D\lambda_3}{\sqrt{(2D\omega-1)^2+4D\omega^2+2D\omega^2+1-2D\omega}} \right), \quad \omega = \frac{\lambda_3}{\lambda_2(n-1)},$$

if  $1 < n < 2$ . On the other hand, there exists no traveling wave when

$$c < \sqrt{\theta_1}Dc_1,$$

where  $\theta_1 = \lambda_2/\lambda_3$ .

**Remark 2.** It is clear from [Theorem 2](#) that the minimum speed is of order  $D$  when  $D \ll 1$ , and the result depends on more detailed information on  $H$  than that of [Theorem 1](#). It will also be clear from the proof of [Theorem 2](#) that if  $H(u, v) \approx uv$ , then a result similar to that of [Theorem 1](#) holds when  $0 < D < 1$ .

We note that assumptions (ii) and (iii) in [Theorem 2](#) imply

$$H(\epsilon u, v) \geq \theta_1 \epsilon H(u, v) \quad \text{on } (0, 1) \times (0, 1), \quad \forall \epsilon \in (0, 1), \quad (1.5)$$

where  $\theta_1 = \lambda_2/\lambda_3$ . This fact will be used in the proof of [Theorem 2](#).

We also study a degenerate case with  $H(u, v)$  modeled after  $u^\alpha v^\beta$  with  $\alpha, \beta > 1$ . The detailed assumptions are as follows.

- (B1)  $H_u(0, 1) = H_v(0, 1) = 0$ ,  $H \in C^{1,\gamma}([0, \infty) \times [0, \infty))$  with  $0 < \gamma < 1$ ;
- (B2) there exists  $\lambda_1 > 0$ , such that

$$\lim_{u \rightarrow 0, v \rightarrow 1} \frac{H(u, v)}{u^\alpha} = \lambda_1.$$

(B3)  $H(u, v)$  is increasing in  $u$  on  $(0, \max(1, D))$ ,  $\forall v \in (0, 1)$  fixed;

(B4) there exist  $\lambda_3 \geq \lambda_2 > 0$ ,  $\alpha, \beta > 1$  such that

$$\lambda_2 u^\alpha v^\beta \leq H(u, v) \leq \lambda_3 u^\alpha v^\beta.$$

Denote the constants

$$M = \frac{(\alpha + \beta - 1)^{\alpha + \beta - 1}}{(\alpha - 1)^{\alpha - 1}(\beta)^\beta} > 1, \quad M_1 = M/\lambda_3.$$

**Theorem 3.** Let  $D > 1$ . Suppose  $H$  satisfies (A1) and (B1)–(B4). There exists a unique traveling wave with speed  $c$  if

(a)

$$c \geq \max \left( \left( \frac{D}{N} \right)^{1/2}, \frac{D - M_1^{1/(\alpha-1)}}{\left( (D-1)(M_1^{1/(\alpha-1)} - 1) \right)^{1/2}} \right)$$

when  $D > M_1^{1/(\alpha-1)}$ , where  $N > 0$  is the solution to

$$N(N+1)^{1+\alpha} \lambda_3 = \frac{1}{c_1^2};$$

(b)  $c \geq c_1 D^{(\alpha+1)/2}$  when  $D \leq M_1^{1/(\alpha-1)}$ .

On the other hand, there exists no traveling wave if  $c < c_1 \sqrt{D}$ .

**Theorem 4.** Let  $0 < D < 1$ ,  $\alpha > 1$ ,  $\beta \geq 2$ . Suppose  $H$  satisfies (A1) and (B1)–(B4). There exists a unique traveling wave with speed  $c$  if

$$c \geq \frac{4D\lambda_3\sqrt{\alpha}}{\sqrt{\lambda_2 + 4\alpha\lambda_3 D}}.$$

But, if  $c < c_1 D^{(\alpha+1)/2}$ , there exists no traveling wave.

The organization of the paper is as follows. In Section 2, we derive some basic properties of the solutions, laying a foundation for more substantial results. In Section 3, we treat the case of  $H_v(0, 1) = 0$ , but  $H_u(0, 1) > 0$ . In Section 4, we study the case  $H(u, v) \approx u^\alpha v^\beta$  with  $\alpha, \beta > 1$ .

## 2. Basic results

First, we collect some easy facts whose proof can be found in many works, for example [2].

**Proposition 1.** Let  $(u, v)$  be a traveling wave solution of (1.1) with  $H$  satisfying (A1). Then,

- (i)  $u' > 0 > v'$  on  $(-\infty, \infty)$ ,
- (ii)  $u + v < 1$ , but  $u + Dv > D$  on  $(-\infty, \infty)$ , if  $0 < D < 1$ ,
- (iii)  $u + v > 1$ , but  $u + Dv < D$  on  $(-\infty, \infty)$ , if  $D > 1$ ,
- (iv)  $c = \int_{-\infty}^{\infty} H(u(z), v(z)) \, dz$ .

Next, let  $w = v'$ , and use (1.3) to replace the 1st equation in (TW), we have

$$\begin{cases} u' = c(1 - u - v) - Dw, \\ v' = w, \\ w' = -\frac{1}{D}(cw + H(u, v)). \end{cases} \quad (2.1)$$

The linearized system at  $(0, 1, 0)$  is

$$\begin{cases} u' = -cu - cv - Dw, \\ v' = w, \\ w' = -\frac{1}{D}(cw + H_u(0, 1)u + H_v(0, 1)v). \end{cases} \quad (2.2)$$

The matrix  $A$  of the linearized system at  $(0, 1, 0)$  has characteristic polynomial

$$f(\nu) \equiv |\nu I - A| = (\nu + c) \left( \nu^2 + \frac{c}{D}\nu + \frac{H_v(0, 1)}{D} \right) - \frac{H_u(0, 1)}{D}(c + D\nu).$$

If  $H_v(0, 1) = 0$ , the three eigenvalues are

$$\nu_1 = -\frac{c}{D}, \quad \nu_{2,3} = -\frac{c \pm \sqrt{c^2 + 4H_u(0, 1)}}{2}.$$

If  $H_u(0, 1) = 0$ ,

$$\nu_1 = -c, \quad \nu_{2,3} = -\frac{c \pm \sqrt{c^2 - 4DH_v(0, 1)}}{2D}.$$

It is clear that if  $H_u(0, 1) = H_v(0, 1) = 0$ , one of the eigenvalues is zero.

In general, assuming  $H_u(0, 1) > 0 > H_v(0, 1)$ .

$$\text{tr}(A) = -c - \frac{c}{D} < 0, \quad \det(A) = \frac{c}{D}(H_u(0, 1) - H_v(0, 1)) > 0.$$

Hence, there exists a positive root. Furthermore,

$$\begin{aligned} \text{at } \nu = -c, \quad f(\nu) &= -\frac{H_u(0, 1)}{D} \left( \frac{c}{D} - c \right) > 0 \text{ } (< 0) \quad \text{if } D > 1 \text{ } (D < 1) \\ \text{at } \nu = -\frac{c}{D}, \quad f(\nu) &= -H_v(0, 1) \left( \frac{c}{D} - c \right) < 0 \text{ } (> 0) \quad \text{if } D > 1 \text{ } (D < 1). \end{aligned}$$

Therefore, there exist two negative real roots and one positive real root.

