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J. Differential Equations 198 (2004) 442–463
Journal of Differential Equations
http://www.elsevier.com/locate/jde

Universal self-similarity of porous media equation with absorption: the critical exponent case

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Received April 22, 2003; revised September 8, 2003

Abstract

In this paper we study the large time behavior of non-negative solutions to the Cauchy problem of $u_t = \Delta u^m - u^q$ in $R^N \times (0, \infty)$, where $m > 1$ and $q = q_c \equiv m + 2/N$ is a critical exponent. For non-negative initial value $u(x,0) = u_0(x) \in L^1(R^N)$, we show that the solution converges, if $u_0(x)(1+|x|)^k$ is bounded for some $k > N$, to a *unique* fundamental solution of $u_t = \Delta u^m$, independent of the initial value, with additional logarithmic anomalous decay exponent in time as $t \to \infty$.

MSC: 35K65; 35K15

Keywords: Large time behavior; Non-linear parabolic equation; Cauchy problem; Self-similar solutions

1. Introduction

In this paper, we consider the N-dimensional Cauchy problem

$$\text{(I)} \quad \begin{cases} u_t = \Delta u^m - u^q & \text{in } R^N \times (0, \infty), \\ u(x,0) = u_0(x) \geqslant 0 & \text{in } R^N, \ u_0(x) \in L^1(R^N) \cap L^\infty(R^N), \end{cases}$$

where $m > 1$ and $q = q_c \equiv m + 2/N$.

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0022-0396/$ - see front matter
doi:10.1016/j.jde.2003.10.022

Our interest is to study the global dynamics of general solution with L^1-initial data having mild decay as $|x|\to\infty$. The purpose is to derive the exact spatial–temporal profile of a solution when $t\to\infty$. When consider an equation like (I) with power-type non-linearity, a typical situation is to show certain type of asymptotic self-similarity determined by simple scaling laws. This is certainly true when the exponent q is not critical, see [8,9,13]. But it is not the case here due to the reason that the exponent q with $q = q_c \equiv m + 2/N$, is critical as we explain below.

What makes the present case interesting and challenge is that the exponent q is critical in the sense that the two competing forces, the diffusion and absorption are perfectly balanced and therefore the simple self-similarity (or scaling) possessed by the equation fails to give useful information to the global dynamics. Therefore, the large time behavior of the solution has more delicate structure which cannot be predicated by the simple scaling law.

Let us make our point clear by providing some background details. For simplicity, we shall only expound the semilinear case of $m = 1$.

Suppose $q > q_c = 1 + 2/N$, $u_0(x) \in L^1(R^N)$ and $u_0(x)|x|^N \to 0$ as $|x|\to\infty$. Let

$$u^k(x,t) = k^N u(kx, k^2 t).$$

Then, u^k satisfies

$$u_t = \Delta u - k^{-\nu} u^q,$$

$$u(x,0) = k^N u_0(kx)$$

with $\nu = N(q-1) - 2 > 0$. That is, the absorption is negligible. Since the $L^1(R^N)$-norm of initial value of u^k is the same as that of u_0, $||u^k(\cdot,0)||_1 = ||u_0||_1$ and $k^N u_0(kx) \to ||u_0||_1 \delta(x)$ as $k\to\infty$, it is imaginable that u^k should converge to a fundamental solution $E_c(x)$ of $u_t = \Delta u$ with mass $c > 0$. Indeed, it was proved in [13], that

$$t^{N/2}|u(x,t) - E_c(x,t)| \to 0 \quad \text{as } t\to\infty.$$

But, when $q \leqslant q_c$, the above argument breaks down since $\nu = (q-1)N - 2 \leqslant 0$. This is because the absorption is too strong to be ignored. As a matter of fact, when $q < q_c$, the large time behavior depends on the more detailed information of the initial value as $|x|\to\infty$. Indeed, suppose

$$\lim_{|x|\to\infty} |x|^\alpha u_0(x) = A, \tag{1.1}$$

where $\alpha = 2/(q-1) > N$. The scaling law appropriate in this case is

$$u^k(x,t) = k^\alpha u(kx, k^2 t).$$

It is easy to see that u_k satisfies the same equation as u but the initial value of u^k, when $k>1$, has larger L^1-norm than u_0. The large time behavior of u reads as follows:

(i) If $A=\infty$ (see [13]), then

$$t^{1/(q-1)}u(x,t)\to\left(\frac{1}{q-1}\right)^{1/(q-1)}\quad \text{as } t\to\infty$$

uniformly in the set of the form $\mathcal{S}_{\mathcal{C}}=\{x\in R^N\colon |x|\leqslant Ct^{1/2}\}$, where $C>0$ is a constant.

(ii) If $A=0$ (see [8,9]), then

$$t^{1/(q-1)}|u(x,t)-W_0(x,t)|\to 0\quad \text{as } t\to\infty,$$

uniformly in the set of the form $\mathcal{S}_{\mathcal{C}}$, where $W_0(x,t)$ is the unique very singular solution of $u_t=\Delta u-u^q$.

(iii) If $0<A<\infty$ (see [9]), then

$$t^{1/(q-1)}|u(x,t)-W_A(x,t)|\to 0\quad \text{as } t\to\infty,$$

uniformly in the set of the form $\mathcal{S}_{\mathcal{C}}$, where $W_A(x,t)$ is the unique self-similar solution of $u_t=\Delta u-u^q$ with asymptotics

$$\lim_{|x|\to\infty} W_A(x,0)|x|^{2/(q-1)}=A.$$

Similar results to the above hold for both $m<1$ and $m>1$, see [6,15,18].

To see why the arguments for $q\neq q_c$ fails for $q=q_c$, we look at the case of $q>q_c$ and that of $q<q_c$ separately.

If we guess the argument for $q>q_c$ is true for $q=q_c$, it leads us to the obvious but not very useful conclusion: u converges to a fundamental solution of the same equation ($v=0$). But, we know [1] that there exists no non-trivial fundamental solution when $q=q_c$. Hence, we may guess $u^k\to 0$ as $k\to\infty$ and we are led to believe

$$t^{1/(q-1)}u(x,t)\to 0\quad \text{as } t\to\infty. \tag{1.2}$$

But, it is not good enough to give the exact picture of large time dynamics, even it is true. Clearly, a more refined estimate is necessary if we want to accurately characterize the large time behavior of solutions for critical exponent case.

