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Blow-up of p-Laplacian evolution equations with variable source power

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Abstract We study the blow-up and/or global existence of the following p-Laplacian evolution equation with variable source power

$$u_t(x,t) = \mathrm{div}(|\nabla u|^{p-2}\nabla u) + u^{q(x)} \quad \text{in} \quad \Omega \times (0,T),$$

where Ω is either a bounded domain or the whole space $\mathbb{R}^N$, and $q(x)$ is a positive and continuous function defined in Ω with $0 < q_- = \inf q(x) \leqslant q(x) \leqslant \sup q(x) = q_+ < \infty$. It is demonstrated that the equation with variable source power has much richer dynamics with interesting phenomena which depends on the interplay of $q(x)$ and the structure of spatial domain Ω, compared with the case of constant source power. For the case that Ω is a bounded domain, the exponent $p-1$ plays a crucial role. If $q_+ > p-1$, there exist blow-up solutions, while if $q_+ < p-1$, all the solutions are global. If $q_- > p-1$, there exist global solutions, while for given $q_- < p-1 < q_+$, there exist some function $q(x)$ and Ω such that all nontrivial solutions will blow up, which is called the Fujita phenomenon. For the case $\Omega = \mathbb{R}^N$, the Fujita phenomenon occurs if $1 < q_- \leqslant q_+ \leqslant p-1+p/N$, while if $q_- > p-1+p/N$, there exist global solutions.

Keywords p-Laplacian, blow-up, variable source power

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1 Introduction

In this paper, we study non-negative solutions of the following p-Laplacian evolution equation with source of variable power:

$$u_t(x,t) = \mathrm{div}(|\nabla u|^{p-2}\nabla u) + u^{q(x)} \quad \text{in} \quad \Omega \times (0,T_{\max}), \tag{1.1}$$

$$u(x,0) = u_0(x) \geqslant 0 \quad \text{in} \quad \Omega, \tag{1.2}$$

where $p > 2$, u_0 is a continuous, bounded and non-zero function, and $q(x)$ is a positive, continuous and bounded function. $T_{\max}$ is the maximum existence time. We denote

$$q_- = \inf_{x\in\Omega} q(x) > 0, \quad q_+ = \sup_{x\in\Omega} q(x) < \infty.$$

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In our problem, Ω is either $\mathbb{R}^N$ or a bounded, smooth, connected domain. In the latter case we assume the Dirichlet boundary condition

$$u(x,t)=0 \quad \text{on} \quad \partial\Omega\times(0,T_{\max}). \tag{1.3}$$

Due to the degeneracy of this parabolic problem we can only define a weak solution of (1.1) and (1.2) (with (1.3) in the Dirichlet problem). The well-posedness and regularity for general p-Laplacian evolution problems has been investigated since the classical work [7], and we refer to [25,37–39], etc., for the further developments. For our problem, the definition of a weak solution and its existence will be discussed in Section 2. Under our assumption that u_0 is continuous, which we assume for convenience, the solution is always continuous. The uniqueness may fail, if $q(x)$ takes values less than 1 in some region of Ω, which causes the source term to be non-Lipschitz on u. Nevertheless, in this case we can prove existence of a maximal solution and a minimal solution and the comparison principle for the maximal solution and minimal solution, respectively. These will also be discussed in Section 2.

Our main interest is in global existence vs. finite time blow-up of solutions. We say a solution of (1.1) and (1.2) blows up at finite time T if

$$\lim_{t\nearrow T}\|u(\cdot,t)\|_\infty=+\infty.$$

Blow-up is an important phenomenon in parabolic problems and has attracted great interest. In the case when $p=2$ and $q(x)$ is a constant q, there are blow-up solutions if and only if $q>1$. So $q_b=1$, which is called the blow-up exponent, is a critical exponent to determine whether the problem has a blow-up solution. Moreover, if $\Omega=\mathbb{R}^N$ there is another critical exponent $q_F=1+2/N$, which is called Fujita exponent (see [13]). If $q>q_F$, then the solution will be global provided that the initial value is small, while for $1<q\leqslant q_F$ every non-trivial solution blows up at finite time. In the Dirichlet case there is no Fujita exponent, since there are always both global and blow-up solutions for $q>1$. It is worth mentioning that after the pioneering work of [13], there emerged fruitful results concerning the blow-up phenomenon of the semilinear heat equations. For example, [19, 26] discussed about the life span of the solution, [22, 23, 30] treated the case of nonlinear boundary condition, [28, 35] investigated the blow-up estimate of the solution. For other important developments on semilinear heat equations, see [4, 10, 12, 16–18, 20, 21, 36].

The case that $p>2$ and $q(x)$ is a constant is different from the case of $p=2$ in several aspects. When $\Omega=\mathbb{R}^N$, there is also the blow-up exponent $q_b=1$, but Fujita exponent becomes $q_F=p-1+p/N$ (see [14]). In [14], it was shown that every solution blows up when $1<q<q_F$. Moreover, one of the authors and his collaborator have done a series of works concerning the blow-up for the critical case when $q=q_F$, and its generalization (see [32–34]). More importantly, the blow-up exponent is $q_b=p-1$ in the bounded domain case (see [27]), which is larger than 1. Moreover, if $q=p-1$, blow-up or global existence depends on the size of domain: if the domain size is large enough, then all non-trivial solutions blow up; but if it is small, all non-trivial solutions exist globally (see [27]).

For a more comprehensive survey, we refer [6] to the role of critical exponents in blow-up theorems in diversified settings. There are also many works dealing with the blow-up for other evolution equations involving with p-Laplacian term (see [1, 3, 8, 29, 31]).

Recently, the case where $q(x)$ is not a constant attracts much attention. In [9], the blow-up and Fujita type phenomenon were discussed when $p=2$. Furthermore, Bai and Zheng [2] dealt with coupled systems in a bounded domain when $p=2$.

The quasi-linear equation we study has a different structure from the semi-linear case of $p=2$, and new ideas and methods are called upon.

Our main results are summarized in the following two theorems.

Theorem 1.1. *Let Ω be a bounded domain in $\mathbb{R}^N$. Then, the following results hold for* (1.1)–(1.3).

(i) *If $q_+>p-1$, then there are solutions that blow up in finite time.*

(ii) *If $q_+<p-1$, then every solution is global.*

(iii) *Given* (q_-, q_+) *with* $q_- < p-1 < q_+$, *there are functions* $q(x)$ *and* Ω *which contains a large ball* $B_L(x_0)$ *such that every solution blows up in finite time. If, on the other hand, the diameter of* Ω *is small enough, then there always exist global non-trivial solutions for any given* $q(x)$.

(iv) *If* $q_- > p-1$, *then there are global solutions regardless of the size of the domain* Ω.

Theorem 1.2. *Let* $\Omega = \mathbb{R}^N$. *Then, the following results are valid for* (1.1)–(1.2).

