



The study of global stability of a diffusive Holling–Tanner predator–prey model



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ABSTRACT

In this paper we study the global stability of diffusive predator–prey system of Holling–Tanner type in a bounded domain $\Omega \subset R^N$ with no-flux boundary condition. By using a novel approach, we establish much improved global asymptotic stability of the unique positive equilibrium solution than works in literature. We also show the result can be extended to more general type of systems with heterogeneous environment.

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1. Introduction

This work is concerned with the study of diffusive Holling–Tanner-type predator–prey system in a bounded domain $\Omega \subset R^N$ with no-flux boundary condition.

$$\begin{cases} u_t = d_1 \Delta u + au - u^2 - \frac{uv}{m+u}, & (x, t) \in \Omega \times (0, \infty) \\ v_t = d_2 \Delta v + bv - \frac{v^2}{\gamma u} & (x, t) \in \Omega \times (0, \infty) \\ \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0, & (x, t) \in \partial \Omega \times (0, \infty) \\ u(x, 0) = u_0(x) > 0, \quad v(x, 0) = v_0(x) \geq 0 (\neq 0) \quad x \in \bar{\Omega}. \end{cases}$$

Here $u(x, t)$ and $v(x, t)$ are the density of prey and predator, respectively, Ω is a bounded domain with smooth boundary $\partial \Omega$, a , b , m and γ are positive constants. We assume throughout this paper that the two diffusion coefficients d_1 and d_2 are positive and equal, but not necessarily constants. From now on, we shall use d to represent the common value. It may depend on both spatial and time variables but strictly positive

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in $\bar{\Omega} \times [0, \infty)$. The no-flux boundary condition is imposed to guarantee that the ecosystem is not disturbed by exterior factors which may influence population flow cross the boundary, and therefore internal forces are the sole reason to generate interesting dynamical behavior of the system.

Remark. We are informed lately by a referee that the model should be called the composite May model according to Robert May since he was the first to use Holling type II functional response in the prey equation with the Leslie formulation for the predator equation.

It is easy to verify that system (1.1) has a unique positive equilibrium (u^*, v^*) , where

$$u^* = \frac{1}{2}(a - m - b\gamma + \sqrt{(a - m - b\gamma)^2 + 4am}), \quad v^* = b\gamma u.$$

The model is a well established one to describe real ecological interactions of various populations such as lynx and hare, sparrow and sparrow hawk, see [1–3] and is widely studied in literature in recent years, see [4–11] and the reference therein. In particular, the following result was proved by Peng and Wang in [10] by the construction of a Lyapunov function and linear analysis.

Theorem (PW). Assume that the parameters $m, a, b, \gamma, d_1, d_2$ are all positive. Then for system (I):

1. The positive equilibrium (u^*, v^*) is locally asymptotically stable if

$$m^2 + 2(a + b\gamma)m + a^2 - 2ab\gamma > 0.$$

2. The positive equilibrium (u^*, v^*) is globally asymptotically stable if

$$m > b\gamma, \quad \text{and} \quad (m + K)[b\gamma + 2(m + u^* + Ka)] > (a + m)b\gamma,$$

where

$$K = \frac{1}{2} \left(a - m + \sqrt{(a - m)^2 + 4a(m - b\gamma)} \right).$$

More recently, Chen and Shi [4] proved a more refined result:

Theorem (CS). Assume that the parameters $m, a, b, \gamma, d_1, d_2$ are all positive. Then for system (I), the positive equilibrium (u^*, v^*) is globally asymptotically stable, if $m > b\gamma$.

The approach in [4] is application of comparison principle and iteration of following scheme: (i) using a lower bound of v in the u equation to get an upper bound of u , (ii) using the upper bound of u to get an upper bound of v from the v equation, (iii) substitute the upper bound of v into u equation to get a lower bound of u , and (iv) using the lower bound of u to get a new lower bound of v from the v equation. The assumption $m > b\gamma$ is necessary to make sure the first lower bound of u is positive.

In this work, we prove a new global stability result for the positive equilibrium by using a novel comparison argument, which is different and more sophisticated from the one used in literature such as [4].

Our main result is as follows.