In fact, for $u_0\in L^1(R^N)$, the identity

$$\int_{R^N}u(x,t)\,dx-\int_{R^N}u_0(x)\,dx=\int_0^t\int_{R^N}u^q(x,t)\,dx\,dt$$

holds and the right-hand side must converge as $t\to\infty$. If $q>q_c$, by a Harnack-type inequality,

$$u(x,t)\leqslant Ct^{-N/2}\quad \text{for } t>0.$$

Consequently,

$$\int_{R^N} u^q(x,t)\,dx \leqslant Ct^{-N(q-1)/2} \int_{R^N} u(x,t)\,dx,$$

where $N(q-1)/2 > 1$ since $q > q_c$. It follows that

$$\int_{R^N} u(x,t)\,dx \to l > 0 \quad \text{as } t \to \infty. \tag{1.3}$$

But, as was shown in [14] that when $q = q_c$,

$$\int_{R^N} u(x,t)\,dx \to 0 \quad \text{as } t \to \infty.$$

By a Harnack-type inequality, it follows that

$$\sup_{x \in R^N} u(x,t) \leqslant Ct^{-N/2} \left(\int_{R^N} u(x,t/2)\,dx \right).$$

Hence, (1.2) is indeed true.

The results for $q < q_c$ also fails to extend to include the $q = q_c$ case, since $2/(q_c - 1) = N$, and u_0 satisfying (1.1) with $A > 0$ is not in $L^1(R^N)$. Whereas the basic setup for $q < q_c$ is u_0 in $L^1(R^N)$.

Nevertheless, together with our discussion of $q > q_c$, it tells us something useful. In particular, it indicates that the decaying rate of $||u(\cdot,t)||_1$ cannot be of the form $t^{-(1+\delta)}$ for any $\delta > 0$. This subtlety seems to suggest that the additional decay should be a lower order term, like a power of $\log t$, which is the key in determining the exact decay of L^1-norm and the asymptotic behavior of solutions for large time. Indeed, the first sharp lower bound involving $\log t$ is derived by Gmira and Veron [13] for the $m = 1$ case. Later on, Galaktionov et al. [11], obtain the large time behavior of solutions of (I) for $m = 1$ under a very special assumption that initial value satisfies

$$u_0(x) = o(\exp(-\delta|x|^2)) \quad \text{as } |x| \to \infty$$

for some $\delta > 0$. The case of $m > 1$ is considered by Galaktionov and Vazquez [12] under the assumption that u_0 has a compact support. In their approach, the construction of super-solution or the derivation of a sharp a priori estimate is valid only when the initial value has a compact support.

To our best knowledge, the only works which deal with the general initial value are that of Bricmont et al. [14], where the method of renormalization group has been used successfully to obtain asymptotic behavior of (I), and a recent work of Herraiz [14]. Both works are on $m = 1$. The work of Herraiz covers much wider class of initial data than that of Bricmont et al., since the latter is a perturbation type argument and the former a more classical type of hard analysis. But it is interesting to note that the method of renormalization group is very powerful which enables the authors of [3–5] to study a wide range of parabolic equations and prove outstanding results. But it seems very hard to generalize that approach to the porous media case.

The key to our approach (I) is to obtain sharp a priori estimate through the construction of suitable super-solutions, and then use maximum principle. But due to the reason that the equation is quasilinear, the detailed analysis is more involved than the semilinear case of $m = 1$. Another complication is that the limiting spatial–temporal profile is a *unique* one, independent of initial value. In addition, since it is of compact support, there is no way to construct a super-solution using no other function than itself to bound a solution with non-compact initial value.

The main result of this paper is the following theorem:

Theorem 1. *Suppose* $m > 1$ *and* $q = m + 2/N$. *If* $u_0(x)$ *satisfies*

$$\limsup_{|x|\to\infty} |x|^K u_0(x) < \infty,$$

where $K > N$, *then the corresponding solution* u *of* (I) *has the following asymptotic behavior*:

$$t^{1/(q-1)}(\log t)^{1/(q-1)}u(x,t) \to G\left(\frac{x}{t^{1/N(q-1)}\log^{(1-m)/2(q-1)}t}\right) \quad as\ t\to\infty$$

uniformly in set of the form $\{x\in R^N\colon |x|\leqslant Ct^{1/N(q-1)}\log^{(1-m)/2(q-1)}t\}$, *where* $G(x)$ *is the* **unique**, *radially symmetric solution of*

$$\Delta u^m + \frac{2}{N(q-1)}\left(\frac{N}{2}u + \frac{x\cdot\nabla u}{2}\right) = 0. \tag{1.4}$$

That is,

$$G(x) = G(x;a_*) = \left(a_* - \frac{(m-1)|x|^2}{2mN(q-1)}\right)_+^{1/(m-1)}$$

with a_* *being* **uniquely** *determined by the property*, *among* $\{G(x;a)\}_{a>0}$, *that*

$$||G||_1 - \frac{2(q-1)}{2+(m-1)N}||G||_q = 0. \tag{1.5}$$

Remark. It is not difficult to see that

$$t^{-1/(q-1)}G(x/t^{1/N(q-1)};a) = t^{-1/(q-1)}\left(a - \frac{(m-1)x^2}{2mN(q-1)t^{2/N(q-1)}}\right)_+^{1/(m-1)}$$

is the Barenblatt–Pattle solution of $u_t = \Delta u^m$.

Remark. It is easy to see from the statement of the theorem that our result is almost the best possible in the sense that our result works for most of L^1 initial values. It is an amazing fact that the *attractor* for almost all L^1 initial values is a *single function.*

Remark. Another novelty is the appearance of additional logarithmic terms both in the decay of solution in time and also the spatial–temporal profile of the limiting function. In particular, the logarithmic anomalous exponent in $(\log t)^{1/(q-1)}$ is the work of subtle balance between diffusion and absorption, the hallmark of the critical exponent case under consideration.

Remark. On a more technical ground, the exact value of a_* is calculated painstakingly in [12] as

$$a_* = \left(\frac{m-1}{2mN(q-1)}\right)^{-(m-1)/2}\left(\frac{NB(N/2, m/(m-1))}{2B(N/2,(m+q-1)/(m-1))}\right)^{(m-1)/2(q-1)}$$

with $q = q_c = m + N/2$, where $B(a,b)$ is the β function. For $m = 1$, the unique G is given explicitly by

$$G^*(y) = \left(\frac{N}{2}\right)^{N/2}(1+2/N)^{N^2/4}\exp(-|y|^2/4). \tag{1.6}$$

We use $a*$ to denote the constant

$$\left(\frac{N}{2}\right)^{N/2}(1+2/N)^{N^2/4}.$$

The organization of this paper is as follows. In Section 2 we prove Theorem 1. In Section 3, we present a new proof for $m = 1$. In Section 4 we show how to generalize our result to more general equations and make a few remarks.