(i) *If* $q_+ \leqslant 1$, *then every solution of* (1.1)–(1.2) *is global.*

(ii) *If* $q_+ > p-1$, *then there exist solutions that blow up in finite time.*

(iii) *If* $q_- > p-1+p/N$, *then there exist global solutions.*

(iv) *If* $1 < q_- \leqslant q_+ \leqslant p-1+p/N$, *then all solutions blow up in finite time.*

The organization of this paper is as follows. In Section 2, we give a precise definition of a weak solution and discuss fundamental properties such as existence, uniqueness and comparison principle. In Section 3, we construct self-similar sub-solutions with interesting properties. In Section 4, we study the bounded domain case and prove Theorem 1.1. In Section 5, we study the whole space case and prove Theorem 1.2. In Section 6, we present some conclusion and discussion.

2 Definition of solutions, existence and uniqueness

In this section, we discuss the definition of weak solutions to our problem, the existence, uniqueness and the comparison principles. For convenience, we denote $S_T = \mathbb{R}^N \times (0,T)$ and $Q_T = \Omega \times (0,T)$ in what follows.

For the case $\Omega = \mathbb{R}^N$, the results here are parallel to those in [39], where a constant power q case is studied. First, we define a weak solution to (1.1) and (1.2) as follows.

Definition 2.1. *A measurable function* $u(x,t)$ *defined in* S_T *is a weak solution of* (1.1)–(1.2) *if for every bounded open set* D *with smooth boundary* ∂D,

$$u \in C(0,T:L^1(D)) \cap L^p_{\mathrm{loc}}(0,T:W^{1,p}(D)) \cap L^\infty_{\mathrm{loc}}(S_T),$$

and satisfies

$$\begin{aligned}\int_D u(x,t)\phi(x,t)dx &+ \int_{t_0}^t \int_D (-u\phi_t + |\nabla u|^{p-2}\nabla u \cdot \nabla\phi)dxd\tau \\ &= \int_{t_0}^t \int_D u^{q(x)}\phi dxd\tau + \int_D u(x,t_0)\phi(x,t_0)dx,\end{aligned} \tag{2.1}$$

for all $0 \leqslant t_0 < t \leqslant T$ *and all testing functions* $\phi \in C^1(\overline{D}\times[0,T])$, $\phi = 0$ *near* $\partial D \times (0,T)$. *Moreover,*

$$\lim_{t\to 0}\int_{B_R} |u(x,t) - u_0(x)|dx = 0, \quad \forall\, R > 0.$$

Weak subsolutions (supersolutions) are defined in the same way except that the "=" in (2.1) is replaced by "⩽" ("⩾") and ϕ is taken to be non-negative.

If $q(x) \equiv q$, a constant, the results for the local existence of the weak solution have been studied in [39] as a special case. In fact, Zhao [39] has considered the following problem:

$$u_t(x,t) = \mathrm{div}(|\nabla u|^{p-2}\nabla u) + \frac{u^q}{(1+|x|)^\alpha} \quad \text{in} \quad \Omega\times(0,T), \tag{2.2}$$

$$u(x,0) = u_0(x) \quad \text{in} \quad \Omega. \tag{2.3}$$

For $\alpha = 0$, a norm $|||\cdot|||_h$ can be defined as

$$|||f|||_h = \sup_{x\in\mathbb{R}^N}\left(\int_{B_1(x)} |f|^h\right)^{\frac{1}{h}}$$

and the existence theorem is as follows:

Theorem 2.2 (See [39]). *Let h be a constant satisfying $h = 1$ if $q < p - 1 + \frac{p}{N}$ or $h > \frac{N}{p}(q - p + 1)$ if $q \geqslant p - 1 + \frac{p}{N}$. For (2.2) and (2.3) with $\Omega = \mathbb{R}^N$, if $|||u_0|||_h < \infty$, then there exist $\gamma = \gamma(N, p, q, h)$ and $T_0 = T_0(\gamma, |||u_0|||_h)$ such that there exists a weak solution u in S_{T_0} satisfying*

(1) $|||u(\cdot, t)|||_h \leqslant \gamma(|||u_0|||_h)$,

(2) $|u(x, t)| \leqslant \gamma t^{-\frac{N}{\kappa_h}}(|||u_0|||_h^{\frac{ph}{\kappa_h}})$,

(3) $|Du(x, t)| \leqslant \gamma t^{-\frac{N+1}{\kappa}} \max(1, |||u_0|||_1^{1+\frac{p-1}{\kappa}})$,

(4) $\int_0^t \int_{B_1(x_0)} |Du|^\sigma dx d\tau \leqslant \gamma(\sigma) t^{1-\frac{\sigma}{p}-\frac{N(2\sigma-p)}{\kappa p}} \cdot (\sup_{0<\tau<t} \int_{B_2(x_0)} u(x, \tau) dx)^{1+\frac{2\sigma-p}{\kappa}}$, *where* $\kappa_h = N(p - 2) + hp$, $\kappa = N(p - 2) + p$, $p - 1 \leqslant \sigma < p - 1 + \frac{1}{N+1}$.

For our problem, let $q = \max(q_+, 1)$ and use the inequality $u^{q(x)} \leqslant 1 + u^q$ and the estimate for "$1 + u^q$" to replace "u^q" in the proof in [39], one can derive the existence and other estimates following exactly the same procedure. The existence result, with apparent modification from [39, Theorem 2.2], can be stated as follows.

Theorem 2.3. *Let $q = \max(q_+, 1)$ and fix a constant h satisfying $h = 1$ if $q < p - 1 + \frac{p}{N}$ or $h > \frac{N}{p}(q - p + 1)$ if $q \geqslant p - 1 + \frac{p}{N}$. For (1.1) and (1.2) with $\Omega = \mathbb{R}^N$, if $|||u_0|||_h < \infty$, then there exist $\gamma = \gamma(N, p, q, h)$ and $T_0 = T_0(\gamma, |||u_0|||_h)$ such that there exists a weak solution u in S_{T_0} satisfying*

(1) $|||u(\cdot, t)|||_h \leqslant \gamma(|||u_0|||_h + 1)$,

(2) $|u(x, t)| \leqslant \gamma t^{-\frac{N}{\kappa_h}}(|||u_0|||_h^{\frac{ph}{\kappa_h}} + 1)$,

(3) $|Du(x, t)| \leqslant \gamma t^{-\frac{N+1}{\kappa}} \max(1, |||u_0|||_1^{1+\frac{p-1}{\kappa}})$,

(4) $\int_0^t \int_{B_1(x_0)} |Du|^\sigma dx d\tau \leqslant \gamma(\sigma) t^{1-\frac{\sigma}{p}-\frac{N(2\sigma-p)}{\kappa p}} \cdot (\sup_{0<\tau<t} \int_{B_2(x_0)} u(x, \tau) dx)^{1+\frac{2\sigma-p}{\kappa}}$,

where $\kappa_h = N(p - 2) + hp, \kappa = N(p - 2) + p, p - 1 \leqslant \sigma < p - 1 + \frac{1}{N+1}$.

Since the proof is highly similar to that in [39] and is long and technical, we omit it here and suggest the interested reader verify it by himself. Here we only point out that in our theorem, since $q(x)$ is no longer a constant, we need more strict condition on u_0 ($|||u_0|||_h < \infty$ for a probably larger h) and in the conclusion we must add a constant to $|||u_0|||_h$ in the estimate above. Since our paper mainly focuses on the behavior of the solution, we do not need the precise results for the existence problem, and the condition that u_0 is bounded is enough to ensure the local existence of a solution.