At the other end, the matrix  $B$  of linearized system at  $(1, 0, 0)$  has a characteristic polynomial

$$g(\nu) \equiv (\nu + c) \left( \nu^2 + \frac{c}{D}\nu + \frac{H_v(1, 0)}{D} \right) - \frac{H_u(1, 0)}{D}(c + D\nu).$$

It follows that

$$\text{tr}(B) = -c - \frac{c}{D} < 0, \quad \det(B) = \frac{c}{D}(H_u(1, 0) - H_v(1, 0)) \leq 0$$

since  $H_u(1, 0) \leq 0$ ,  $H_v(1, 0) \geq 0$ .

One clear case for which all eigenvalues are explicit is  $D = 1$ , where

$$\nu_1 = -c, \quad \nu_{2,3} = -\frac{c \pm \sqrt{c^2 + 4H_u(1, 0) - 4H_v(1, 0)}}{2}.$$

Another is  $H_u(1, 0) = 0$ , where

$$\nu_1 = -c, \quad \nu_{2,3} = -\frac{c \pm \sqrt{c^2 - 4DH_v(1, 0)}}{2D}.$$

It is like the KPP case when  $H_v(1, 0) > 0$ , which necessitates  $c \geq 2\sqrt{D}H_v(1, 0)$  for the existence of traveling wave.

For general case, it is clear that there is no positive real eigenvalue since  $g(0) \geq 0$  and  $g'(\nu) > 0$  for all  $\nu > 0$ . It is also easy to verify that there is a negative real eigenvalue between  $\min(-c, -c/D)$  and  $\max(-c, -c/D)$ . Furthermore, by the expression of  $\text{tr}(B)$ , if  $\nu = 0$  is not a root, either there exists a conjugate pair with negative real parts, or two more negative real roots.

Now, we turn to Eq. (1.4) for the special case of  $D = 1$  for more details. It is clear that  $F(v) = H(1 - v, v)$  satisfies  $F(0) = F(1) = 0$  and  $F(v) > 0$  on  $(0, 1)$ . In addition,  $F'(0) \geq 0$  and  $F'(1) \leq 0$ . By classical theory, there exists  $c_1 > 0$  such that traveling wave exists for every  $c \geq c_1$ . By a simple substitution,  $S(y) = 1 - v(z)$ ,  $y = cz$ , the equation becomes

$$S_{yy} + S_y = \frac{1}{c^2}H(S, 1 - S),$$

or by using  $S$  as an independent variable, and denote  $P(S) = S_y$ ,

$$PP' + P = \frac{1}{c^2}H(S, 1 - S), \quad S \in (0, 1), \quad (2.3)$$

which we shall use as a reference case for comparison argument.

For our traveling wave problem, let  $s = 1 - v$ ,  $q = \frac{Du}{c^2}$ ,  $y = \frac{cz}{D}$ ,  $\sigma = \frac{D}{c}$ , then

$$q_y = \frac{D^2}{c^3}u_z, \quad s_y = \frac{D}{c}s_z, \quad s_{yy} = \frac{D^2}{c^2}s_{zz}.$$

Using the second equation of (TW) and (1.3), we have the system

$$\begin{cases} s_{yy} + s_y = \frac{D}{c^2}H\left(\frac{c^2}{D}q(y), 1 - s(y)\right), \\ q_y = \sigma^2(s_y + s) - Dq. \end{cases}$$

Since  $s_y > 0$ , we use  $s$  as the independent variable which enables us to reduce the above third order system to second order. Let  $R = s_y$ ,  $Q = q$ , we obtain the working system, which is equivalent to (TW),

$$(I) \begin{cases} RQ' = \sigma^2(R + s) - DQ, & \forall s \in (0, 1), \\ RR' + R = \frac{D}{c^2}H\left(\frac{c^2}{D}Q, 1 - s\right), & \forall s \in (0, 1), \\ Q(s) > Q(0) = 0, \quad R(s) > R(0) = 0, & \forall s \in (0, 1). \end{cases} \quad (2.4)$$

$(u, v)$  is a traveling wave to (1.1) if  $R(1) = 0$ .

### 3. The case of $H_u(0, 1) > 0$ , $H_v(0, 1) = 0$

In this section, we study the case  $H_u(0, 1) > 0$ ,  $H_v(0, 1) = 0$  which signifies non-degeneracy of equilibrium point  $(0, 1)$ . By our analysis in Section 2 there is one dimensional unstable manifold which is the unique path followed by the traveling wave coming out of  $(0, 1)$ . In addition, we have the following result.

**Lemma 1.** *Let  $D > 0$  and  $\sigma > 0$ , (I) has a unique solution on  $(0, 1)$  with the property*

$$R(s) = \mu s + O(s^2), \quad Q(s) = \frac{\mu(\mu + 1)}{H_u(0, 1)}s + O(s^2) \quad \text{as } s \searrow 0, \quad (3.5)$$

where  $\mu = (-D + \sqrt{D^2 + 4\sigma^2 H_u(0, 1)})/2$  is the positive root of  $\mu^2 + D\mu - \sigma^2 H_u(0, 1) = 0$ . Moreover,  $Q'(s) > 0$  on  $(0, 1)$  and  $(Q, R)$  is a traveling wave iff  $R(1) = 0$ .

**Proof.** Since the procedure is similar to that of Lemma 2.2 in [2], we shall just give a outline of proof. The existence of a solution with the asymptotic behavior (3.5) for  $0 < s \ll 1$  can be obtained by classical theory and simple computation. In addition, it can be proved that a solution can be extended as long as  $R(s) > 0$ , and it can be shown in sequence  $Q(s) > 0$  before  $R(s)$  becomes zero,  $R(s)$ (therefore  $Q(s)$ )  $> 0$  on  $(0, 1)$ . Next,

$$R(\sigma^2(R+s) - DQ)' = -D(\sigma^2(R+s) - DQ) + \frac{D\sigma^2}{c^2}H\left(\frac{c^2}{D}Q, 1-s\right)$$

and

$$\sigma^2(R+s) \approx \frac{\mu(\mu+1)}{H_u(0,1)}(\mu+D)s > DQ(s) \approx D\frac{\mu(\mu+1)}{H_u(0,1)}s$$

for  $0 < s \ll 1$ . Grownwall's inequality then yields  $\sigma^2(R(s)+s) - DQ(s) > 0$  on  $(0, 1)$ . Hence,

$$Q'(s) > 0, \quad Q(s) < D^{-1}\sigma^2(R(s)+s) \quad \text{on } (0, 1).$$

It is obvious  $(R+s)' \geq 0$  and  $Q$  grows at most linearly in  $s$ , and thus  $H(Qc^2/D, 1-s)$  is uniformly bounded on  $(0, 1)$ . This shows both  $\lim_{s \nearrow 1} Q(s)$  and  $\lim_{s \nearrow 1} R(s)$  are finite. The rest follows from Lemma 2.2 in [2].  $\square$

In what follows, we show Theorem 1 by a sequence of lemmas.