2. Proof of Theorem 1

The key here is to construct a suitable super-solution and then use the maximum principle to obtain a sharp upper-bound. For final convergence analysis, we use the elegant and simple dynamical system argument in [12] to prove the desired result, rather than resort to a somewhat equivalent argument.

To simplify our calculation, we start by making a variable transformation

$$v = mu^{m-1}/(m-1).$$

Then the equation for v is

$$v_t = (m-1)v\Delta v + |\nabla v|^2 - c(m,q)v^p, \tag{2.1}$$

where $p = (q+m-2)/(m-1) > 2$, $p-1 = (q-1)/(m-1)$ and $c(m,q) = (m-1)[(m-1)/m]^{q-1}$. We shall construct our super-solution using (2.1) rather than the original equation.

Lemma 1. *The function*

$$A(x,t) = ((T+t)\log(T+t))^{-1/(p-1)}$$

$$\times \left(a - \frac{c_0|x|^2}{(T+t)^{2/N(q-1)}\log^{-1/(p-1)}(T+t)} \frac{\log(T+t)}{\log(T+t)+b} \right)_+$$

is a super-solution of (2.1) *provided* a, b *are properly chosen and* $T \geqslant T_0(m,N)$. *Here the constant* $c_0 = 1/2N(q-1)$. *Furthermore, define* $s = \log(T+t)$ *and* $z = |x|s^{1/2(p-1)}/e^{s/N(q-1)}$, *we have*

$$A_t - (m-1)A\Delta A + |\nabla A|^2 + c(m,q)A^p$$

$$\geqslant e^{-s/(p-1)}s^{-1/(p-1)} \begin{cases} \dfrac{c(m,q)a^p}{2(s+b)} \left(\dfrac{k-1}{k}\right)^p & \text{if } \dfrac{sc_0z^2}{s+b} \leqslant a/k, \\ \dfrac{a}{2(s+b)} & \text{if } a/k \leqslant \dfrac{sc_0z^2}{s+b} \leqslant a, \end{cases} \tag{2.2}$$

where $k = 1 + 2(p-1)c_0$.

Proof. We assume $s > 0$. It is easy to calculate that if $a - c_0z^2s/(b+s) > 0$,

$$A_t = e^{-sp/(p-1)}s^{-p/(p-1)} \left(-\frac{1+s}{p-1}\left(a - \frac{c_0z^2s}{s+b}\right)_+ + \frac{2c_0z^2s^2}{N(q-1)(b+s)} \right.$$

$$\left. - \frac{c_0z^2s}{(p-1)(b+s)} - \frac{c_0z^2bs}{(b+s)^2} \right),$$

$$\nabla A = e^{-sp/(p-1)}s^{-p/(p-1)} \left(-\frac{2c_0|x|s}{e^{2s/N(q-1)}s^{-1/(p-1)}(b+s)} \right) \frac{x}{|x|},$$

$$\Delta A = e^{-sp/(p-1)}s^{-p/(p-1)} \left(-\frac{2c_0Ns}{e^{2s/N(q-1)}s^{-1/(p-1)}(b+s)} \right).$$

Hence,

$$I(A) \equiv \Big(A_t - (m-1)A\Delta A + |\nabla A|^2 + c(m,q)A^p\Big)e^{sp/(p-1)}s^{p/(p-1)}$$
$$= -\frac{1+s}{p-1}\left(a - \frac{c_0 z^2 s}{s+b}\right)_+ + \frac{c_0 z^2 s}{(b+s)}\left(\frac{2s}{N(q-1)} - \frac{1}{(p-1)} - \frac{1}{(b+s)}\right)$$
$$+ \frac{2c_0 N(m-1)s^2}{(b+s)}\left(a - \frac{c_0 z^2 s}{s+b}\right)_+ - \frac{4c_0^2 z^2 s^3}{(b+s)^2} + c(m,q)\left(a - \frac{c_0 z^2 s}{s+b}\right)_+^p$$
$$= -a\frac{s(b+1)+b}{(s+b)(p-1)} + \frac{bc_0 s z^2}{(s+b)^2}\left(s[2(m-1)c_0 N + 4c_0] - \frac{p}{p-1}\right)$$
$$+ c(m,q)\left(a - \frac{c_0 z^2 s}{s+b}\right)_+^p. \tag{2.3}$$

To prove A is a super-solution, we only need to show $I(A) \geqslant 0$ when $\left(a - \frac{c_0 z^2 s}{s+b}\right) > 0$. Now, we fix $b = 2k/c_0(p-1)$. We show that if a is chosen to be larger than some $a(m,N)$, the lemma holds.

Case 1: $c_0 z^2 s/(b+s) < a/k$. In this case

$$c(m,q)\left(a - \frac{c_0 z^2 s}{s+b}\right)_+^p > c(m,q)\left(\frac{k-1}{k}\right)^p a^p.$$

Since $(s(b+1)+b)/(s+b) < b+1$,

$$\frac{c(m,q)}{2}\left(\frac{k-1}{k}\right)^p a^p > a\frac{(b+1)}{p-1},$$

if a is suitably large. By the inspection of (2.3), we see that $I(A) > 0$ when $s > s_1(m,q,n)$.

Case 2: $a/k < c_0 z^2 s/(b+s) < a$. In this case, we shall use the second term in (2.3) to bound the first term. Note that $2(m-1)c_0 N = 1/(p-1)$,

$$I(A) \geqslant \frac{a}{(s+b)(p-1)}\left(-s(b+1) - b + \frac{sb}{k} + \frac{4b(p-1)sc_0}{k} - \frac{p}{k}\right)$$
$$\geqslant \frac{a}{(s+b)(p-1)}\left(3s - b - \frac{p}{k}\right) > 0$$

if $s > s_2(m,q,N)$. This complete the proof of lemma. □

Remark. The fact that $A(x,t)$ is a super-solution is already demonstrated in [13], but since we need more detailed information such as the inequality in (2.2), we include a proof here. The shortcoming of a super-solution like $A(x,t)$ is that it cannot be used to control solutions with initial data with non-compact support. The next lemma shows, nevertheless, it can be used as a foundation to construct one which can control solutions of (I) whose initial data is $O((1+|x|)^{-K}$ with $K>N$.

Lemma 2. *Suppose* $\beta > N(m-1)$, $\tau = [\beta - N(m-1)]/4N(q-1)$ *and* $E>0$, *then*

$$w(x,t) = A(x,t) + (T+t)^{-1/(p-1)} \log^{-1/(p-1)}(T+t) f(x,t),$$

where

$$f(x,t) = \begin{cases} Ee^{-\tau s} & \text{if } |z| \leqslant C, \\ EC^{\beta}|z|^{-\beta} Ee^{-\tau s} & \text{if } |z| \geqslant C \end{cases}$$

is a super-solution of (2.1) *provided* $a = a(m,N,E)$ *is properly chosen and* $T \geqslant T_0(m,N)$. *Here* $C>0$ *is an arbitrary but fixed constant and* z *is as in Lemma* 1.