The regularity of the weak solution can be derived using the results in [7]. In our case, where the initial data is continuous, the solution will be C^α with some $0 < \alpha < 1$ in S_T and continuous up to $t = 0$.

For the uniqueness and the comparison principle, we notice that if $q_- \geqslant 1$, then the reaction term $f(x, s) = s^{q(x)}$ is continuous in both variables and locally Lipschitz with respect to s. Once again following exactly the same proof as in [39, Section 8], we can prove the comparison principle and consequently the uniqueness in the class of $\mathfrak{R}$, in which every function has the following properties:

(1) $\sup_{x\in\mathbb{R}^N} \int_{B_1(x)} |u(y, t)| dy \leqslant C$,

(2) $\sup_{x\in\mathbb{R}^N} |u(x, t)| \leqslant Ct^{-\delta}$,

(3) $\sup_{x\in\mathbb{R}^N} \frac{|Du(x,t)|}{(1+|x|)^{\frac{2}{p-2}}} \leqslant Ct^{-\delta_1}$,

for $t \in (0, T)$. Here C, δ and δ_1 are positive constants, and

$$\delta < \frac{1}{\lambda - 1}, \quad \delta_1 < \frac{1}{p - 2}, \quad \lambda = \max\{q, p - 1\}.$$

Lemma 2.4. *Suppose $q_- \geqslant 1$ and w is a supersolution of (2.1) with initial value w_0. If v is a subsolution of (2.1) with initial value v_0, $w_0(x) \geqslant v_0(x)$ in $\mathbb{R}^N$ and w and v belong to the class $\mathfrak{R}$, then $w \geqslant v$ in $\mathbb{R}^N \times (0, T)$.*

We will give the detailed proof of this lemma in Appendix, and the reader can check how the methods in [39] can be applied in our case.

Remark 2.5. It can be easily seen that the weak solution we get from Theorem 2.3 is in the class $\mathfrak{R}$. Moreover, in the rest of this article, the supersolution and subsolution we construct always satisfy the condition of the comparison principle.

If $q_- < 1$, we still have the existence of weak solutions but uniqueness is not true in general. In this case we use the method in [5] to construct a maximal solution by taking the limit

$$\overline{u} = \lim_{\varepsilon \to 0} u^{(\varepsilon)},$$

where $u^{(\varepsilon)}$ is the unique solution to our problem with initial value $u^{(\varepsilon)}(x, 0) = u_0(x) + \varepsilon$, and the reaction $f(x, s) = s^{q(x)}$ replaced by

$$f_{(\varepsilon)}(x, s) = \begin{cases} s^{q(x)}, & \text{if} \quad s \geqslant \varepsilon \quad \text{or} \quad q(x) \geqslant 1, \\ \varepsilon^{q(x)-1} s, & \text{if} \quad s < \varepsilon \quad \text{and} \quad q(x) < 1. \end{cases}$$

We point out that for the problem for $u^{(\varepsilon)}$, the existence of solution is valid because $f_{(\varepsilon)}(x, s) \leqslant s^{q(x)}$. Also, $u^{(\varepsilon)}$ satisfies the properties listed in Theorem 2.2. Moreover, since $f_{(\varepsilon)}(x, s)$ is locally Lipschitz, the comparison principle can be applied to the solutions, subsolutions and supersolutions in the class $\mathfrak{R}$. Thus we get a non-increasing sequence of positive functions. The existence time is then uniformly bounded from below.

A minimal solution is obtained by taking limits for similar problems that approximate (1.1) from below. To be precise, let

$$\underline{u} = \lim_{\varepsilon \to 0} u_{(\varepsilon)},$$

where $u_{(\varepsilon)}$ is the unique solution to the problem (1.1) with f replaced by $f_{(\varepsilon)}$ and with the same initial data. It is not difficult to verify that the maximal solution and the minimal solution are both weak solutions, furthermore every solution u of the problem (1.1) satisfies

$$0 \leqslant \underline{u} \leqslant u \leqslant \overline{u}.$$

It is easy to prove the following results of the comparison principle related to the maximal solution and minimal solution.

(1) If z is a supersolution of (1.1) with initial value $z_0 \geqslant u_0$, then $z \geqslant \underline{u}$.

(2) If z is a supersolution of (1.1) with initial value $z_0 \geqslant u_0$ and satisfies $z \geqslant \mu > 0$, then $z \geqslant \overline{u}$.

(3) If z is a subsolution of (1.1) with initial value $z_0 \leqslant u_0$, then $z \leqslant \overline{u}$.

(4) If z is a subsolution of (1.1) with initial value $z_0 \leqslant u_0$ and satisfies $z \geqslant \mu > 0$, then $z \leqslant \underline{u}$.

Next, we turn to the discussion for the case that Ω is a bounded domain. In this case the results can be directly derived from those in [38].

Definition 2.6. A measurable function $u(x, t)$ defined in Q_T is a weak solution of (1.1)–(1.3) if

$$u \in L^\infty(Q_T) \cap L^p(0, T : W_0^{1,p}(\Omega))$$

satisfies

$$u_t \in L^2(Q_T),$$

and

$$\int_\Omega u(x, t)\phi(x, t)dx + \int_0^t \int_\Omega (-u\phi_t + |\nabla u|^{p-2}\nabla u \cdot \nabla \phi)dxd\tau$$
$$= \int_0^t \int_\Omega u^{q(x)}\phi dxd\tau + \int_\Omega u_0(x)\phi(x, 0)dx, \tag{2.4}$$

for any $t \in (0, T]$ and each testing function $\phi \in C^1(\overline{\Omega} \times [0, T])$ with $\phi = 0$ near $\partial\Omega \times (0, T)$.

The existence theorem can be stated as follows (see [38]).

Theorem 2.7. *If* $u_0 \in L^\infty(\Omega) \cap H_0^p(\Omega)$, *then there exists* T *such that* (1.1)–(1.3) *has a solution* $u \in Q_T$.

If $q_- \geqslant 1$, the comparison principle is also valid. Weak subsolutions (supersolutions) are defined by replacing the "=" in (2.4) by "$\leqslant$" ("$\geqslant$") and ϕ is taken to be non-negative.

Lemma 2.8. *Suppose $q_- \geqslant 1$ and w is a supersolution of* (2.4) *with initial value w_0. If v is a subsolution of* (2.4) *with initial value v_0, $w_0 \geqslant v_0 \geqslant 0$ in Ω and $w \geqslant v \geqslant 0$ on $\partial\Omega \times (0,T)$, then $w \geqslant v \geqslant 0$ in $\Omega \times (0,T)$.*

Remark 2.9. The proof of the comparison principle above is contained in the proof of [38, Theorem 2.1], where the uniqueness of solution is proved. In fact, what we need to prove here is $(v-w)_+ = 0$ in Q_T instead of $v - w = 0$.