**Lemma 2.** Suppose  $D > 1$  and  $X^T M X < 0$ , where  $X = (Q'(0)c^2/D, -1)^T$  and  $M$  the Hessian matrix of  $H(u, v)$  at  $(0, 1)$  and  $H(u, v) \leq H_u(0, 1)u$  on  $(0, D) \times (0, 1)$ , then

$$Q(s) < \frac{\mu(\mu+1)}{H_u(0,1)}s, \quad R(s) < \mu s \quad \text{on } (0, 1).$$

**Proof.** First we show that for  $0 < s \ll 1$ .

$$Q(s) \approx \frac{\mu(\mu+1)}{H_u(0,1)}s + \xi_1 s^2, \quad R(s) \approx \mu s + \xi_2 s^2$$

with  $\xi_1, \xi_2 < 0$ . Denote  $G(s) = \frac{D}{c^2}H(c^2Q(s)/D, 1-s)$ , computation demonstrates that

$$G'(0) = H_u(0, 1)Q'(0), \quad G''(0) = \frac{D}{c^2}X^T M X + 2H_u(0, 1)\xi_1.$$

Hence, for  $0 < s \ll 1$ , we get from the second equation of (I),

$$RR' + R \approx \mu(\mu+1)s + (3\mu+1)\xi_2 s^2 \approx \mu(\mu+1)s + H_u(0, 1)\xi_1 s^2 + \frac{1}{2}\frac{D}{c^2}X^T M X s^2,$$

and consequently,

$$(3\mu+1)\xi_2 - H_u(0, 1)\xi_1 = \frac{1}{2}\frac{D}{c^2}X^T M X. \quad (3.6)$$

Similarly, using the first equation of (I), we have

$$(2\mu+D)\xi_1 = \left(\sigma^2 - \frac{\mu(\mu+1)}{H_u(0,1)}\right)\xi_2 = \frac{\mu(D-1)}{H_u(0,1)}\xi_2. \quad (3.7)$$

Substituting (3.7) into (3.6) yields

$$H_u(0, 1)((3\mu+1)(2\mu+D) - \mu(D-1))\xi_2 = \frac{1}{2}\frac{D}{c^2}X^T M X \cdot (2\mu+D) < 0.$$

This gives  $\xi_1, \xi_2 < 0$ . The rest is a standard argument, see Lemma 3.2 of [2]. In particular,

$$\begin{aligned} R(R - \mu s)' &= -(1 + \mu)R + \frac{D}{c^2}H(c^2Q(s)/D, 1 - s) \\ &= -(1 + \mu)(R - \mu s) - \mu(\mu + 1)s + \frac{D}{c^2}H(c^2Q(s)/D, 1 - s) \\ &\leq -(1 + \mu)(R - \mu s) - \mu(\mu + 1)s + Q(s)H_u(0, 1) \\ &\leq -(1 + \mu)(R - \mu s), \end{aligned}$$

by  $H(u, v) \leq H_u(0, 1)u$ . This proves the lemma.  $\square$

**Corollary 1.** Suppose  $H$  satisfies the assumptions of Lemma 2, and in addition, it is an increasing function of  $u$  on  $(0, D)$  for every  $v \in (0, 1)$  fixed and for  $0 < \epsilon < D$ ,

$$H(\epsilon u, v) \leq \lambda_0 \epsilon H(u, v) \quad \text{on } (0, 1) \times (0, 1), \quad (3.8)$$

where  $\lambda_0 > 0$  is a fixed number. Then, there exists traveling wave when

$$\frac{\mu(\mu + 1)}{H_u(0, 1)} \leq \frac{1}{\lambda_0 c_1^2},$$

that is, when

$$c \geq c_1 \frac{\sqrt{D\lambda_0}}{\sqrt{1 - (1 - \frac{1}{D}) \frac{L-1}{L+1}}},$$

where

$$L = \sqrt{1 + \frac{4H_u(0, 1)}{\lambda_0 c_1^2}}.$$

**Proof.** By Lemma 2,  $Q(s) < \frac{\mu(\mu+1)}{H_u(0,1)}s$ , and by (3.8),

$$\frac{D}{c^2}H\left(\frac{c^2}{D}Q(s), 1 - s\right) \leq \lambda_0 \frac{\mu(\mu + 1)}{H_u(0, 1)}H(s, 1 - s).$$

It is clear by simple comparison with (2.3) that when

$$\frac{\mu(\mu + 1)}{H_u(0, 1)} \leq \frac{1}{\lambda_0 c_1^2}, \quad R(1) = 0,$$

proving the corollary.  $\square$

**Lemma 3.** Suppose  $D \geq 1$  and  $H(u, v)$  is an increasing function of  $u$  on  $(0, D)$ ,  $\forall v \in (0, 1)$  fixed. Then,  $DQ(s) \geq \sigma^2 s$  on  $(0, 1)$  and consequently, there exists no traveling wave when  $c < \sqrt{D}c_1$ .

**Proof.** Since  $Q(s) = \sigma^2 s$  on  $(0, 1)$  when  $D = 1$ , we consider only the case of  $D > 1$ . If  $D > 1$ ,

$$R[DQ - \sigma^2 s]' = -D[DQ - \sigma^2 s] + (D - 1)\sigma^2 R > -D[DQ - \sigma^2 s] \quad \text{on } (0, 1)$$

and, when  $0 < s \ll 1$ ,

$$DQ(s) = \frac{D\mu(\mu + 1)}{H_u(0, 1)}s + O(s^2) > \sigma^2 s.$$

By Grownwall's inequality, we obtain  $DQ(s) \geq \sigma^2 s$  on  $(0, 1)$ . By Lemma 2,  $Q(s) < \sigma^2$  and consequently,  $c^2Q(s)/D < D$  on  $(0, 1)$ . Since  $H(u, v)$  is an increasing function of  $u$  on  $(0, D)$ , for any  $v \in (0, 1)$  fixed, it



follows that

$$RR' + R = \frac{D}{c^2} H\left(\frac{c^2}{D} Q(s), 1-s\right) \geq \frac{D}{c^2} H(s, 1-s).$$

By direct comparison, when  $D/c^2 > 1/c_1^2$ , that is, when  $c < \sqrt{D}c_1$ , there exists no traveling wave.  $\square$

**Remark 3.** If we replace the condition in Lemma 3 that “ $H(u, v)$  is an increasing function of  $u$  on  $(0, D)$ ,  $\forall v \in (0, 1)$  fixed” by

$$H(\alpha u, v) \geq \Lambda \alpha H(u, v) \quad \text{for any } 1 < \alpha < D \text{ on } (0, 1) \times (0, 1)$$

for some  $\Lambda > 0$ , then the non-existence result of Lemma 3 can be modified to

$$\text{There exists no traveling wave for } c < \sqrt{\Lambda D}c_1.$$

**Proof of Theorem 1.** The theorem follows directly from Lemmas 1–3.  $\square$

Now, we turn out attention to  $D < 1$ .

**Lemma 4.** Suppose  $0 < D < 1$ ,  $H(u, v)$  is an increasing function of  $u$  on  $(0, 1)$ ,  $\forall v \in (0, 1)$  fixed, and has the property that  $\forall \epsilon \in [0, 1]$ ,

$$H(\epsilon s, 1-s) \geq \theta_1 \epsilon H(s, 1-s) \quad \text{on } [0, 1],$$

where  $\theta_1 > 0$  is a number. Then,

$$\sigma^2 s < Q(s) < \frac{\sigma^2}{D} s.$$

Consequently, there exists no traveling wave if  $c < \sqrt{\theta_1 D}c_1$ .