Remark. It is clear from the composition of $w(x,t)$ that it can be used to bound a solution of (2.1) with initial data $O((1+|x|)^{-\beta})$ with $\beta > N(m-1)$. A direct consequence of Lemma 2 is the following

Corollary 1. *Suppose* u *is a solution of* (I) *with non-negative initial data* $u_0(x) = O((1+|x|)^K)$ *as* $|x| \to \infty$ *with* $K>N$, *then* $u(x,t)$ *can be bounded by a constant multiple of* $w^{1/(m-1)}(x,t)$, *and in particular*, *we have the bounds*

$$u(x,t) \leqslant M(T+t)^{-1/(q-1)} \log^{-1/(q-1)}(T+t),$$

$$||u(\cdot,t)||_1 \leqslant M(\log(T+t))^{-(2+(m-1)N)/2(q-1)},$$

where $M>0$ *is a constant.*

Proof of Lemma 2. Direct calculation reveals that

$$\begin{aligned} &w_t - (m-1)w\Delta w - |\nabla w|^2 + c(m,q)w^p \\ &\quad = (T+t)^{-1/(p-1)} \log^{-1/(p-1)}(T+t) J(z,s), \end{aligned} \tag{2.4}$$

where

$$\begin{aligned} J(z,s) = {} & \bar{A}_t - \frac{f}{p-1}\left(1+\frac{1}{s}\right) - \frac{zf_z}{N(q-1)} - \tau f - (m-1)\bar{A}\Delta_z\bar{A} - (m-1)\bar{A}\Delta_z f \\ & - (m-1)f\Delta_z\bar{A} - (m-1)f\Delta_z f - \bar{A}_z^2 - 2\bar{A}_z f_z - f_z^2 + c(m,q)(\bar{A}+f)^p. \end{aligned} \tag{2.5}$$

Here every term with overbar "–" is the original term times $[(T+t)\log(T+t)]^{-1/(p-1)}$. Regrouping the terms and replacing $(\bar{A}+f)^p$ by $\bar{A}^p$, we have

$$J(z,s) \geqslant I(A) + J_1 - (m-1)J_2 - J_3, \tag{2.6}$$

where

$$\begin{aligned} J_1 &= -\frac{f}{p-1}\left(1+\frac{1}{s}\right) - \frac{zf_z}{N(q-1)} - \tau f, \\ J_2 &= \bar{A}\Delta_z f + f\Delta_z \bar{A} + f\Delta_z f, \\ J_3 &= 2\bar{A}_z f_z + f_z^2. \end{aligned}$$

We now consider different ranges of z.

First, $z \leqslant C$. In this case $f_z = \Delta_z f = 0$.

$$J(z,s) \geqslant I(A) - \frac{Ee^{-\tau s}}{p-1}\left(1+\frac{1}{s}\right) - \tau Ee^{-\tau s} \geqslant \frac{a}{2(s+b)} - \frac{Ee^{-\tau s}}{p-1}\left(1+\frac{1}{s}\right) - \tau Ee^{-\tau s}$$

by Lemma 1. If we choose a a large number such that $a \geqslant E+1$, $J(z,s) > 0$ for $s > s_1(m,N,\tau)$.

Next we consider the case where $z > C$. It is easy to see that

$$J_1 = \left(\frac{\beta}{N(q-1)} - \frac{1}{p-1}\left(1+\frac{1}{s}\right) - \tau\right) EC^\beta e^{-\tau\beta} z^{-\beta} > 0.$$

if $s \geqslant s_2(N,m,\tau)$. Furthermore,

$$\begin{aligned} J_2 &= \left(\left(a - \frac{c_0 z^2 s}{s+b}\right)_+ \beta(\beta+2-N)C^\beta z^{-2} - \frac{2c_0 NS}{s+b} \right.\\ &\quad \left. + \beta(\beta+2-N)EC^\beta z^{-\beta-2}e^{-\tau s}\right) EC^\beta z^{-\beta} e^{-\tau s} \\ &\leqslant D_1\left(az^{-2} + 1 + Ez^{-\beta-2}e^{-\tau s}\right) Ez^{-\beta} e^{-\tau s}, \end{aligned}$$

where $D_1 = D_1(N,m,\beta,C)$. Similarly,

$$\begin{aligned} J_3 &= \left(\frac{4c_0 sb}{s+b} + \beta^2 EC^\beta z^{-\beta-2} e^{-\tau s}\right) EC^\beta z^{-\beta} e^{-\tau s} \\ &\leqslant D_2\left(1 + Ez^{-\beta-2}e^{-\tau s}\right) Ez^{-\beta} e^{-\tau s}, \end{aligned}$$

where $D_2 = D_2(N,m,\beta,C)$. Now we look at $I(A)$. If $sc_0 z^2/(s+b) \leqslant a/k$, then by Lemma 1,

$$I(A) \geqslant \frac{a^p}{2(s+b)}\left(\frac{k-1}{k}\right)^p.$$

Hence, $J \geqslant I(A) + J_1 - (m-1)J_2 - J_3 > 0$ if $a > M(E+1)$, with $M = M(N, m, \beta, C)$ a big positive constant and $s \geqslant s_3(m, N, \tau)$.

If $a/k < sc_0 z^2/(s+b) < a$, $z^{-2} < c_0 k/a$.

$$J_2 \leqslant D_3 \left(a \frac{k-1}{k} \frac{c_0 k}{a} + 1 + E \frac{c_0 k}{a} e^{-\tau s} \right) E e^{-\tau s}$$

$$\leqslant D_3 (k+1) E z^{-\beta} e^{-\tau s}$$

with a positive constant $D_3 = D_3(N, m, \beta, C)$. Likewise,

$$J_3 \leqslant D_4 (k+1) E z^{-\beta} e^{-\tau s}.$$

Therefore, $J \geqslant I(A) + J_1 - (m-1)J_2 - J_3 > 0$ if $s \geqslant s_4(m, N, \tau)$, since $I(A) \geqslant a/2(s+b)$ by Lemma 1.