Theorem 1. Suppose $d = d(x, t)$ is strictly positive, bounded and continuous in $\bar{\Omega} \times [0, \infty)$, a, b, γ and m are positive constants, $\gamma^{-1} > a/(m + a)$, then the positive equilibrium solution (u^*, v^*) is globally asymptotically stable in the sense that every solution to (I) satisfies

$$\lim_{t \rightarrow \infty} (u, v) = (u^*, v^*) \quad \text{uniformly in } \Omega.$$

Remark. The above result covers more ground than the result of [10] or [4]. In particular, if $\gamma \leq 1$, for all choices of a , b , m we have global asymptotic stability, or when $b \geq (\gamma - 1)a/\gamma$, our assumption is weaker than $m > b\gamma$.

Remark. The method we use here is more flexible than the Lyapunov function method and more powerful than that used in [4], and the results cover more general settings such as when the Laplace operator is replaced by a uniform elliptic operator. It means we can cover cases with heterogeneous environment.

Let

$$\mathcal{L}u = \sum_{i,j=1}^N a_{ij}(x) \frac{\partial^2 u}{\partial x_i \partial x_j}$$

be a uniform elliptic operator in Ω with continuous coefficients $a_{ij}(x)$, $i, j = 1, \dots, N$. Then, we can show a result similar to [Theorem 1](#) for the following initial–boundary value problem:

$$(II) \begin{cases} u_t = \mathcal{L}u + au - u^2 - \frac{uv}{m+u}, & (x, t) \in \Omega \times (0, \infty) \\ v_t = \mathcal{L}v + bv - \frac{v^2}{\gamma u} & (x, t) \in \Omega \times (0, \infty) \\ \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0, & (x, t) \in \partial\Omega \times (0, \infty) \\ u(x, 0) = u_0(x) > 0, \quad v(x, 0) = v_0(x) \geq 0 (\neq 0) & x \in \bar{\Omega}. \end{cases}$$

Theorem 2. Suppose a, b, γ, m are positive constants satisfying the assumption in [Theorem 1](#) and \mathcal{L} a uniform elliptic operator in Ω with continuous coefficients. Then, the unique positive equilibrium (u^*, v^*) of (II) is globally asymptotically stable.

The organization of the paper is that we shall prove [Theorem 1](#) in Section 2. In Section 3, we shall discuss how to generalize our results to more general setting.

We note by passing that there are also important works related to the model studied in this paper, such as [12–15].

2. Proof of [Theorem 1](#)

We would like to start by clarifying some simple facts on the system.

Proposition 1. Suppose $d_1 = d_2 > 0$ are constants.

- (1) $u^* > a - b\gamma$.
- (2) (u^*, v^*) is locally asymptotically stable if $\gamma^{-1} > a/(m + a)$.
- (3) $\gamma^{-1} > a/(m + a)$ implies $u^* > a - b$.

Proof. It is easy to verify that

$$(a - m - b\gamma)^2 + 4am = (a + m - b\gamma)^2 + 4mb\gamma > (a + m - b\gamma)^2$$

and therefore,

$$u^* > \frac{a - m - b\gamma + a + m - b\gamma}{2} = a - b\gamma.$$

This proves the first statement.

For (ii), from the proof of Theorem 2.1 in [10] it follows, when $d_1 = d_2$, that (u^*, v^*) is locally asymptotically stable iff

$$2u^* - (a - m - b\gamma) > 0 \quad \text{and} \tag{2.1}$$

$$2(u^*)^2 - (a - m - b)u^* + bm > 0. \tag{2.2}$$

If $\gamma \leq 1$, both are trivially true by (i) and the expression of u^* .

If $\gamma > 1$, we only need to show u^* is larger than the largest positive root of the quadratic function in (2.2) under the condition that $a > m + b$, which is only possible if $\gamma < 2$ by the condition $\gamma^{-1} > a/(m + a)$. Using the assumption that $\gamma^{-1} > a/(m + a)$, it is easy to verify that

$$u^* > a - b\gamma \geq \frac{a - m - b}{2}$$

if $\gamma a \geq (2\gamma - 1)b$. The only case we need to consider is $\gamma a < (2\gamma - 1)b$. But, it contradicts $a > m + b > (\gamma - 1)a + b$. Hence, (ii) holds.