**Proof.** It follows from Lemma 1 that for  $0 < s \ll 1$ ,

$$Q(s) = \frac{\mu(\mu+1)}{H_u(0,1)} s + O(s^2) > \sigma^2 s.$$

In addition,

$$\begin{aligned} R[Q - \sigma^2 s]' &= \sigma^2(R + s) - DQ - \sigma^2 R \\ &= -D(Q - \sigma^2 s) + \sigma^2(1 - D)s \\ &> -D(Q - \sigma^2 s), \quad \text{on } (0, 1) \end{aligned}$$

and Gronwall's inequality gives  $Q(s) > \sigma^2 s$  on  $(0, 1)$ . An argument similar to the one in Lemma 3 yields  $Q(s) < \sigma^2 s/D$  on  $(0, 1)$ . Consequently,

$$R'R + R = \frac{D}{c^2} H\left(\frac{c^2}{D} Q(s), 1-s\right) \geq \frac{D}{c^2} H(Ds, 1-s) \geq \theta_1 \frac{D^2}{c^2} H(s, 1-s).$$

Hence, it follows from simple comparison that there exists no traveling wave when  $c < \sqrt{\theta_1 D}c_1$ .  $\square$

**Lemma 5.** Suppose  $0 < D < 1$ ,  $H(u, v) \geq \lambda_2 uv^n$  with  $n \geq 1$  on  $(0, 1) \times (0, 1)$ , where  $\lambda_2 > 0$ , then

$$Q(s)(1-s)^{n/2} \leq \eta(R(s) + s) \quad \text{on } (0, 1)$$

for any

$$\eta \geq \eta_1 \equiv \frac{-D + \sqrt{D^2 + 4\lambda_2 \sigma^2}}{2\lambda_2}.$$

**Proof.** We only need to show the inequality for  $\eta > \eta_1$ . It is easy to verify that the inequality holds when  $0 < s \ll 1$  since  $\eta_1 \geq \mu/H_u(0, 1)$ . Furthermore,

$$\begin{aligned} & R[(1-s)^{n/2}Q(s) - \eta(R(s) + s)]' \\ &= (1-s)^{n/2}(\sigma^2(R(s) + s) - DQ(s)) - \frac{n}{2}(1-s)^{(n-2)/2}Q(s)R(s) - \frac{\eta D}{c^2}H\left(\frac{c^2}{D}Q(s), 1-s\right) \\ &\leq (1-s)^{n/2}(\sigma^2(R+s) - DQ) - \eta\lambda_2(1-s)^nQ \quad (\text{by our assumption } H(u, v) \geq \lambda_2 uv^n) \\ &= [-D - \eta\lambda_2(1-s)^{n/2}][(1-s)^{n/2}Q - \eta(R+s)] + (R+s)[(1-s)^{n/2}(\sigma^2 - \eta^2\lambda_2) - D\eta] \\ &\leq [-D - \eta\lambda_2(1-s)^{n/2}][(1-s)^{n/2}Q - \eta(R+s)]. \end{aligned}$$

The lemma follows from Gronwall's inequality.  $\square$

**Remark 4.** If  $\lambda_2 = H_u(0, 1)$ , then the lemma is valid for any  $\eta \geq \frac{\mu}{H_u(0, 1)}$ .

**Lemma 6.** Suppose the assumption of [Lemma 5](#) holds and

$$H(u, v) \leq \lambda_3 uv^n \quad \text{on } (0, 1) \times (0, 1),$$

for some  $\lambda_3 > 0$  and  $n \geq 2$ . Then, there exists a traveling wave if  $\lambda_3\eta_1 \leq 1/4$ .

**Proof.** We show  $R(1) = 0$  directly. Let  $\eta > \eta_1$ .

$$\begin{aligned} R(2R - s(1-s))' &= 2RR' - R(1-2s) \\ &= -2R - R(1-2s) + \frac{2D}{c^2}H\left(\frac{c^2}{D}Q(s), 1-s\right) \\ &\leq -R(3-2s) + 2\lambda_3Q(s)(1-s)^n \\ &= (2R - s(1-s))\left(s - \frac{3}{2} + \lambda_3\eta(1-s)^{n/2}\right) \\ &\quad + s(1-s)\left(s - \frac{3}{2} + 2\lambda_3\eta(1-s)^{(n-2)/2} + \lambda_3\eta(1-s)^{n/2}\right) \\ &\leq (2R - s(1-s))\left(s - \frac{3}{2} + \lambda_3\eta(1-s)^{n/2}\right) \end{aligned}$$

if  $s - \frac{3}{2} + 2\lambda_3\eta(1-s)^{(n-2)/2} + \lambda_3\eta(1-s)^{n/2} \leq 0$  on  $(0, 1)$ . It is easy to verify that if  $\lambda_3\eta \leq 1/4$ , the inequality holds. This proves the lemma.  $\square$

The case of  $0 < D < 1$  and  $1 < n < 2$  needs more elaborate argument mainly because the result of [Lemma 5](#) is not sufficient for us to show existence of traveling wave.

**Lemma 7.** Let  $0 < D < 1$ ,  $1 < n < 2$  and assume

$$\lambda_2 uv^n \leq H(u, v) \leq \lambda_3 uv^n \quad \text{on } (0, 1) \times (0, 1), \quad \lambda_3 \geq \lambda_2 > 0,$$

then

- (i)  $Q(s) > \eta(R(s) + s)$ ,  $\forall \eta < \eta_2 \equiv \frac{-D + \sqrt{D^2 + 4\lambda_3\sigma^2}}{2\lambda_3}$  on  $(0, 1)$ ;
- (ii)  $R(s) - \theta s < 0$ ,  $\forall \theta > \lambda_3\eta_1$  on  $(0, 1)$ , where  $\eta_1$  is defined in [Lemma 5](#); and
- (iii)  $Q(s) \geq \bar{\eta}s$  on  $[0, 1]$ , where  $\bar{\eta} = (\eta_1\lambda_3 + 1)\sigma^2/(\eta_1\lambda_3 + D)$ .

**Proof.** Since  $H_u(0, 1) \leq \lambda_3$ ,  $\eta_2 \leq \mu/H_u(0, 1)$ , (i) holds for  $0 < s \ll 1$ .

$$\begin{aligned} R[Q - \eta(R + s)]' &= \sigma^2(R + s) - DQ - \frac{\eta D}{c^2} H\left(\frac{c^2}{D} Q(s), 1 - s\right) \\ &\geq \sigma^2(R + s) - DQ - \eta \lambda_3 (1 - s)^n Q \\ &\geq [-D - \eta \lambda_3 (1 - s)^n][Q - \eta(R + s)] + (R + s)(-D\eta - \eta^2 \lambda_3 (1 - s)^n + \sigma^2) \\ &\geq [-D - \eta \lambda_3 (1 - s)^n][Q - \eta(R + s)]. \end{aligned}$$

This proves (i).