If $q_- \leqslant 1$, we can construct the maximal and minimal solutions as in the case $\Omega = \mathbb{R}^N$. Here

$$\overline{u} = \lim_{\varepsilon \to 0} u^{(\varepsilon)},$$

where $u^{(\varepsilon)}$ is the solution of (1.1) in Ω with the reaction term replaced by $f_{(\varepsilon)}$, initial value $u^{(\varepsilon)}(x,0) = u_0(x) + \varepsilon$ and boundary condition $u^{(\varepsilon)}|_{\partial\Omega} = \varepsilon$, i.e.,

$$\underline{u} = \lim_{\varepsilon \to 0} u_{(\varepsilon)},$$

where $u_{(\varepsilon)}$ is the solution of (1.1) in Ω with the reaction term replaced by $f_{(\varepsilon)}$, initial value $u^{(\varepsilon)}(x,0) = u_0(x)$ and boundary condition $u^{(\varepsilon)}|_{\partial\Omega} = 0$.

Also,

$$0 \leqslant \underline{u} \leqslant u \leqslant \overline{u}.$$

For the bounded domain case, the comparison principles related to the maximal solution and minimal solution are similar to the above (1)–(4) for the whole space case. The only thing we should notice is that for supersolution z, it should also satisfy $z \geqslant u$ on $\partial\Omega$, and for subsolution z, $z \leqslant u$ on $\partial\Omega$.

3 Self-similar subsolutions

In this section, we introduce some important self-similar subsolutions to (1.1), which are very helpful to derive a lower bound of our solutions.

First, let us recall the Barenblatt solution, which is given by

$$u_S^a(x - x_0, t - t_0) = [\tau + (t - t_0)]^{-\frac{N}{(p-2)N+p}} V_S^a\left(\frac{|x - x_0|}{[\tau + (t - t_0)]^{\frac{1}{(p-2)N+p}}}\right),$$

where

$$V_S^a(r) = A[a - r^{\frac{p}{p-1}}]_+^{\frac{p-1}{p-2}},$$
$$A = \left(\frac{p-2}{p}\right)^{\frac{p-1}{p-2}} \left(\frac{1}{(p-2)N+p}\right)^{\frac{1}{p-2}}$$

and $\tau > 0$ and $a > 0$ are arbitrary constants.

We know that u_S^a is a weak solution of the equation

$$u_t = \operatorname{div}(|\nabla u|^{p-2}\nabla u), \tag{3.1}$$

so it is a subsolution of (1.1). We notice that if we fix $\tau = 1$, then $u_S^a \leqslant A a^{\frac{p-1}{p-2}}$ and the support of $u_S^a(x - x_0, t_0)$ is $B_R(x_0)$, where $R = a^{\frac{p-1}{p}}(1 + t_0)^{\frac{1}{(p-2)N+p}}$. Letting $a \to 0$, the value and the diameter of the support of $u_S^a(x - x_0, t_0)$ tend to zero. Therefore, as long as our initial data $u_0(x) \not\equiv 0$, it is larger than some $u_S^a(x - x_0, t_0)$. A comparison argument implies that the solution u of (1.1), with initial data u_0, is larger than $u_S^a(x - x_0, t - t_0)$, as long as it is defined. (Notice that although we may not be able to use directly the comparison principle to (1.1), it is easy to see u is a supersolution to (3.1), where the comparison principle is true.) Thus we obtain the following lemma.

Lemma 3.1. *Assume u is a global solution to* (1.1)–(1.2). *Then for any open subset Ω_1 of Ω, with $\overline{\Omega_1}$ compact, there is a finite time $t_0 > 0$ such that $u(x,t) > 0$ in Ω_1 when $t \geqslant t_0$.*

Proof. The case that $\Omega = \mathbb{R}^N$ does not need to be proved because the support of u_S^a is spreading to $\mathbb{R}^N$ as $t \to \infty$ and $u \geqslant u_S^a$ for some $a > 0$.

In the bounded domain case, without loss of generality we assume $u_0(x) > 0$ in some ball $B(x_0, \delta_1)$. Let $\overline{x} \in \Omega$ be another point. First, we show that there is a finite time $\overline{t}$ and a neighborhood $V_{\overline{x}}$ of $\overline{x}$ such that $u(x, \overline{t}) > 0$ in $V_{\overline{x}}$. Since Ω is connected, there exists a continuous curve Γ connecting x_0 and $\overline{x}$. Denote $2\delta_2 = \mathrm{dist}(\Gamma, \partial\Omega)$ and $\delta = \min(\delta_1, \delta_2)$. Let $x_1 = \Gamma \cap \partial B(x_0, \delta/2), \ldots, x_k = \Gamma \cap \partial B(x_{k-1}, \delta/2), \ldots$ such that $x_k \neq x_{k-2}$. It is clear that $\overline{x} \in B(x_n, \delta/2)$ for some n. Since $\overline{B(x_1, \delta/4)} \subset B(x_0, \delta)$, we have $u_0(x) > 0$ in $\overline{B(x_1, \delta/4)}$. Choose a small such that supp $u_S^a(x - x_0, 0) \subset B(x_1, \delta/4)$, and $\|u_S^a(x - x_0, 0)\|_\infty \leqslant \min_{x \in B(x_1, \delta/4)} u_0(x)$, then u is a weak supersolution to (3.1) in $B(x_1, 2\delta)$ with zero boundary condition. The comparison principle implies that there exists $\tau_1 > 0$ such that $u(x, \tau_1) > 0$ in $B(x_1, \delta)$. Thus $u(x, \tau_1) > 0$ in $B(x_2, \delta/2)$. Repeating the above procedure, by finite steps, there exists a finite time $\overline{t}$ such that $u(x, \overline{t}) > 0$ in $B(x_n, \delta/2)$.

We note that if $u(x, t_0) > 0$, then for all $t > t_0$, $u(x, t) > 0$. This follows from the fact that we can always compare $u(\cdot, t_0)$ with some u_S^a, and if the value of u_S^a is smaller, the time interval in which the comparison principle holds is larger.

Since $\overline{\Omega_1}$ is compact, the conclusion follows from a finite covering argument. □

Next, we use V_S^a to construct a blow-up subsolution with $a > 0$ fixed and we write it as V_S.

Lemma 3.2. *Suppose $q_- > 1$, let $\alpha_1 = \frac{1}{q_+ - 1}$, $\beta_1 = \frac{q_+ - p + 1}{p(q_+ - 1)}$, $\alpha_2 = \frac{1}{q_- - 1}$ and $\beta_2 = \frac{q_- - p + 1}{p(q_- - 1)}$. Then there exist constants C_1 and C_2, depending only on p, q_+, q_- and a, such that $u_1 = (T - t)^{-\alpha_1} C_1 V_S(\frac{C_2|x|}{(T-t)^{\beta_1}})$ is a subsolution to* (1.1) *when $T - t \geqslant 1$, while $u_2 = (T - t)^{-\alpha_2} C_1 V_S(\frac{C_2|x|}{(T-t)^{\beta_2}})$ is a subsolution to* (1.1) *when $T - t \leqslant 1$.*

Proof. Let q be a constant with $q_- \leqslant q \leqslant q_+$. Denote

$$\alpha = \frac{1}{q - 1}, \quad \beta = \frac{q - p + 1}{p(q - 1)}.$$

Define

$$w(r) = C_1 V_S(C_2 r)$$

and

$$\widetilde{u}(x, t) = (T - t)^{-\alpha} w\left(\frac{|x|}{(T - t)^\beta}\right), \quad r = \frac{|x|}{(T - t)^\beta}.$$