The last statement follows from simple computation which shows that the inequality, under the assumption $a > b$, and $\gamma > 1$ is equivalent to $m > (\gamma - 1)(a - b)$. This completes the proof of proposition. \square

Let $w = \frac{v}{u}$. It is easy to compute

$$\begin{aligned} w_t &= \frac{v_t}{u} - \frac{u_t v}{u^2}, & \nabla w &= \frac{\nabla v}{u} - \frac{\nabla u}{u^2} v, \\ \Delta w &= \frac{\Delta v}{u} - \frac{v \Delta u}{u^2} - \frac{2 \nabla u \cdot \nabla v}{u^2} + \frac{2 |\nabla u|^2}{u^3} v. \end{aligned}$$

The equation satisfied by w is

$$\begin{aligned} w_t - d \Delta w &= \mu \frac{v}{u} \left(b - \frac{v}{\gamma u} \right) - \frac{v}{u} \left(a - u - \frac{v}{m + u} \right) + \frac{2d}{u} \nabla u \cdot \nabla w \\ &= w \left(b - a + u + w \left(-\gamma^{-1} + \frac{u}{m + u} \right) \right) + \frac{2d}{u} \nabla u \cdot \nabla w. \end{aligned} \tag{2.3}$$

Lemma 2.1. *Suppose $\gamma^{-1} > a/(m + a)$ and $\varepsilon_1 > 0$ small. There exists T sufficiently large such that when $t \geq T$,*

$$u \leq \bar{u}_2(\varepsilon_1) \equiv \frac{a - m + \sqrt{(a + m)^2 - 4b\gamma \underline{u}_1}}{2} + O(\varepsilon_1), \quad \text{in } \Omega,$$

where

$$\underline{u}_1 \equiv \frac{a - m - \bar{w}_1 + \sqrt{(a - m - \bar{w}_1)^2 + 4am}}{2}, \quad \bar{w}_1 = \frac{(b - a + \bar{u}_1)(m + \bar{u}_1)}{\gamma^{-1}(m + \bar{u}_1) - \bar{u}_1},$$

$\bar{u}_1 \equiv a$.

Proof. Since $v \geq 0$, it is clear that u satisfies

$$u_t - d \Delta u \leq u(a - u) \quad \text{in } \Omega \times (0, \infty).$$

By a simple comparison argument and the well established fact that any positive solution of

$$\begin{cases} u_t - d \Delta u = u(a - u), & \text{in } \Omega \times (0, \infty) \\ \frac{\partial u}{\partial \nu} = 0, & \text{on } \partial \Omega \times (0, \infty) \end{cases}$$

converges to a uniformly as $t \rightarrow \infty$, we obtain that $\forall \varepsilon_1 > 0, \exists t_1 > 0$ such that if $t \geq t_1$,

$$u(x, t) < \bar{u}_1(\varepsilon_1) \equiv a + \frac{\varepsilon_1}{5} \quad \text{in } \Omega. \quad (2.4)$$

Then, for $t \geq t_1$,

$$w_t - d_1 \Delta w \leq w \left(b - a + \bar{u}_1(\varepsilon_1) + w \left(-\gamma^{-1} + \frac{\bar{u}_1(\varepsilon_1)}{m + \bar{u}_1(\varepsilon_1)} \right) \right) + \frac{2d}{u} \nabla u \cdot \nabla w.$$

We assume ε_1 is sufficiently small so that $\gamma^{-1}(m + \bar{u}_1(\varepsilon_1)) > \bar{u}_1(\varepsilon_1)$. Since any positive solution $W(t)$ of the ODE

$$W_t = W \left(b - a + \bar{u}_1(\varepsilon_1) + W \left(-\gamma^{-1} + \frac{\bar{u}_1(\varepsilon_1)}{m + \bar{u}_1(\varepsilon_1)} \right) \right)$$

converges to the stable equilibrium point

$$W_0 = \frac{(b - a + \bar{u}_1(\varepsilon_1))(m + \bar{u}_1(\varepsilon_1))}{\gamma^{-1}(m + \bar{u}_1(\varepsilon_1)) - \bar{u}_1(\varepsilon_1)}, \quad (2.5)$$

a simple comparison argument yields that $\exists t_2 > t_1$ such that if $t \geq t_2$,

$$w(x, t) \leq \bar{w}_1(\varepsilon_1) \equiv W_0 + \frac{\varepsilon_1}{5} \quad \text{in } \Omega. \quad (2.6)$$