For (ii), again, it is clear that the inequality holds for  $0 < s \ll 1$ . Furthermore,

$$\begin{aligned} R[R - \theta s]' &= -(1 + \theta)R + \frac{D}{c^2} H\left(\frac{c^2}{D} Q(s), 1 - s\right) \\ &\leq -(1 + \theta)R + \eta_1 \lambda_3 (R + s)(1 - s)^{n/2} \quad (\text{by Lemma 5}) \\ &\leq [R - \theta s] \left( -(1 + \theta) + \eta_1 \lambda_3 (1 - s)^{n/2} \right) + (1 + \theta)s(-\theta + \eta_1 \lambda_3 (1 - s)^{n/2}) \\ &\leq [R - \theta s] \left( -(1 + \theta) + \eta_1 \lambda_3 (1 - s)^{n/2} \right). \end{aligned}$$

A standard Grownwall's inequality implies (ii).

The proof of the last statement proceeds in a similar way. For any  $\theta > \eta_1 \lambda_3$ , let  $\eta = (\theta + 1)\sigma^2/(\theta + D)$ .

$$\begin{aligned} R[Q - \eta s]' &= \sigma^2(R + s) - DQ - \eta R \\ &= -D[Q - \eta s] + (\sigma^2 - \eta)(R - \theta s) + (-D\eta + \sigma^2(\theta + 1) - \theta\eta)s \\ &\geq -D[Q - \eta s]. \end{aligned}$$

This, plus the obvious fact that  $Q(s) - \eta s > 0$  for  $0 < s \ll 1$ , shows  $Q(s) - \eta s > 0$  on  $[0, 1]$ . Let  $\theta \rightarrow \eta_1 \lambda_3$ , we get the desired result.  $\square$

**Lemma 8.** Suppose the assumptions of Lemma 7 hold. Then,

$$R(s) \geq \lambda_2 \eta_2 s(1 - s)^n \quad Q(s)(1 - s)^{n-1} \leq \bar{\xi}(R + s)$$

on  $[0, 1]$ , where

$$\bar{\xi} = \max \left( (n - 1)\eta_2, \frac{\sigma^2}{\lambda_2 \eta_2 (n - 1) + D} \right).$$

**Proof.** For any  $\eta < \eta_2$ ,  $R(s) \geq \lambda_2 \eta s(1 - s)^n > 0$  when  $0 < s \ll 1$ . In addition, for  $\nu > 0$ ,

$$\begin{aligned} R[R - \nu s(1 - s)^n]' &= -R - \nu(1 - s)^n R + \frac{D}{c^2} H\left(\frac{c^2}{D} Q(s), 1 - s\right) + n\nu s(1 - s)^{n-1} R \\ &\geq -R - \nu(1 - s)^n R + \lambda_2 Q(s)(1 - s)^n \\ &\geq -R - \nu(1 - s)^n R + \lambda_2 \eta(R + s)(1 - s)^n \\ &= -(1 + (\nu - \lambda_2 \eta)(1 - s)^n)(R - \nu s(1 - s)^n) + s(1 - s)^n (-\nu - \nu^2(1 - s)^n + \lambda_2 \eta \nu(1 - s)^n + \lambda_2 \eta) \\ &\geq -(1 + (\nu - \lambda_2 \eta)(1 - s)^n)(R - \nu s(1 - s)^n), \end{aligned}$$

on  $(0, 1)$ , if  $\nu \leq \lambda_2 \eta$ . Thus,  $R(s) \geq \lambda_2 \eta s(1 - s)^n$  on  $[0, 1]$ , and upon taking a limit  $\eta \rightarrow \eta_2$  the first result is proved.

The second result follows the same procedure as in Lemma 5, but we use the first result to get this refined bound. For  $\xi > 0$ ,

$$\begin{aligned} & R((1-s)^{n-1}Q - \xi(R+s))' \\ &= (1-s)^{n-1}(\sigma^2(R+s) - DQ) - (n-1)(1-s)^{n-2}QR - \frac{\xi D}{c^2}H\left(\frac{c^2}{D}Q(s), 1-s\right) \\ &\leq (1-s)^{n-1}(\sigma^2(R+s) - DQ) - (n-1)\lambda_2\eta_2s(1-s)^{2n-2}Q - \xi\lambda_2Q(1-s)^n \\ &= (-D - (n-1)\lambda_2\eta_2(1-s)^{n-1} - \xi\lambda_2(1-s))((1-s)^{n-1}Q - \xi(R+s)) \\ &\quad + (R+s)(1-s)^{n-1}(\sigma^2 - D\xi(1-s)^{1-n} - (n-1)\lambda_2\eta_2\xi s - \xi^2\lambda_2(1-s)^{2-n}). \end{aligned}$$

It is easy to see that  $\xi^2\lambda_2(1-s)^{2-n} \geq (n-1)\lambda_2\eta_2\xi(1-s)$ , when  $\xi \geq (n-1)\eta_2$ , and

$$D\bar{\xi}(1-s)^{1-n} + (n-1)\lambda_2\eta_2\bar{\xi}s + \bar{\xi}^2\lambda_2(1-s)^{2-n} \geq \sigma^2.$$

The result then follows from Grownwall's inequality.  $\square$

**Proof of Theorem 2.** The non-existence part is proved in Lemma 4. The existence for  $n \geq 2$  follows directly from Lemma 6. We show the existence for  $1 < n < 2$  using Lemmas 7 and 8. For  $\eta > 0$ ,

$$\begin{aligned} R(R - \eta s(1-s))' &= -R + \frac{D}{c^2}H\left(\frac{c^2}{D}Q(s), 1-s\right) - \eta(1-2s)R \\ &\leq -R(1 + \eta(1-2s)) + \lambda_3Q(s)(1-s)^n \\ &\leq -R(1 + \eta(1-2s)) + \lambda_3\bar{\xi}(R+s)(1-s) \quad (\text{by Lemma 8}) \\ &= (R - \eta s(1-s))(-1 - \eta(1-2s) + \lambda_3\bar{\xi}(1-s)) \\ &\quad + \eta s(1-s)\left(-1 - \eta(1-2s) + \lambda_3\bar{\xi}(1-s) + \frac{\lambda_3\bar{\xi}}{\eta}\right) \\ &\leq (R - \eta s(1-s))(-1 - \eta(1-2s) + \lambda_3\bar{\xi}(1-s)) \end{aligned}$$

if

$$L(s) \equiv -1 - \eta(1-2s) + \lambda_3\bar{\xi}(1-s) + \frac{\lambda_3\bar{\xi}}{\eta} \leq 0 \quad \text{on } [0, 1]$$

for some  $\eta > \lambda_3\bar{\xi}$ . It is possible when  $\lambda_3\bar{\xi} \leq 1/4$ . This proves the theorem.  $\square$

#### 4. The degenerate case

In this section, we shall treat a special class of functions  $H(u, v)$  with the property that  $H_u(0, 1) = H_v(0, 1) = 0$  as well as  $H_u(1, 0) = H_v(1, 0) = 0$ . In particular, we shall study first  $H(u, v) = u^\alpha v^\beta$  with  $\alpha, \beta > 1$ , then a more general situation later. Notice that the function is not  $C^2$  at either  $(0, 1)$  or  $(1, 0)$  if  $1 < \alpha, \beta < 2$ .