If $sc_0 z^2/(s+b) \geqslant a$, then $A \equiv 0$ and $z^{-2} < c_0/a < 1/E$, $z^{-\beta} < (c_0/a)^{\beta/2} \ll 1$.

$$J_2 = \beta(\beta + 2 - N) E C^\beta z^{-\beta-2} e^{-\tau s} \times E C^\beta z^{-\beta} e^{-\tau s}$$

$$\leqslant D_5 E z^{-\beta} e^{-2\tau s},$$

with a positive constant $D_5 = D_5(N, m, \beta, C)$. Similarly,

$$J_3 \leqslant D_6 E z^{-\beta} e^{-2\tau s}.$$

Hence,

$$J \geqslant J_1 - (m-1)J_2 - J_3 \geqslant \left(\frac{\beta}{N(q-1)} - \frac{1}{p-1}\left(1 + \frac{1}{s}\right) - \tau - mD_7 e^{-\tau s} \right) E C^\beta z^{-\beta} e^{-2\tau s} > 0$$

if $s \geqslant s_4(m, N, \tau)$. Thus, the function w is indeed a super-solution. $\square$

Now, we come to the final ingredient of our proof-the dynamical system setup of [12]. For completeness and easy reference in our proof of Theorem 1, we describe briefly the setup in [12] by stating the key Theorem 3 there as follows.

Suppose a general dynamical system is given by the evolution equation

$$u_t = A(u) \tag{2.7}$$

and a perturbation by

$$u_t = B(t, u). \tag{2.8}$$

Theorem 3 (Galaktionov and Vazquez [12]). *The ω-limit sets for the solution of* (2.8) *in the class Φ are contained in the ω-limit sets Ω for the solution of* (2.7) *under assumption* (H1)–(H3) *below. Consequently, the orbits approach uniformly Ω as* $t \to \infty$.

(H1) *The class* Φ *of solutions* $u \in C([0,\infty):X)$ *of* (2.8) *is defined for* $t>0$ *with values in a complete metric space* X (*with distance* d). *The assumption is that the orbits* $\{u(t)\}_{t>0}$ *are relatively compact in* X. *Moreover, if*

$$u^{\tau}(t) = u(t+\tau), \quad t,\tau > 0,$$

then we assume the sets $\{u^{\tau}\}_{\tau>0}$ *are relatively compact in* $L^{\infty}_{\text{loc}}([0,\infty):X)$.

(H2) B *is a small perturbation of* A *in the sense that given a solution* $u \in \Phi$ *of* (1.8), *if for a sequence* $\{t_j \to \infty\}$, $u(t+t_j)$ *converges to a function* $v(t)$ *in* $L^{\infty}_{\text{loc}}([0,\infty):X)$ *as* $j\to\infty$, *then* v *is a solution of* (2.7).

(H3) *The* ω*-limit set of* (2.7) in X,

$$\Omega = \{ f \in X\colon\ \exists u \in C([0,\infty):X) \text{ solution of (2.7) and a sequence } t_j \to \infty \text{ such that } u(t_j) \to f \}$$

is non-empty, compact and uniformly stable *in the sense that for every* $\varepsilon>0$, $\exists \delta>0$ *such that if* u *is any solution of* (2.7) *with* $d(u(0),\Omega) \leqslant \delta$ *then*

$$d(u(t),\Omega) \leqslant \varepsilon \quad \text{for every } t>0.$$

Proof of Theorem 1. The proof is based upon the observation that with the sharp upper bound in Lemma 2, the argument in [12], in particular the Theorem 3 there is valid for our situation.

We start by making a variable transformation. If $u(x,t)$ is a solution of (I), then, for any $T>0$, denoting $s=\log(T+t)$,

$$v(y,s) = ((T+t)\log(T+t))^{-1/(q-1)} u(x,t), \quad y = \frac{x}{(T+t)^{1/N(q-1)} \log^{(1-m)/2(q-1)}(T+t)}$$

is a solution of

$$v_s = \Delta v^m + \frac{1}{N(q-1)}(Nv + y\cdot\nabla v) + \frac{1}{s(q-1)}\left(v + \frac{1-m}{2} y\cdot\nabla v\right) - \frac{v^q}{s}. \tag{2.9}$$

Furthermore, by Lemma 2 u can be bounded by a super-solution, and the u bound there is directly translated into the following estimate:

$$v(y,s) \leqslant G(y;a) + E\min(1,|y|^{-k})e^{-\tau s}, \tag{2.10}$$

A lower bound for v is easy and is supplied by Lemma 2.3 in [12] which is

$$G(y;a_-) \leqslant v(y,s) \tag{2.11}$$

for all $s \geqslant s_0$, where $a_- > 0$.

From now on, we concentrate on solutions with fixed $K>N$ and their bounds in (2.10) and (2.11) are given by fixed E, $a=a_+$ and a_- with $0<a_-<a=a_+$.

Following [12], we let

$$X = \{f \in L^1(B)\colon\ G(y; a_-) \leqslant f(y) \leqslant G(y, a_+) \text{ a.e. in } B\}, \quad B = B_{a+}(0) \subset R^N.$$

It is well known [10] that the ω-limit set Ω of unperturbed equation

$$v_s = \Delta v^m + \frac{1}{N(q-1)}(Nv + y \cdot \nabla v) \tag{2.12}$$

with initial value $v_0 \in X$ is the Barenblatt–Pattle profile $G(y; a)$ with the same L^1-norm as v_0. Ω is apparently compact in X.

Let Φ be the class of functions $v \in C([0, \infty); Y)$, where

$$Y = \{f \in L^1(R^N)\colon\ G(y; a_-) \leqslant f \leqslant G(y; a_+) + E \min(1, |y|^{-k})\}.$$

The compactness of solution orbit $\{v(s)\}_{s \geqslant M}$ of (2.9) in L^1, with M a fixed positive number, follows from interior regularity (cf. [2,7]) and the uniform bound (2.10). Furthermore, the family is compact in $C(B_R)$ for any $R > 0$. Moreover, let $v^\tau(s) = v(s + \tau)$, $s \geqslant M$ and $\tau > 0$. The fact that the set $\{v^\tau\}_{\tau > 0}$ is relatively compact in $L^\infty_{\mathrm{Loc}}([0, \infty) : Y)$ follows from the same reasoning. This verifies (H1) in Theorem 3 of [12].

It is clear that given a solution $v \in \Phi$, if for a sequence $\{t_j \to \infty\}$, $v(t + t_j)$ converges to a function $w(t)$ in $L^\infty_{\mathrm{Loc}}([0, \infty) : Y)$ as $j \to \infty$, then w is a solution of (2.12) and $w(t) \in C([0, \infty); X)$ as a consequence of estimates (2.10) and (2.11). This validates (H2) in Theorem 3 of [13]. Assumption (H3) is automatically satisfied, see [12].

Hence, by Theorem 3 in [12], the ω-limit set of (2.9) is contained in Ω.