We compute directly that

$$\begin{aligned} \widetilde{u}_t - \mathrm{div}(|\nabla\widetilde{u}|^{p-2}\nabla\widetilde{u}) - (\widetilde{u})^{q(x)} = (T - t)^{-\alpha-1}(&\alpha w + \beta w' r - (|w'|^{p-2}w')' \\ &- (N - 1)|w'|^{p-2}w'/r - (T - t)^{(\alpha+1)-\alpha q(x)} w^{q(x)}). \end{aligned} \tag{3.2}$$

We need to have the right-hand side of (3.2) $\leqslant 0$ when $T - t \geqslant 1, \alpha = \alpha_1$ and when $T - t \leqslant 1, \alpha = \alpha_2$. In any case, $(T - t)^{\alpha+1-\alpha q(x)} \geqslant 1$, so it is sufficient to show

$$\alpha w + \beta w' r - (|w'|^{p-2}w')' - (N - 1)|w'|^{p-2}w'/r - w^{q(x)} \leqslant 0. \tag{3.3}$$

Taking $w = C_1 V_S, w' = C_1 C_2 V_S'$ into (3.3) and noticing that V_S satisfies

$$(|V_S'|^{p-2}V_S')' + (N - 1)|V_S'|^{p-2}V_S'/r + N\kappa V_S + \kappa V_S' r = 0,$$

with $\kappa = 1/((p - 2)N + p)$, we have that (3.3) is equivalent to

$$C_1^{q(x)-1} V_S^{q(x)} - (N\kappa C_1^{p-2} C_2^p + \alpha) V_S - (\kappa C_1^{p-2} C_2^p + \beta) V_S' r \geqslant 0. \tag{3.4}$$

Denote $s = r^{\frac{p}{p-1}}$. It is known that

$$V_S = A(a - s)_+^{\frac{p-1}{p-2}}, \quad V_S' r = -A\frac{p}{p - 2}(a - s)_+^{\frac{1}{p-2}} s.$$

Therefore, it is sufficient that

$$C_1^{q(x)-1}A^{q(x)}(a-s)^{\frac{(p-1)q(x)-1}{p-2}} + (\kappa C_1^{p-2}C_2^p+\beta)A\frac{p}{p-2}s \\ -(N\kappa C_1^{p-2}C_2^p+\alpha)A(a-s) \geqslant 0, \quad s\in(0,a). \tag{3.5}$$

It is clear that (3.5) is true when

$$s \geqslant s_1 := \frac{a(N\kappa C_1^{p-2}C_2^p+\alpha)}{(\kappa C_1^{p-2}C_2^p+\beta)\frac{p}{p-2}+(N\kappa C_1^{p-2}C_2^p+\alpha)}$$

or

$$s \leqslant s_2 := a - \left(\frac{N\kappa C_1^{p-2}C_2^p+\alpha}{C_1^{q-1}A^{q-1}}\right)^{\frac{p-2}{(p-1)(q-1)}}.$$

So it remains to show that there exist constants C_1 and C_2 such that $s_1 \leqslant s_2$. By elementary calculation, we can see that since $1 < q_- \leqslant q \leqslant q_+$, if we let $C_1 \to \infty, C_1^{p-2}C_2^p \to \infty$ and $C_1^{p-q_--1}C_2^p \to 0$, $s_1 \leqslant s_2$ will be satisfied. This will do if we take $C_2 = C_1^{-\gamma}$, $\frac{p-q_--1}{p} < \gamma < \frac{p-1}{p}$.

The lemma is then proved. □

The next lemma is a direct corollary of Lemma 3.2.

Lemma 3.3. *Denote D_1 to be the maximum of $V_S(r;a)$ and D_2 to be the radius of the support of $V_S(r;a)$. Consider* (1.1) *and* (1.2) *in $\mathbb{R}^{\mathbb{N}}$ with $q_- > 1$. If there exist constants $\delta > 0, \varepsilon > 0$ such that $u_0 \geqslant \varepsilon$ in the ball $\{x \mid |x| \leqslant \delta\}$, and*

$$\delta\varepsilon^{\frac{q_+-p+1}{p}} \geqslant \frac{D_2}{C_2}(C_1D_1)^{\frac{q_+-p+1}{p}}, \quad \frac{C_1D_1}{\varepsilon} \geqslant 1,$$

then u must blow up. Here C_1 and C_2 are the constants which appeared in Lemma 3.2.

Proof. Let $\alpha_1 = \frac{1}{q_+-1}$, $\beta_1 = \frac{q_+-p+1}{p(q_+-1)}$, $\alpha_2 = \frac{1}{q_--1}$, $\beta_2 = \frac{q_--p+1}{p(q_--1)}$ as in Lemma 3.2 and

$$T = \left(\frac{C_1D_1}{\varepsilon}\right)^{\frac{1}{\alpha_1}} \geqslant 1.$$

We compare the solution $u(x,t)$ with

$$u_1 = (T-t)^{-\alpha_1}C_1V_S\left(\frac{C_2|x|}{(T-t)^{\beta_1}}\right)$$

in $[0, T-1]$. When $t = 0$, the maximum of u_1 is

$$T^{-\alpha_1}C_1D_1 = \varepsilon,$$

and the radius of the support of u_1 is

$$\frac{T^{\beta_1}D_2}{C_2} = \left(\frac{C_1D_1}{\varepsilon}\right)^{\frac{\beta_1}{\alpha_1}}\frac{D_2}{C_2} = \left(\frac{C_1D_1}{\varepsilon}\right)^{\frac{q_+-p+1}{p}}\frac{D_2}{C_2} \leqslant \delta,$$

which implies $u_0 \geqslant u_1(x,0)$. By Lemma 3.2 and the comparison principle, $u(x,t) \geqslant u_1(x,t)$ when $t \in [0, T-1]$.

Next, we compare the solution $u(x,t)$ with

$$u_2 = (T-t)^{-\alpha_2}C_1V_S\left(\frac{C_2|x|}{(T-t)^{\beta_2}}\right)$$

in $[T-1, T)$. When $t = T-1$,

$$u_2(x, T-1) = C_1V_S(C_2|x|) = u_1(x, T-1) \leqslant u(x, T-1).$$

By Lemma 3.2 and the comparison principle, $u(x,t) \geqslant u_2(x,t)$ when $t \in [T-1, T)$, which implies u blows up before T. □

4 The bounded domain case

In this section, we give the proof of Theorem 1.1, which deals with the case that Ω is bounded. First we prove some lemmas.