4.1.  $H(u, v) = u^\alpha v^\beta$  with  $\alpha, \beta > 1$

The system (I) now takes the form

$$(II) \begin{cases} RQ' = \sigma^2(R+s) - DQ, & \forall s \in (0, 1), \\ RR' + R = \left(\frac{c^2}{D}\right)^{\alpha-1} Q^\alpha(s)(1-s)^\beta, & \forall s \in (0, 1), \\ Q(s) > Q(0) = 0, \quad R(s) > R(0) = 0, & \forall s \in (0, 1). \end{cases} \quad (4.9)$$

The main results on (II) are the following theorems.

**Theorem 5.** Suppose  $D > 1$ ,  $\alpha, \beta > 1$ . There exists a unique traveling wave with speed  $c$  if

(a)

$$c \geq \max \left( \left( \frac{D}{N} \right)^{1/2}, \frac{D - M^{1/(\alpha-1)}}{((D-1)(M^{1/(\alpha-1)} - 1))^{1/2}} \right)$$

when  $D > M^{1/(\alpha-1)}$ , where  $N > 0$  is the solution to

$$N(N+1)^{1+\alpha} = 1/c_1^2 \quad \text{and} \quad M = \frac{(\alpha + \beta - 1)^{\alpha+\beta-1}}{(\alpha-1)^{\alpha-1}(\beta)^\beta} > 1;$$

(b)  $c \geq c_1 D^{(\alpha+1)/2}$  when  $D \leq M^{1/(\alpha-1)}$ .

On the other hand, there exists no traveling wave if  $c < c_1 \sqrt{D}$ .

**Theorem 6.** Suppose  $0 < D < 1$ ,  $\alpha > 1$  and  $\beta \geq 2$ . There exists a unique traveling wave with speed  $c$  if

$$c \geq \frac{4D\sqrt{\alpha}}{\sqrt{1+4\alpha D}}. \quad (4.10)$$

But, there exists no traveling wave if  $c < c_1 D^{(\alpha+1)/2}$ .

Recall that for  $H(u, v) = u^\alpha v^\beta$  with  $\alpha, \beta > 1$ , the three eigenvalues at the equilibrium point  $(0, 1, 0)$  of (2.1) are

$$\nu = -c, \quad \nu = -\frac{c}{D}, \quad \nu = 0.$$

A type of center manifold argument, see [14–16] shows there is a unique positive solution coming out of  $(0, 1, 0)$ , whose asymptotic behavior we describe in more detail in what follows.

**Lemma 9.** There is a unique solution  $(Q(s), R(s))$  of (II) on  $(0, 1)$  with the property that

$$Q(s) = \frac{\sigma^2}{D}s + \frac{D(D-1)}{c^4}s^\alpha, \quad R(s) = \frac{\sigma^2}{D}s^\alpha + \mu s^\Delta \quad \text{as } s \searrow 0,$$

where

$$(\mu, \Delta) = \begin{cases} \left( -\frac{\beta D}{c^2}, \alpha + 1 \right) & \text{if } \alpha > 2, \\ \left( -\frac{D\beta}{c^2} - \frac{\alpha D}{c^4}, 3 \right) & \text{if } \alpha = 2, \\ \left( -\frac{\alpha D}{c^4}, 2\alpha - 1 \right) & \text{if } 1 < \alpha < 2. \end{cases}$$

Furthermore,  $Q'(s) > 0$ ,  $R(s) > 0$  on  $(0, 1)$ , and it is a traveling wave iff  $R(1) = 0$ .

**Proof.** The asymptotic behavior as  $s \searrow 0$  is by elementary computation, the rest follows from the same line of argument as in the proof of Lemma 1, which we omit the details.  $\square$

**Remark 5.** For the constant  $M$  which appears in Theorem 5, it is easy to verify that the function  $s^{\alpha-1}(1-s)^\beta$  has a maximum value  $M^{-1}$  on  $[0, 1]$ .

**Lemma 10.** Let  $D > 1$ . Let  $(Q(s), R(s))$  be a solution of (II) on  $(0, 1)$ , then

- (i)  $\frac{\sigma^2}{D}s < Q(s) < \sigma^2 s$  on  $(0, 1)$ ;
- (ii)  $Q(s) < \lambda(1 + \lambda)s$ ,  $R(s) < \lambda s$  on  $(0, 1)$ ,

where

$$\lambda = \begin{cases} \frac{-1 + \sqrt{1 + 4\sigma^2}}{2} \text{ is the positive root of } \lambda(1 + \lambda) = \sigma^2, & \text{if } D \leq M^{1/(\alpha-1)}, \\ \frac{-D + \sqrt{D^2 + 4\sigma^2}}{2} \text{ solves } \lambda(D + \lambda) = \sigma^2, & \text{if } D > M^{1/(\alpha-1)} \text{ and } \sigma^2 \leq \Sigma_0, \end{cases}$$

with

$$\Sigma_0 = \frac{D^2(D-1)(M^{1/(\alpha-1)} - 1)}{(D - M^{1/(\alpha-1)})^2}.$$

**Proof.** (i) is a routine calculation. For (ii), simple computation shows

$$\begin{aligned} & R(Q - \lambda(1 + \lambda)s)' \\ &= \sigma^2(R + s) - DQ - \lambda(1 + \lambda)R \\ &= -D(Q - \lambda(1 + \lambda)s) + (\sigma^2 - \lambda(1 + \lambda))(R - \lambda s) + (1 + \lambda)s(\sigma^2 - \lambda D - \lambda^2) \end{aligned} \quad (4.11)$$

and

$$\begin{aligned} R(R - \lambda s)' &= (1 + \lambda)R + \left(\frac{c^2}{D}\right)^{\alpha-1} Q^\alpha(s)(1-s)^\beta \\ &= -(1 + \lambda)(R - \lambda s) - \lambda(1 + \lambda)s + \left(\frac{c^2}{D}\right)^{\alpha-1} Q^\alpha(s)(1-s)^\beta. \end{aligned} \quad (4.12)$$

Case I:  $D \leq M^{1/(\alpha-1)}$ . With the choice of  $\lambda$ , we have both inequalities valid for  $0 < s \ll 1$ . Moreover, if  $R(s) - \lambda s < 0$  holds on  $(0, s_0)$  with  $0 < s_0 < 1$ ,  $R(s) - \lambda s = 0$  at  $s_0$ ,

$$R(Q - \lambda(1 + \lambda)s)' + D(Q - \lambda(1 + \lambda)s) < 0 \quad \text{on the same interval,}$$

by (4.11). This implies  $Q(s) - \lambda(1 + \lambda)s < 0$  holds on  $(0, s_0]$ . Consequently,

$$\begin{aligned} R(R - \lambda s)' &\leq -(1 + \lambda)(R - \lambda s) - \lambda(1 + \lambda)s + (\lambda(1 + \lambda)s)^\alpha \left(\frac{c^2}{D}\right)^{\alpha-1} (1-s)^\beta \\ &\leq -(1 + \lambda)(R - \lambda s) + \lambda(1 + \lambda)s \left(-1 + (\lambda(1 + \lambda))^{\alpha-1} \left(\frac{\sigma^2}{D}\right)^{1-\alpha} M^{-1}\right) \\ &< -(1 + \lambda)(R - \lambda s) \quad \text{on } (0, s_0]. \end{aligned}$$

We reach a contradiction. Similarly,  $Q(s) - \lambda(1 + \lambda)s$  cannot be zero at any point  $s_0 \in (0, 1)$ . This means both inequalities must hold on  $(0, 1)$ .