Consequently, $v \leq \bar{w}_1(\varepsilon_1)u$ in Ω when $t \geq t_2$, and

$$\begin{aligned} u_t - d \Delta u &\geq u(a - u) - \frac{\bar{w}_1(\varepsilon_1)}{m + u} u^2 \\ &= \frac{u}{m + u} ((a - u)(m + u) - \bar{w}_1(\varepsilon_1)u). \end{aligned}$$

The quadratic equation

$$(a - u)(m + u) - \bar{w}_1(\varepsilon_1)u = 0$$

has only one positive root,

$$R = \frac{a - m - \bar{w}_1(\varepsilon_1) + \sqrt{(a - m - \bar{w}_1(\varepsilon_1))^2 + 4am}}{2},$$

which is a stable equilibrium point of corresponding ODE

$$u_t = \frac{u}{m + u} ((a - u)(m + u) - \bar{w}_1(\varepsilon_1)u)$$

and it attracts every positive solution. This, in turn, by comparison, implies there exists $t_3 > t_2$ such that if $t \geq t_3$,

$$u \geq \underline{u}_1(\varepsilon_1) \equiv \frac{a - m - \bar{w}_1(\varepsilon_1) + \sqrt{(a - m - \bar{w}_1(\varepsilon_1))^2 + 4am}}{2} - \frac{\varepsilon_1}{5} \quad \text{in } \Omega. \quad (2.7)$$

The above inequality, when used in the v equation, gives

$$v_t - d \Delta v \geq bv - \frac{v^2}{\gamma \underline{u}_1(\varepsilon_1)} \quad \text{in } \Omega \times [t_3, \infty).$$

Hence, there exists $t_4 > t_3$ such that if $t \geq t_4$,

$$v \geq \underline{v}_1(\varepsilon_1) = b\gamma \underline{u}_1(\varepsilon_1) - \frac{\varepsilon_1}{5} \quad \text{in } \Omega. \quad (2.8)$$

Substitute $v \geq \underline{v}_1(\varepsilon_1)$ into the u equation, we obtain

$$u_t - d \Delta u \leq au - u^2 - \frac{u \underline{v}_1(\varepsilon_1)}{m + u} \quad \text{in } \Omega \times [t_4, \infty).$$

A direct application of comparison principle then yields there exists $t_5 > t_4$ such that if $t \geq t_5$,

$$u \leq \bar{u}_2(\varepsilon_1) \equiv \frac{a - m + \sqrt{(a + m)^2 - 4v_1(\varepsilon_1)}}{2} + \frac{\varepsilon_1}{5} \quad \text{in } \Omega. \tag{2.9}$$

Simple computation using (2.4)–(2.9) shows the expression of $\bar{u}_2(\varepsilon_1)$ and that of $\underline{u}_1(\varepsilon_1)$ and $\bar{w}_1(\varepsilon_1)$ are valid. This proves the lemma. \square

By repeating the above procedure, for any positive integer n , there exists T sufficiently large such that when $t \geq T$,

$$\begin{aligned} u &\leq \bar{u}_{n+1}(\varepsilon_1) \equiv \frac{a - m + \sqrt{(a + m)^2 - 4v_n(\varepsilon_1)}}{2} + \frac{\varepsilon_1}{5}, \quad \text{and} \\ u &\geq \underline{u}_n(\varepsilon_1) \equiv \frac{a - m - \bar{w}_n + \sqrt{(a - m - \bar{w}_n)^2 + 4am}}{2} + \frac{\varepsilon_1}{5} \end{aligned}$$

uniformly in Ω , where

$$\underline{v}_n(\varepsilon_1) = b\gamma\underline{u}_n - \frac{\varepsilon_1}{5}, \quad \bar{w}_n = \frac{(b - a + \bar{u}_n(\varepsilon_1))(m + \bar{u}_n(\varepsilon_1))}{\gamma^{-1}(m + \bar{u}_n(\varepsilon_1)) - \bar{u}_n(\varepsilon_1)} + \frac{\varepsilon_1}{5}.$$