Lemma 4.1. *Let $D \subset \Omega$ be an open domain. Suppose one of the following conditions holds*: (1) $q(x) \leqslant \sigma < 1$ *for* $x \in D$; (2) $1 < \gamma_2 \leqslant q(x) \leqslant \gamma_2 < p-1$ *for* $x \in D$, *and u is a global solution, then for any compact set $\widetilde{D} \subset D$, there exist constants t_0, $\delta > 0$ such that $u(x,t) \geqslant \delta$ in $\widetilde{D}$ for $t \geqslant t_0$.*

Proof. We can find a constant r such that for all $x_0 \in \widetilde{D}, B_r(x_0) \subset D$. If we can prove that for each $B_r(x_0)$, there exist constants t_0, $\delta > 0$ such that $u(x,t) \geqslant \delta$ in $B_{\frac{r}{2}}(x_0)$ for $t \geqslant t_0$, then by finite covering, the conclusion of the lemma holds. Next, we prove it for the following two cases:

Case 1. $q(x) \leqslant \sigma < 1$ for $x \in D$.

By Lemma 3.1, there exists a constant t_0 independent of x_0, such that

$$\mu = \min\{u(x,t) : x \in B_r(x_0), t_0 \leqslant t \leqslant t_0 + 1\} > 0.$$

For $\varepsilon < \mu, \eta > 0, \alpha = \frac{1}{1-\sigma}$, we consider the following function

$$\widetilde{w} = \varepsilon + \eta t^{\alpha} \varphi_1\left(\frac{x - x_0}{r}\right),$$

where φ_1 is the first eigenfunction of p-Laplacian in B_1 with $\varphi_1(0) = 1$.

We compare $\widetilde{w}(x,t)$ and $u(x,t+t_0)$ in $B_r(x_0) \times (0,1]$. Since both functions are strictly away from zero, by the discussion in Section 2, the comparison principle holds.

On the parabolic boundary of $B_r(x_0) \times (0,1]$, we have

$$\widetilde{w} = \varepsilon < \mu \leqslant u,$$

thus it remains to verify that $\widetilde{w}$ is a subsolution, i.e.,

$$\begin{aligned}&\widetilde{w}_t - \operatorname{div}(|\nabla \widetilde{w}|^{p-2} \nabla \widetilde{w}) - \widetilde{w}^{q(x)} \\ &\quad = \eta \alpha t^{\alpha-1} \varphi_1 + (\eta t^{\alpha})^{p-1} \frac{\lambda_1}{r^p} \varphi_1^{p-1} - (\eta t^{\alpha} \varphi_1 + \varepsilon)^{q(x)}.\end{aligned}$$

If we take $\eta = \min(1, (\alpha + \frac{\lambda_1}{r^p})^{-\alpha})$ then the above expression $\leqslant 0$ and $\widetilde{w}$ is a subsolution.

Then we have $u(x,t+t_0) \geqslant \widetilde{w}(x,t)$, which implies

$$u(x, t+t_0) \geqslant c t^{\alpha} \quad \text{for} \quad x \in B_{\frac{r}{2}}(x_0), \quad 0 \leqslant t \leqslant 1, \quad c = \eta \min_{B_{\frac{1}{2}}} \varphi_1$$

and

$$u(x, t+t_0) \geqslant \delta \quad \text{for} \quad x \in B_{\frac{r}{2}}(x_0), \quad t_0 + \frac{1}{2} \leqslant t \leqslant t_0 + 1.$$

Since the above comparison still holds if we replace t_0 by any $t_0' > t_0$, we have $u(x,t) \geqslant \delta$ for $x \in B_{\frac{r}{2}}(x_0)$, $t \geqslant t_0 + \frac{1}{2}$, and this case is proved.

Case 2. $1 < \gamma_2 \leqslant q(x) \leqslant \gamma_1 < p - 1$ for $x \in D$.

Again, by Lemma 3.1, there exists time t_0 independent of x_0, $u(x,t_0) \geqslant \varepsilon_0$ in $B_r(x_0)$.

By Lemma 3.2,

$$u_1 = (T-t)^{-\frac{1}{\gamma_1 - 1}} C_1 V_S\left(\frac{C_2|x - x_0|}{(T-t)^{\frac{\gamma_1 - p + 1}{p(\gamma_1 - 1)}}}\right)$$

is a subsolution to (1.1) when $T - t \geqslant 1$, while

$$u_2 = (T-t)^{-\frac{1}{\gamma_2 - 1}} C_1 V_S\left(\frac{C_2|x - x_0|}{(T-t)^{\frac{\gamma_2 - p + 1}{p(\gamma_2 - 1)}}}\right)$$

is a subsolution when $T-t\leqslant 1$.

Take T sufficiently large such that

$$C_1T^{-\frac{1}{\gamma_1-1}}a^{\frac{p-1}{p-2}}\leqslant\varepsilon_0,\quad \frac{1}{C_2}a^{\frac{p-1}{p}}T^{\frac{\gamma_1-p+1}{p(\gamma_1-1)}}\leqslant r,$$

where a is the constant in V_S, and C_1 and C_2 are the constants in Lemma 3.2. Thus $u_1(x,0)\leqslant u(x,t_0)$ in $B_r(x_0)$.

Since $1<\gamma_2\leqslant q(x)\leqslant\gamma_1<p-1$, the supports of u_1 and u_2 are both spreading as t increases. So the comparison principle will hold before the support of u_1 or u_2 expands beyond $\partial B_r(x_0)$. To be precise, if $\frac{1}{C_2}a^{\frac{p-1}{p}}\geqslant r$, the comparison will stop when $t=t_1$, where t_1 satisfies

$$T-t_1=\left(\frac{C_2r}{a^{\frac{p-1}{p}}}\right)^{\frac{p(\gamma_1-1)}{\gamma_1-p+1}}\geqslant 1.$$

In this case we compare $u(x,t+t_0)$ with u_1 in $B_r(x_0)\times(0,t_1]$, and when $t=t_1$,

$$u(x,t_1+t_0)\geqslant C_1\left(\frac{C_2r}{a^{\frac{p-1}{p}}}\right)^{\frac{p}{p-1-\gamma_1}}V_S\left(a^{\frac{p-1}{p}}\frac{|x-x_0|}{r}\right).$$

Then we have $u(x,t_1+t_0)\geqslant\delta_1$ in $B_{\frac{r}{2}}(x_0)$.

If $\frac{1}{C_2}a^{\frac{p-1}{p}}<r$, the comparison will hold until $t=t_2$, where t_2 satisfies

$$T-t_2=\left(\frac{C_2r}{a^{\frac{p-1}{p}}}\right)^{\frac{p(\gamma_2-1)}{\gamma_2-p+1}}<1.$$

In this case we first compare $u(x,t+t_0)$ with u_1 in $B_r(x_0)\times(0,T-1]$. Notice that when $t=T-1$, $u_1(x,t)=u_2(x,t)$, so we can continue to compare $u(x,t+t_0)$ with $u_2(x,t)$ until $t=t_2$, and

$$u(x,t_2+t_0)\geqslant C_1\left(\frac{C_2r}{a^{\frac{p-1}{p}}}\right)^{\frac{p}{p-1-\gamma_2}}V_S\left(a^{\frac{p-1}{p}}\frac{|x-x_0|}{r}\right).$$

Then we have $u(x,t_2+t_0)\geqslant\delta_2$ in $B_{\frac{r}{2}}(x_0)$.

Therefore, we can always have that for some time t_3 and $\delta>0$, $u(x,t_3+t_0)\geqslant\delta$ in $B_{\frac{r}{2}}(x_0)$.