Case II:  $D > M^{1/(\alpha-1)}$ . It is easy to check that  $\sigma^2 \leq \Sigma_0$  is equivalent to  $\lambda(1 + \lambda)D < \sigma^2 M^{1/(\alpha-1)}$ , and consequently,

$$\left(-1 + (\lambda(1 + \lambda))^{\alpha-1} \left(\frac{\sigma^2}{D}\right)^{1-\alpha} M^{-1}\right) < 0.$$

Repeating the same argument as in Case I, we prove the lemma.

**Proof of Theorem 5.** The non-existence part is standard. By using

$$Q(s) \geq \frac{\sigma^2}{D}s = \frac{D}{c^2}s,$$

there exists no traveling wave if

$$\left(\frac{c^2}{D}\right)^{\alpha-1} \cdot \left(\frac{c^2}{D}\right)^{-\alpha} > \frac{1}{c_1^2}$$

which is exactly  $c < c_1\sqrt{D}$ . For the existence part, we treat  $1 < D \leq M^{1/(\alpha-1)}$  and  $D > M^{1/(\alpha-1)}$  separately.

If  $1 < D \leq M^{1/(\alpha-1)}$ , simply use  $Q(s) \leq \sigma^2 s$ , we have

$$RR' + R \leq \left(\frac{c^2}{D}\right)^{\alpha-1} \cdot \left(\frac{D}{c}\right)^{2\alpha} s^\alpha (1-s)^\beta = \left(\frac{D^{\alpha+1}}{c^2}\right) s^\alpha (1-s)^\beta.$$

There existence of traveling wave if

$$\left(\frac{D^{\alpha+1}}{c^2}\right) \leq \frac{1}{c_1^2} \iff c \geq c_1 D^{(\alpha+1)/2}.$$

For  $D > M^{1/(\alpha-1)}$ , first we note

$$c \geq \frac{(D - M^{1/(\alpha-1)})}{((D-1)(M^{1/(\alpha-1)} - 1))^{1/2}}$$

is the same as  $\sigma^2 \leq \Sigma_0$ . Second,  $\sigma^2/D \leq N$  yields

$$\lambda(1+\lambda) = \sigma^2 - (D-1)\lambda = \sigma^2 \left(1 - \frac{2(D-1)}{D + \sqrt{D^2 + 4\sigma^2}}\right) \leq (N+1)\frac{\sigma^2}{D}.$$

This is because  $\sqrt{D^2 + 4\sigma^2} \leq D + 2\sigma^2/D$  and therefore

$$\left(1 - \frac{2(D-1)}{D + \sqrt{D^2 + 4\sigma^2}}\right) \leq 1 - \frac{(D-1)}{D + \sigma^2/D} = \frac{1}{D} \frac{1+N}{1+\frac{N}{D}} \leq (N+1)\frac{1}{D}.$$

By comparison, there exists a traveling wave if

$$\left(\frac{\sigma^2}{D}\right)^{1-\alpha} (N+1)^\alpha \left(\frac{\sigma^2}{D}\right)^\alpha \leq \frac{1}{c_1^2},$$

which is the same as

$$c \geq \left(\frac{D}{N}\right)^{1/2} = \sqrt{D}c_1(N+1)^{-\alpha/2}. \quad \square$$

In what follows we treat the case of  $0 < D < 1$ .

**Lemma 11.** Let  $0 < D < 1$ . Let  $(Q(s), R(s))$  be a solution of (II) on  $(0, 1)$ , then on  $(0, 1)$ ,

$$(i) \quad \sigma^2 s < Q(s) < \frac{\sigma^2}{D}s, \quad (ii) \quad R(s) - \frac{\sigma^2}{D}s^\alpha < 0,$$

$$(iii) \quad Q(s) \geq \bar{\eta}s, \quad \text{where } \bar{\eta} = \frac{\frac{\sigma^2}{D} + 1}{\frac{\sigma^2}{D} + D}\sigma^2.$$

**Proof.** (i) is obvious. We proceed to prove (ii). For  $\eta > 0$ ,

$$(R(s) - \eta s^\alpha)' = -1 + \frac{1}{R(s)} \left(\frac{c^2}{D}\right)^{\alpha-1} Q^\alpha(s)(1-s)^\beta - \eta \alpha s^{\alpha-1}.$$

It follows from Lemma 9 that for  $\eta = \sigma^2/D$ , the inequality holds for  $0 < s \ll 1$ . If  $R(s) - \frac{\sigma^2}{D}s^\alpha = 0$  at  $s = s_0 > 0$ , and  $R(s) - \frac{\sigma^2}{D}s^\alpha < 0$  on  $(0, s_0)$ , then  $(R(s) - \frac{\sigma^2}{D}s^\alpha)' \geq 0$  at the point. But, since  $Q(s) < \frac{\sigma^2}{D}s$ ,

$$\left(R(s) - \frac{\sigma^2}{D}s^\alpha\right)' \leq -1 + (1-s)^\beta - \frac{\sigma^2}{D}\alpha s^{\alpha-1} < 0, \quad \text{at } s = s_0,$$

a contradiction. This shows (ii) is valid.

For (iii), we can proceed exactly as in Lemma 7 by using (ii). The lemma is proved.  $\square$

**Lemma 12.** Let  $0 < D < 1$  and  $(Q(s), R(s))$  be a solution of (II) on  $(0, 1)$ . Then, on  $(0, 1)$ ,

- (i)  $Q(s) > \eta(R(s) + s), \forall \eta \leq (-D + \sqrt{D^2 + 4\sigma^2})/2$ ,
- (ii)  $(1-s)^{\beta/2}Q^\alpha(s) - \eta(R(s) + s) < 0$  if

$$\eta \geq \eta_3 \equiv \frac{2\alpha \left(\frac{\sigma^2}{D}\right)^\alpha D}{\alpha D + \sqrt{\alpha^2 D^2 + 4\alpha\sigma^2}}, \quad (4.13)$$

- (iii)  $R(s) - \theta s < 0$ , where  $\theta = 2\alpha\sigma^2/(\alpha D + \sqrt{\alpha^2 D^2 + 4\alpha\sigma^2})$ .

**Proof.** (i) can be proved by using the same procedure as in Lemma 7, applying the fact that  $Q(s) < \sigma^2 s/D$ . We omit the details. The proof of (ii) is similar to that of Lemma 5.

$$\begin{aligned} I &\equiv R[(1-s)^{\beta/2}Q^\alpha(s) - \eta(R(s) + s)]' \\ &= \alpha Q^{\alpha-1}(s)(1-s)^{\beta/2}(\sigma^2(R(s) + s) - DQ(s)) - \frac{\beta}{2}(1-s)^{(\beta-2)/2}Q^\alpha(s)R(s) - \eta \left(\frac{c^2}{D}\right)^{\alpha-1} Q^\alpha(s)(1-s)^\beta \\ &\leq \left(-\alpha D - \eta(1-s)^{\beta/2} \left(\frac{c^2}{D}\right)^{\alpha-1}\right) \left((1-s)^{\beta/2}Q^\alpha(s) - \eta(R(s) + s)\right) \\ &\quad + (R + s) \left(\sigma^2\alpha(1-s)^{\beta/2}Q^{\alpha-1}(s) - \eta^2(1-s)^{\beta/2} \left(\frac{c^2}{D}\right)^{\alpha-1} - \alpha D\eta\right) \\ &\leq \left(-\alpha D - \eta \left(\frac{c^2}{D}\right)^{\alpha-1} (1-s)^{\beta/2}\right) \left((1-s)^{\beta/2}Q^\alpha(s) - \eta(R + s)\right) \end{aligned}$$

when  $\eta$  satisfies (4.13). Taking into account of  $\alpha > 1$  and Lemma 9, this shows (ii).