It is clear that when $\varepsilon_1 = 0$, we have

$$\bar{u}_{n+1} = \frac{a - m + \sqrt{(a + m)^2 - 4b\gamma\underline{u}_n}}{2}, \quad \underline{u}_n = \frac{a - m - \bar{w}_n + \sqrt{(a - m - \bar{w}_n)^2 + 4am}}{2}$$

and

$$\underline{v}_n = b\gamma\underline{u}_n, \quad \bar{w}_n = \frac{(b - a + \bar{u}_n)(m + \bar{u}_n)}{\gamma^{-1}(m + \bar{u}_n(\varepsilon_1)) - \bar{u}_n}, \quad \bar{v}_n = \bar{u}_n\bar{w}_n, \quad n = 1, 2, \dots$$

with $\bar{u}_1 = a > u^*$, $\bar{w}_1 > b\gamma$ and $\underline{u}_1 < u^*$. It is easy to see that $u^* < \bar{u}_2 < \bar{u}_1$, \bar{w}_n is an increasing function of \bar{u}_n as long as $(m + \bar{u}_n)\gamma^{-1} - \bar{u}_n > 0$. This, together with $\bar{u}_2 < \bar{u}_1$, implies $\underline{u}_2 > \underline{u}_1$. A simple induction then shows, under the assumption of Theorem 1, $\{\bar{u}_n\}$ is a decreasing sequence and $\{\underline{u}_n\}$ is an increasing sequence with

$$\lim_{n \rightarrow \infty} \bar{u}_n = \lim_{n \rightarrow \infty} \underline{u}_n = u^*.$$

Consequently,

$$\lim_{n \rightarrow \infty} \bar{v}_n = \lim_{n \rightarrow \infty} \underline{v}_n = v^*.$$

Now, we show $\lim_{t \rightarrow \infty} (u, v) = (u^*, v^*)$ uniformly in Ω .

Remark. The above procedure depends critically on two inequalities, (i) $\gamma^{-1} > a/(m+a)$ and (ii) $u^* > a - b$. It is shown the second condition follow from the first in Proposition 1.

Proof of Theorem 1. $\forall \varepsilon > 0$, there exists $n_0 > 1$ such that when $n \geq n_0$,

$$|\bar{u}_n - u^*| + |\underline{u}_n - u^*| < \frac{\varepsilon}{4}. \tag{2.10}$$

Choose $\varepsilon_1 > 0$ sufficiently small such that

$$|\bar{u}_{n_0}(\varepsilon_1) - \bar{u}_{n_0}| + |\underline{u}_{n_0}(\varepsilon_1) - \underline{u}_{n_0}| < \frac{\varepsilon}{4} \tag{2.11}$$

and the same to $\underline{v}_n(\varepsilon_1)$, \underline{v}_n , $\bar{v}_n(\varepsilon_1)$, \bar{v}_n and v^* . Furthermore, there exists $t_M \gg 1$ such that when $t \geq t_M$,

$$\underline{u}_{n_0}(\varepsilon_1) \leq u(x, t) \leq \bar{u}_{n_0}(\varepsilon_1) \quad \text{in } \Omega.$$

Hence, by (2.10) and (2.11), when $t \geq t_M$,

$$|u(x, t) - u^*| < \varepsilon \quad \text{in } \Omega.$$

This proves $\lim_{t \rightarrow \infty} u(x, t) = u^*$ uniformly in Ω . Similarly, $\lim_{t \rightarrow \infty} v(x, t) = v^*$ uniformly in Ω . \square

3. Generalization and future works

It is easy to see that the proof of [Theorem 2](#) follows exactly the same line of argument as in [Theorem 1](#) and we omit the details.

The method we develop in this work is new and can be applied to many interesting reaction–diffusion type models where the stability of a unique positive equilibrium solution is a key issue to be studied. For example, the famous Gierer–Meinhardt system is an interesting model worth of looking into.

It will be interesting to see how can we incorporate other interesting features such as time delay into our scheme.

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