Since u_1 is increasing as t increases, we can repeat the above comparison for all $t\geqslant t_0$, thus $u(x,t)\geqslant\delta$ for $x\in B_{\frac{r}{2}}(x_0)$, $t\geqslant t_0+t_3$.

The lemma is then proved. □

Corollary 4.2. *If there exists a domain $D\subset\Omega$ such that $q(x)$ satisfies in D the conditions in Lemma 4.1, then for any compact set $\widetilde{\Omega}\subset\Omega$, there exist t_0 and $\delta>0$ such that $u(x,t)\geqslant\delta$ in $\widetilde{\Omega}$ for $t\geqslant t_0$.*

Proof. By Lemma 4.1, we have that for a compact domain $\widetilde{D}\subset D$, there exist constants t_1, $\delta_1>0$ such that $u(x,t)\geqslant\delta$ in $\widetilde{D}$ for $t\geqslant t_1$.

By Lemma 3.1, if we compare $u(x,t+t_1)$ with some u_S^a which has the support in $\widetilde{D}$ and not larger than δ_1, we obtain that there exist some time t_2 and $\delta>0$ such that $u(x,t_1+t_2)\geqslant\delta$ in $\widetilde{\Omega}$.

Since the comparison argument in Lemma 4.1 applies to any $t_2'\geqslant t_2$, the corollary is proved by taking $t_0=t_1+t_2$. □

Lemma 4.3. *If $u(x,t)\geqslant\delta>0$ in a ball $B_R(0)$ with radius $R\geqslant 1$ when $t\geqslant t_0$ and $q(x)\leqslant\sigma<p-1$ in $B_R(0)$, then there exist constants $c>0$ and t_1 such that $u(x,t)\geqslant cR^{\frac{p}{p-1-\sigma}}$ in $B_{\frac{R}{2}}(0)$ when $t\geqslant t_1$.*

Proof. Consider the following problem:

$$\begin{cases} v_t=\operatorname{div}(|\nabla v|^{p-2}\nabla v)+v^{q(x)} & \text{in}\quad B_R(0)\times(t_0,\infty),\\ v=\delta & \text{on}\quad \partial B_R(0)\times(t_0,\infty),\\ v=\delta & \text{for}\quad B_R(0)\times\{t_0\}. \end{cases}\tag{4.1}$$

It is clear that $v \geqslant \delta > 0$, and $u \geqslant v$ on $B_R(0) \times \{t_0\}$ and on $\partial B_R(0) \times (t_0, \infty)$. We can apply the comparison principle to obtain $u \geqslant v$ for $t \geqslant t_0$.

On the other hand, with A large, we have that

$$\overline{w}(x,t) = A - A^{\alpha}|x|^{\frac{p}{p-1}}$$

is a supersulotion of (4.1), where $\frac{\sigma}{p-1} < \alpha < 1$. This implies v is uniformly bounded. Moreover, there exists a Lyapunov functional given by

$$F(v) = \frac{1}{p}\int_{B_R(0)} |\nabla v|^p - \int_{B_R(0)} \frac{|v|^{q(x)+1}}{q(x)+1},$$

which satisfies

$$\frac{d}{dt}F(v)(t) = -\int_{B_R(0)} |v_t|^2 dx \leqslant 0.$$

By the weak compactness of the unit ball in $W^{1,p}(B_R(0))$, we conclude that, for every sequence $t_j \to \infty$, we can extract a subsequence, still denoted by t_j, such that

$$v(x,t_j) \rightharpoonup V(x) \quad \text{in} \quad W^{1,p}(B_R(0))$$

and then $v(x,t_j) \to V(x)$ strongly in $L^p(B_R(0))$ and $V(x) \geqslant \delta$.

Since $F(v)$ is bounded from below, it is obvious that $v_t \to 0$ in $L^2(B_R(0))$ as $t \to \infty$, which means that $V(x)$ is a stationary solution of (4.1). Note that here $V(x)$ may depend on the sequence $\{t_j\}$.

We now prove that $V(x)$ is unique and therefore is independent of the sequence $\{t_j\}$. Consider the following convex set in $W^{1,p}(B_R(0))$:

$$\Gamma = \{v \in W^{1,p}(B_R(0)) \mid v \text{ is bounded and } v > 0\}.$$

It is easy to verify that $F(v)$ has certain convex property, i.e., for all $0 < \lambda < 1, u, v \in \Gamma$, u is not a constant multiple of v,

$$F((\lambda u^p + (1-\lambda)v^p)^{\frac{1}{p}}) < \lambda F(u) + (1-\lambda)F(v).$$

Therefore a stationary solution $V(x) \in \Gamma$ must be a minimizer of $F(v)$ in Γ with boundary condition $V(x) = \delta$, which is unique. Thus $V(x)$ is the limit of every convergent sub-sequence $\{v(t_j)\}_1^{\infty}$, which means that

$$\lim_{t\to\infty} v(x,t) = V(x) \quad \text{a.e.}$$

By classical regularity theory [7], $v(x,t)$ is Hölder continuous and the C^{α} norm of x variable is uniformly bounded for $t \geqslant t_0$. Then by Arzela-Ascoli theorem, $v(x,t)$ converges uniformly to $V(x)$ and for a sufficiently large t_0, $v(x,t) \geqslant \frac{1}{2}V(x), t \geqslant t_0$.

Now we estimate the size of $V(x)$.

Set $V(x) = R^{\frac{p}{p-1-\sigma}}\widetilde{V}(\frac{x}{R})$, if $R \geqslant 1$, $\widetilde{V}$ satisfies

$$\begin{aligned} &-\operatorname{div}(|\nabla\widetilde{V}|^{p-2}\nabla\widetilde{V}) \geqslant \widetilde{V}^{q(Rx)} \quad \text{in} \quad B_1,\\ &\widetilde{V} = \delta R^{-\frac{p}{p-1-\sigma}} \quad \text{on} \quad \partial B_1. \end{aligned}$$

We consider the following functional on the Sobolev space $W_0^{1,p}(B_1(0))$:

$$E(v) = \frac{1}{p}\int_{B_1(0)} |\nabla v|^p - \int_{B_1(0)} \frac{|v|^{\sigma+1}}{\sigma+1}.$$

Similar to the above arguement, $E(v)$ has a unique positive minimizer ϕ, which satisfies

$$\begin{cases} -\operatorname{div}(|\nabla\phi|^{p-2}\nabla\phi) = \phi^{\sigma} & \text{in} \quad B_1,\\ \phi = 0 & \text{on} \quad \partial B_1. \end{cases}$$

By the classical regularity theory $\phi \in C^{\alpha}(\overline{B_1(0)})$. Set $z = \eta\phi$ with $\eta = \min(1, \|\phi\|_{\infty}^{-1})$. Then z satisfies $z \leqslant 1$ and $-\mathrm{div}(|\nabla z|^{p-2}\nabla z) \leqslant z^{\sigma}$ in $B_1(0)$. Next, we show that $\widetilde{V} \geqslant z$ in $B_1(0)$.