We now turn to (iii).

$$\begin{aligned} R(R - \theta s)' &= -(1 + \theta)R + \left(\frac{c^2}{D}\right)^{\alpha-1} Q^\alpha(s)(1-s)^\beta \\ &\leq -(1 + \theta)R + \theta(R + s)(1-s)^{\beta/2} \quad (\text{by (ii)}) \\ &= \left(-(1 + \theta) + \theta(1-s)^{\beta/2}\right) (R - \theta s) + \theta s \left(-(1 + \theta) + (\theta + 1)(1-s)^{\beta/2}\right) \\ &< \left(-(1 + \theta) + \theta(1-s)^{\beta/2}\right) (R - \theta s). \end{aligned}$$

By using Lemma 9, we get the result by a standard argument.  $\square$

**Proof of Theorem 6.** The non-existence follows directly from  $Q(s) > \sigma^2 s$ . For the existence, we show there exists  $\eta > 0$  such that  $R(s) - \eta s(1-s) \leq 0$  on  $(0, 1)$  when  $\theta \leq 1/4$ , which is exactly (4.10).

$$(R - \eta s(1-s))' = -1 + \frac{1}{R} \left(\frac{c^2}{D}\right)^{\alpha-1} Q^\alpha(s)(1-s)^\beta - \eta(1-s) + \eta s$$



$$\leq -1 + \theta(1-s)^{\beta/2} + \theta(1-s)^{\beta/2} \frac{s}{R} - \eta(1-s) + \eta s.$$

At the point where  $R(s) - \eta s(1-s) = 0$ ,

$$(R - \eta s(1-s))' \leq -1 + \theta(1-s)^{\beta/2} + \frac{\theta}{\eta}(1-s)^{(\beta-2)/2} - \eta(1-2s) < 0$$

with an appropriate choice of  $\eta$ ,  $\forall s \in (0, 1)$ , which is possible if  $\theta \leq 1/4$ . Hence,  $R(s) - \eta s(1-s) \leq 0$  on  $(0, 1)$ .  $\square$

#### 4.2. The general case

Now, we study the general case and prove [Theorems 3](#) and [4](#).

We shall use, in various places, in addition to (A1), one or all of the assumptions (B1)–(B4) as listed in [Section 1](#).

Recall the constant  $M_1 = M/\lambda_3$ , where

$$M = \frac{(\alpha + \beta - 1)^{\alpha+\beta-1}}{(\alpha - 1)^{\alpha-1}\beta^\beta}.$$

Since the line of argument is very much similar to that of [Lemmas 9–12](#), we shall only state the main technical lemmas and omit the detailed demonstration.

**Lemma 13.** Suppose  $D > 0$  and  $H$  satisfies (A1), (B1) and (B2). Then, there is a unique solution  $(Q(s), R(s))$  of (I) on  $(0, 1)$ . Moreover, it has the property that

$$Q(s) = \frac{\sigma^2}{D}s + o(s), \quad R(s) = \frac{\lambda_1\sigma^2}{D}s^\alpha + o(s^\alpha) \quad \text{as } s \searrow 0.$$

In addition,  $Q'(s) > 0$ ,  $R(s) > 0$  on  $(0, 1)$ , and it is a traveling wave iff  $R(1) = 0$ .

**Lemma 14.** Let  $D > 1$  and  $H$  satisfies assumptions (A1), (B1)–(B3). Then, a solution  $(Q(s), R(s))$  of (I) on  $(0, 1)$  has the property that

$$\begin{aligned} \text{(i)} \quad & \frac{\sigma^2}{D}s < Q(s) < \sigma^2 s \quad \text{on } (0, 1); \\ \text{(ii)} \quad & Q(s) < \lambda(1 + \lambda)s, \quad R(s) < \lambda s \quad \text{on } (0, 1), \end{aligned}$$

where

$$\lambda = \begin{cases} \frac{-1 + \sqrt{1 + 4\sigma^2}}{2} \text{ is the positive root of } \lambda(1 + \lambda) = \sigma^2, & \text{if } D \leq M_1^{1/(\alpha-1)}, \\ \frac{-D + \sqrt{D^2 + 4\sigma^2}}{2} \text{ solves } \lambda(D + \lambda) = \sigma^2, & \text{if } D > M_1^{1/(\alpha-1)} \text{ and } \sigma^2 \leq \Sigma_1, \end{cases}$$

with

$$\Sigma_1 = \frac{D^2(D-1)(M_1^{1/(\alpha-1)} - 1)}{(D - M_1^{1/(\alpha-1)})^2}.$$

**Proof of Theorem 3.** It follows directly from [Lemmas 13](#) and [14](#) using the familiar procedure as in [Theorem 5](#).  $\square$

**Lemma 15.** Suppose  $0 < D < 1$  and  $H$  satisfies (A1), (B1)–(B4). Let  $(Q(s), R(s))$  be a solution of (I) on  $(0, 1)$ , then on  $(0, 1)$ ,

$$\begin{aligned} & \text{(i) } \sigma^2 s < Q(s) < \frac{\sigma^2}{D} s, \quad \text{(ii) } R(s) - \lambda_3 \frac{\sigma^2}{D} s^\alpha < 0; \\ & \text{(iii) } Q(s) \geq \bar{\eta} s \quad \text{where } \bar{\eta} = \frac{\frac{\lambda_3 \sigma^2}{D} + 1}{\frac{\lambda_3 \sigma^2}{D} + D} \sigma^2. \quad \square \end{aligned}$$

**Lemma 16.** Let  $0 < D < 1$  and  $H$  satisfies (A1), (B1)–(B4). Let  $(Q(s), R(s))$  be a solution of (I) on  $(0, 1)$ . Then, on  $(0, 1)$ ,

$$\begin{aligned} & \text{(i) } Q(s) > \eta(R(s) + s), \forall \eta \leq (-D + \sqrt{D^2 + 4\lambda_3 \sigma^2})/2\lambda_3 \\ & \text{(ii) } (1-s)^{\beta/2} Q^\alpha(s) - \eta(R(s) + s) < 0 \text{ if} \end{aligned}$$

$$\eta \geq \eta_3 \equiv \frac{2\alpha \left(\frac{\sigma^2}{D}\right)^\alpha D \lambda_2}{\alpha D + \sqrt{\alpha^2 D^2 + 4\alpha \lambda_2 \sigma^2}}. \quad (4.14)$$

$$\text{(iii) } R(s) - \theta s < 0, \text{ where } \theta = 2\alpha \lambda_3 \sigma^2 / (\alpha D + \sqrt{\alpha^2 D^2 + 4\alpha \lambda_2 \sigma^2}).$$

**Proof of Theorem 4.** It follows directly from Lemmas 15 and 16 using the familiar procedure as in Theorem 6.  $\square$

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