Now we show $v(y, s) \to G(y; a_*)$ uniformly in y as $s \to \infty$ under the assumption that $||v(\cdot, s)||_1 \to I_0 > 0$ as $s \to \infty$. This is the content of Lemma 5.1 in [12]. Our case is only slight different from the proof of Lemma 5.1 in [12], which is caused by v not having a compact support. But, again, (2.10) assures that $v(y, s) \to 0$ uniformly outside B as $s \to \infty$. Therefore, the exact argument in Lemma 5.1 in [12] applies.

The convergence of $||v(\cdot, s)||_1$ as $s \to \infty$ can be demonstrated using the compactness of orbit $\{v(s)\}_{s \geqslant M}$ in $C(B)$ and the rapid decay to zero outside B by Lemma 2. We refer the interested reader to Proposition 5.2 in [12]. This completes the proof of Theorem 1. □

Remark. An interesting fact which comes out of our proof of Theorem 1, in particular out of the super-solution we constructed is that the algebraic decay in space of the initial value does have a material effect in the convergence rate of the solution to the limiting profile. More specifically, the convergence rate is proportional to $K - N$, where K is the decay rate of initial value at $|x| = \infty$. There is an apparent slow down of the convergence when K is close to N. This may explain again why our case is much more involved than the case when the initial value is of compact support.

3. The semilinear case of $m = 1$

In this section, we present a new proof for the semilinear case $m = 1$ of (I). Our contribution here is a more transparent demonstration how a super-solution can be constructed after the equation is transformed by a self-similar change of variable, and by taking advantage of the structure of the transformed equation. Another benefit is our result applies to more general initial data than the known results.

Theorem 2. *Suppose $m = 1$ and $q = 1 + 2/N$. If $u_0(x)$ satisfies*

$$\limsup_{|x|\to\infty} |x|^k u_0(x) < \infty,$$

where $k > N$, then the corresponding solution u of (I) *has the following asymptotic behavior*:

$$t^{N/2}(\log t)^{N/2} u(x,t) \to G\left(\frac{x}{t^{1/2}}\right) \quad \text{as } t \to \infty$$

uniformly in set of the form $\{x \in R^N\colon |x| \leqslant Ct^{1/2}\}$, where $G(x)$ is the **unique**, *radially symmetric solution of*

$$\Delta u + \frac{2}{N(q-1)}\left(\frac{N}{2}u + \frac{x\cdot\nabla u}{2}\right) = 0. \tag{3.13}$$

That is,

$$G(y) = \left(\frac{N}{2}\right)^{N/2} (1 + 2/N)^{N^2/4} \exp(-|y|^2/4).$$

As in the case of $m > 1$, the crucial step is to construct a suitable super-solution with sharp bound. First, we make a change of variables. Let

$$s = \log(t+T), \quad y = \frac{x}{(T+t)^{1/2}}, \quad v(y,s) = [(T+t)\log(T+t)]^{1/(q-1)} u(x,t),$$

then we obtain the equation for v as:

$$v_s = \Delta v + \frac{1}{2}(Nv + y\cdot\nabla v) + \frac{1}{s}\left(\frac{N}{2}v - v^q\right). \tag{3.14}$$

Define

$$I(v) \equiv v_s - \Delta v - \frac{1}{2}(Nv + y\cdot\nabla v) - \frac{1}{s}\left(\frac{N}{2}v - v^q\right). \tag{3.15}$$

It is easy to see that a super-solution to (3.14) is one satisfying $I(v) \geqslant 0$.

We construct a super-solution in the form of

$$V(y,s) = \Psi(y)\left(1+\frac{f(y)}{s}\right), \tag{3.16}$$

by choosing f properly, where $\Psi(y) = \Psi_M(y) = M\exp(-|y|^2/4)$ with $M > 0$ a constant.

Simple calculation shows

$$V_s = -\frac{f\Psi}{s^2}, \quad \nabla V = \nabla\Psi\left(1+\frac{f(y)}{s}\right) + \frac{\Psi}{s}\nabla f,$$
$$\Delta V = \Delta\Psi\left(1+\frac{f(y)}{s}\right) + \frac{2\nabla\Psi\cdot\nabla f}{s} + \frac{\Psi}{s}\Delta f$$

and

$$I(V) = -\frac{f\Psi}{s^2} - \left(1+\frac{f(y)}{s}\right)\left(\Delta\Psi + \frac{1}{2}(N\Psi + y\cdot\nabla\Psi\right)$$
$$-\frac{\Psi}{s}\left(\Delta f + \frac{1}{2}y\cdot\nabla f - y\cdot\nabla f + \frac{N}{2} - \Psi^{q-1}\right) - \frac{\Psi}{s^2}\frac{N}{2}f + \frac{\Psi^q}{s}\left(\left(1+\frac{f}{s}\right)^q - 1\right).$$

Since,

$$\Delta\Psi + \tfrac{1}{2}(N\Psi + y\cdot\nabla\Psi) = 0,$$

$$I(V) = -\frac{\Psi}{s}\left(\Delta f + \frac{1}{2}y\cdot\nabla f - y\cdot\nabla f + \frac{N}{2} - \Psi^{q-1}\right)$$
$$+\frac{f\Psi}{s^2}\left(-\frac{N+2}{2} + \Psi^{q-1}\left(\frac{\left(1+\frac{f}{s}\right)^q - 1}{\frac{f}{s}}\right)\right)$$
$$\equiv -\frac{1}{s}I_1 + \frac{1}{s^2}I_2. \tag{3.17}$$

From now on, we assume f is radial, and consider the following equation for f:

$$\Psi\left(f'' + \frac{N-1}{r} - \frac{r}{2}f'\right) + \frac{N}{2}\Psi - \Psi^q = -I_M G(r), \tag{3.18}$$

where $r = |y|$, G is a positive function satisfying

$$\int_0^\infty r^{N-1}G(r)\,dr = 1$$

and

$$I_M \equiv M^q \int_0^\infty r^{N-1} e^{-qr^2/4}\, dr - \frac{MN}{2} \int_0^\infty r^{N-1} e^{-r^2/4}\, dr = \int_{R^N} \Psi_M^q\, dx - \frac{N}{2} \int_{R^N} \Psi_M\, dx.$$

The first assumption we make on G is:

$$\text{(A1)} \quad \exists L > N \text{ such that } rG'(r) + LG(r) > 0, \quad \text{for all } r > r_1 > 0. \tag{3.19}$$

That is, G will not have very rapid decay at $r = \infty$. Indeed, an integration of $rG'(r) + LG(r)$ implies $r^L G(r) > r_1^L G(r_1)$ for all $r > r_1$.

An integration of (3.18) yields that

$$f' r^{N-1} \Psi = \int_0^r \Psi^q s^{N-1}\, ds - \frac{N}{2} \int_0^r \Psi s^{N-1}\, ds - I_M \int_0^r s^{N-1} G(s)\, ds. \tag{3.20}$$

It is clear that $\exists r_2 > 0$, independent of M, such that

$$\Psi^q(r) - \frac{N}{2} \Psi(r) - I_M G(r) < 0 \quad \text{for all } r > r_2 > 0.$$

Hence, $f' > 0$ for all $r > r_2$, since the right-hand side of (3.20) has a limit equal to zero as $r \to \infty$. An integration of (3.20), after being divided by $r^{N-1}\Psi$ shows

$$f(r) = f(0) + \int_0^r \Psi^{-1}(\sigma) \sigma^{1-N} d\sigma \left(\int_0^\sigma \left(\Psi^q(s) - \frac{N}{2} \Psi(s) - I_M G(s) \right) s^{N-1}\, ds \right) \equiv J(r).$$

Clearly, we can make $f > 0$ for all $r > 0$ by making $f(0)$ large enough. An equivalent form of the above equation is

$$f(r) e^{-r^2/4} r^N \left(\int_r^\infty s^{N-1} G(s)\, ds \right)^{-1} = \frac{J(r)}{e^{r^2/4} r^{-N} \left(\int_r^\infty s^{N-1} G(s)\, ds \right)}.$$

It is easy to show that the right-hand side has a limit which equals to I_M. Therefore,

$$\frac{I_M}{2} \int_r^\infty s^{N-1} G(s)\, ds < f(r) e^{-r^2/4} r^N < 2 I_M \int_r^\infty s^{N-1} G(s)\, ds \tag{3.21}$$

for $r > r_3$, where r_3 is independent of M.

Given the form of the super-solution we try to construct and (3.17), which now takes the form

$$I(v) = \frac{1}{s} I_M G(r) + \frac{f\Psi}{s^2} \left(-\frac{N+2}{2} + \Psi^{q-1} \left(\frac{\left(1 + \frac{f}{s}\right)^q - 1}{\frac{f}{s}} \right) \right), \tag{3.22}$$

it is very important to compare $f\Psi$ with $G(r)$. By (A1),

$$L\int_r^\infty G(s)s^{N-1}\,ds > -\int_r^\infty G'(s)s^N\,ds = G(r)r^N + N\int_r^\infty G(s)s^{N-1}\,ds.$$

Consequently,

$$\int_r^\infty G(s)s^{N-1}\,ds > \frac{G(r)r^N}{L-N}.$$

This, in combination of (3.21), shows

$$f(r)\Psi(r) > \frac{I_M G(r)}{2(L-N)}.$$

for $r > r_3$. Next, we assume

$$\text{(A2)}\quad \exists C>0 \text{ such that } \int_r^\infty G(s)s^{N-1}\,ds < CG(r)r^N \quad \text{for all } r > r_4 > 0. \tag{3.23}$$

This implies, using (3.21),

$$f(r)\Psi(r) < 2CI_M G(r) \quad \text{for all } r > R \equiv \max(r_3, r_4).$$

Now we can finally show that

$$V(y,s) = \Psi(y)\left(1 + \frac{f(y)}{s}\right),$$

with f satisfying (3.18) and G satisfying $(A1)$ and $(A2)$ is a super-solution for $s > s_1(m, N, G)$, when M is sufficiently large.

It is clear from (3.22) that $I(v) > 0$ if $\Psi^{q-1} > N/2$, since

$$\frac{\left(1+\frac{f}{s}\right)^q - 1}{\frac{f}{s}} > q = 1 + \frac{2}{N}.$$

But, if $\Psi^{q-1} \leqslant N/2$, then

$$e^{-r^2/4} < \left(\frac{N}{M^{q-1}}\right)^{1/(q-1)},$$

or $r > r_M$, where $r_M \to \infty$ as $M \to \infty$. Hence,

$$I(v) \geqslant \frac{I_M G(r)}{s} - \frac{N+2}{s^2} f\Psi \geqslant \frac{f\Psi}{2Cs} - \frac{N+2}{s^2} f\Psi > 0$$

if $r > R$ and $s > s_1$, where s_1 is independent of M. Therefore, v is a super-solution for $s > s_1$.

Proof of Theorem 2. Since the proof follows the same procedure as that of Theorem 1, we shall be brief.

Let u be a solution of (I) satisfying the assumption of Theorem 2. Let $v(y,s)$ be the function after self-similar transformation. Then, clearly we can choose $G(r) = C(1+r)^{-K}$ for constructing a super-solution V as in (3.16) with M sufficiently large. Furthermore,

$$v(y,s) \leqslant V(y,s) \leqslant E\left(e^{-|y|^2/4} + \frac{(1+|y|)^{-K}}{s}\right), \tag{3.24}$$

where $E > 0$ is a suitable constant. A lower bound for v is obtained in [13] which is

$$me^{-|y|^2/4} \leqslant v(y,s) \tag{3.25}$$

for all $s \geqslant s_0$, where $m > 0$.

From now on, we concentrate on solutions with fixed $K > N$ and their bounds in (2.10) and (2.11) are given by fixed C, $m > 0$ with $0 < m < E$.

Let

$$X = \{f \in L^1(R^N):\ me^{-|y|^2/4} \leqslant f(y) \leqslant Ee^{-|y|^2/4} \text{ a.e. in } R^N\}.$$

It is well known that the ω-limit set of unperturbed equation

$$v_s = \Delta v + \frac{1}{N(q-1)}(Nv + y \cdot \nabla v) \tag{3.26}$$

with initial value $v_0 \in X$ is $ae^{-|y|^2/4}$ with the same L^1-norm as v_0. This is because (3.26) is nothing but the self-similar transformed heat equation $u_t = \Delta u$. Therefore, ω-limit sets with initial value in X is:

$$\Omega = \{ae^{-|y|^2/4}:\ m \leqslant a \leqslant E\},$$

which is clearly a compact set, and asymptotically stable because the L^1 contraction of heat equation. Hence, (H3) holds.

Let Φ be the class of functions $v \in C([0,\infty); Y)$, where

$$Y = \left\{ f \in L^1(R^N):\ me^{-|y|^2/4} \leqslant f \leqslant E(e^{-|y|^2/4} + (1+|y|)^{-K}) \right\}.$$

The compactness of solution orbit $\{v(s)\}_{s \geqslant M}$ of (3.14) in L^1, with M a fixed positive number, follows from interior regularity and the uniform bound (3.24). Furthermore, the family is compact in $C(B_R)$ for any $R > 0$, Moreover, let $v^\tau(s) =$

$v(s+\tau)$, $s \geqslant M$ and $\tau > 0$. The fact that the set $\{v^\tau\}_{\tau>0}$ is relatively compact in $L^\infty_{\text{Loc}}([0,\infty):Y)$ follows from the same reasoning. This verifies (H1) in Theorem 3 of [12].

It is clear that given a solution $v \in \Phi$, if for a sequence $\{t_j \to \infty\}$, $v(t+t_j)$ converges to a function $w(t)$ in $L^\infty_{\text{Loc}}([0,\infty):Y)$ as $j \to \infty$, then w is a solution of (3.26) and $w(t) \in C([0,\infty);X)$ as a consequence of estimates (3.24) and (3.25). This validates (H2) in Theorem 3 of [13].

Hence, by Theorem 3 in [12], the ω-limit set of (3.14) is contained in Ω.

Now we show $v(y,s) \to G^*(y)$ (as given in (1.6)) uniformly in y in any finite ball B_R with radius $R>0$, as $s \to \infty$ under the assumption that $||v(\cdot,s)||_1 \to I_0 > 0$ as $s \to \infty$, where I_0 is the L^1-norm of $G^*(y)$.

Since Ω consists functions of the form of $H_a \equiv ae^{-|y|^2/4}$, there is a unique a with the property that its L^1-norm is the given limit I_0. Therefore, $v(y,s) \to G^*(y)$ as $s \to \infty$. The uniform convergence in any finite ball B_R follows from (3.24) and regularity, and thus equi-continuity of solutions.

We now show the convergence of $w(s) \equiv ||v(\cdot,s)||_1$ to I_0 as $s \to \infty$. Suppose the contrary, then there exist sequences $s_j \to \infty$ and $\bar{s}_k \to \infty$ such that $v(y,s_j) \to H_{a_1}$ and $v(y,\bar{s}_k) \to H_{a_2}$ uniformly in any finite ball B_R as $j,k \to \infty$, where $a_1 \neq a_2$. Since $||H_{a_1}||_1$ and $||H_{a_2}||_1$ must be different, then one of them is different from $a*$.

Suppose $a* < a_2$, $a_1 < a_2$. Fix an arbitrary $a' \in (a_1,a_2)$, $a' > a*$. Then, by the continuity of $w(s)$ and its oscillatory property near $s=\infty$ there exists a sequence $s'_j \to \infty$ as $j \to \infty$ such that $||v(\cdot,s'j)||_1 = ||H_{a'}||_1$ and

$$dw(s'_j)/ds \geqslant 0 \quad \text{for all } j.$$

Then Theorem 3 in [12] implies

$$v(s'_j) \to H_{a'} \quad \text{uniformly in any finite ball } B_R \text{ as } j \to \infty.$$

It is clear from an integration of (3.14) that

$$s'_j \frac{dw}{ds}(s'_j) = \frac{N}{2} w(s'_j) - \int_{R^N} v^q(y,s'_j)\,dy$$

which is negative for all j large by uniform convergence in any finite ball B_R, uniform smallness at $y=\infty$ and the fact

$$\frac{N}{2}\int_{R^N} H_{a'}(y)\,dy - \int_{R^N} H^q_{a'}(y)\,dy < 0,$$

since $a' > a*$. So, we have a contradiction. This completes the proof of Theorem 2. $\square$

4. Extensions and remarks

Though it seems that the scaling invariant property of power non-linearities is essential to our approach, the result of Theorem 1 can be extended to equations which are small perturbation of Eq. (I). For instance, consider the equation

$$u_t = \Delta u^m - f(u), \quad (x,t) \in R^N \times (0,\infty) \tag{4.27}$$

with f a continuous function $R^+ \to R^+$ and $f(0) = 0$. In addition, we assume

$$u(x,0) = u_0(x) \geqslant 0$$

and there exists $M > 0$ and $K > N$ such that $u_0(x) \leqslant M(1+|x|)^{-K}$. It is well known from the works of Kalashnikov and Kersner [16,17] that there exists a unique solution $u \in C([0,\infty) : L^1(R^N))$ under our assumption of f and our initial value. Moreover, one can deduce from the non-negativity of f and the Maximum Principle that such a solution is bounded above by the solution of $u_t = \Delta u^m$ with the same initial data. In consequence, u will decay as $t \to \infty$ with at least the rate $O(t^{-N/[N(m-1)+2]})$. If we make additional assumption such as

$$f(s) \geqslant C_1 s^{q_c}, \quad \text{for } 0 \leqslant s \leqslant C_2$$

with $C_1 > 0$ and $C_2 > 0$, then

$$v(x,t) = u\left(\sqrt{C_1}x, C_1 t\right)$$

will be a sub-solution of (I): $u_t = \Delta u^m - u^{q_c}$ whenever u is a solution of (4.27), when $t \geqslant 1$. Again, by the Maximum Principle, a decay rate of $O((t \log t)^{-1/(q-1)})$ is readily obtained as $t \to \infty$. In particular, we deduce from Theorem 1 that for any $\varepsilon > 0$ there exists $T = T(\varepsilon) > 0$ such that when $t \geqslant T$,

$$u(x,t) \leqslant \left(\frac{t}{C_1}\log(t/C_1)\right)^{-1/(q_c-1)} (G(y;a_*) + \varepsilon),$$

where $y = xt^{-1/N(q_c-1)}(\log(t/C_1))^{(m-1)/2(q_c-1)} C_1^{(m-1)/(q_c-1)}$ and a_* is as in Theorem 1.

In a similar way, we can get lower bound from below by assuming

$$f(s) \leqslant C_2 s^{q_c} \quad \text{for } 0 \leqslant s \leqslant C_2.$$

In summary, we get the following result.

Theorem 3. *Let u be a solution of* (4.27) *with the initial condition as specified above. In addition, suppose*

$$\lim_{u \to 0} \frac{f(u)}{u^{q_c}} = 1,$$

then Theorem 1 *holds.*

Remark. it is clear that if the above limit is $C > 0$, then a result which is a simple scaling of Theorem 1 is valid.

It is probably more interesting to study how the perturbation of diffusion term will affect the underlying result. For example, a good candidate will be the equation $u_t = \Delta\phi(u) - f(u)$ with ϕ a small perturbation of u^m. We believe the analysis will be more involved.

Another interesting extension is to consider the P-Laplacian counterpart of (I):

$$u_t = \mathrm{div}(|\nabla u|^{p-2}\nabla u) - u^q$$

with $p > 2$ and $q = p - 1 + p/N$. The case of initial value with compact support is considered in [12]. But to cover more general situation with non-compact-support initial value with moderate decay as $|x| \to \infty$, which is sufficient to guarantee L^1-integrability but not much more, to derive a sharp estimate similar to one in (2.10) is the key. But to our best knowledge, such work has not appeared. We are sure it can be worked out, but the calculation seems to be formidable.

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