Suppose it is not true. We set

$$\tau_0 = \sup\{\tau \geqslant 1 \mid \tau\widetilde{V} - z \text{ takes some negative values in } B_1(0)\}.$$

Then $\tau_0\widetilde{V} \geqslant z$ in $B_1(0)$ and attains a null minimum at some point in $B_1(0)$ (Note that $\tau_0\widetilde{V} > z$ on $\partial B_1(0)$). On the other hand,

$$\begin{aligned} -\mathrm{div}(|\nabla(\tau_0\widetilde{V})|^{p-2}\nabla(\tau_0\widetilde{V})) &\geqslant \tau_0^{p-1}\widetilde{V}^{q(Rx)} \\ &\geqslant (\tau_0\widetilde{V})^{q(Rx)} \geqslant z^{q(Rx)} \geqslant z^{\sigma} \geqslant -\mathrm{div}(|\nabla z|^{p-2}\nabla z) \end{aligned}$$

with the boundary condition $\tau_0\widetilde{V} \geqslant \tau_0\delta R^{-\frac{p}{p-1-\sigma}} > 0 = z$ on $\partial B_1(0)$. We conclude that $\tau_0\widetilde{V} > z$ on $\overline{B_1(0)}$, which leads to a contradiction. Thus, $\widetilde{V} \geqslant z$ in $B_1(0)$.

Next, when $x \in B_{\frac{R}{2}}(0), t > t_0$,

$$u(x,t) \geqslant \frac{1}{2}V(x) = \frac{1}{2}R^{\frac{p}{p-1-\sigma}}\widetilde{V}\left(\frac{x}{R}\right) \geqslant \frac{1}{2}R^{\frac{p}{p-1-\sigma}} \min_{x\in B_{\frac{1}{2}}(0)} z(x) = cR^{\frac{p}{p-1-\sigma}}. \qquad \square$$

Proof of Theorem 1.1. (i) If $q_+ > p-1$, we show that u is larger than a blow-up subsolution if the initial data is large enough.

Without loss of generality, we suppose $B_{r_0}(0) \subset \Omega$ and $q(x) \geqslant \gamma > p-1$ in $B_{r_0}(0)$. Since $\gamma > p-1$, $(T-t)^{\frac{\gamma-p+1}{p(\gamma-1)}} \to 0$ when $t \to T$. Thus we can take t_0 sufficiently close to T such that the support of $u_\gamma = (T-t)^{-\frac{1}{\gamma-1}}C_1V_S(\frac{C_2|x|}{(T-t)^{\frac{\gamma-p+1}{p(\gamma-1)}}})$ is contained in $B_{r_0}(0)$ when $t_0 \leqslant t < T$.

By Lemma 3.2, u_γ is a subsolution to (1.1). Therefore as long as $u_0(x) \geqslant u_\gamma(x,t_0)$, by the comparison principle we have $u(x,t) \geqslant u_\gamma(x, t+t_0)$. Then u must blow up in finite time.

(ii) If $q_+ < p-1$, for any initial data we construct a global supersolution larger than u.

Let $\Omega' \supset \overline{\Omega}$ be a smooth bounded connected domain and ψ be a non-negative first eigenfunction of p-Laplacian in Ω', i.e.,

$$\begin{aligned} &-\mathrm{div}(|\nabla\psi|^{p-2}\nabla\psi) = \lambda_1\psi^{p-1}, \\ &\psi|_{\partial\Omega'} = 0, \end{aligned} \tag{4.2}$$

where λ_1 is the first eigenvalue of p-Laplacian in Ω'.

We know that $\psi > 0$ in Ω', thus $\inf_\Omega \psi \geqslant \delta > 0$. We take $\varphi = K\psi$ with K sufficiently large such that $K\delta > 1, \varphi \geqslant u_0$ and $-\lambda_1\varphi^{p-1} + \varphi^{q_+} \leqslant 0$ in Ω, then

$$\begin{aligned} \mathrm{div}(|\nabla\varphi|^{p-2}\nabla\varphi) + \varphi^{q(x)} &= -\lambda_1\varphi^{p-1} + \varphi^{q(x)} \\ &\leqslant -\lambda_1\varphi^{p-1} + \varphi^{q_+} \leqslant 0. \end{aligned}$$

Thus, by the comparison principle, the maximum solution $\overline{u}(x,t) \leqslant \varphi(x)$ and $\overline{u}$ is global. Then every solution u is global.

(iii) Let $q_- < p-1 < q_+$.

We will construct functions $q(x)$ which take q_+ and q_- as their supremum and infimum and domains Ω such that all solutions blow up.

First, we suppose $q(x) \geqslant \gamma > p-1$ in some ball $B_{r_0}(x_0)$. As in the proof of Theorem 1.1(i), we see that as long as u is larger than some $M > 0$, depending on r_0 and γ, in $B_{r_0}(x_0)$ at time t_0, then u must blow up.

Since $q_- < p-1$ and $q(x)$ is continuous, the assumption of Corollary 4.2 is satisfied. We suppose that $q(x) \leqslant \sigma < p-1$ in $B_R(0) \subset \Omega$, where R is a large number to be determined. By Corollary 4.2, there exist constants t_0, $\delta > 0$ such that $u(x,t) \geqslant \delta$ in $B_R(0)$ for $t \geqslant t_0$. By Lemma 4.3, $u(x,t) \geqslant cR^{\frac{p}{p-1-\sigma}}$ in $B_{\frac{R}{2}}(0)$ when t is sufficiently large.

Let $L > 0$ be such that $B_R(0) \cup B_{r_0}(x_0) \subset B_{\frac{L}{2}}(0) \subset B_L(0) \subset \Omega$. Then $u \geqslant w$, where w is a solution of the following problem:

$$\begin{cases} w_t = -\mathrm{div}(|\nabla w|^{p-2}\nabla w) & \text{in} \quad B_L(0) \setminus B_{\frac{R}{2}}(0), \\ w = 0 & \text{on} \quad \partial B_L(0), \\ w = cR^{\frac{p}{p-1-\sigma}} & \text{on} \quad \partial B_{\frac{R}{2}}(0), \\ w(x,0) = u(x,t_0) & \text{in} \quad B_L(0) \setminus B_{\frac{R}{2}}(0). \end{cases}$$

We know that w converges uniformly to the unique stationary solution given by

$$r(x) = cR^{\frac{p}{p-1-\sigma}} \frac{|x|^\alpha - L^\alpha}{(\frac{R}{2})^\alpha - L^\alpha},$$

where $\alpha = \frac{p-N}{p-1}$. For t large enough,

$$u(x,t) \geqslant w(x,t) \geqslant r(x) - \varepsilon \geqslant \frac{c}{2} R^{\frac{p}{p-1-\sigma}}$$

in $B_{\frac{L}{2}}(0) \setminus B_{\frac{R}{2}}(0) \supset B_{r_0}(x_0)$. Then if R is large enough, u must blow up.

Next, we prove that if the diameter of Ω is sufficiently small then there exist global solutions. It is not hard to verify that

$$g(x) = 1 - \frac{(p-1)|x|^{\frac{p}{p-1}}}{pN^{\frac{1}{p-1}}}$$

is a supersolution. Then if $\Omega \subset B_r(x_0), r < (\frac{p}{p-1} N^{\frac{1}{p-1}})^{\frac{p-1}{p}}$, the solution is global when the initial data is less than $g(x)$.

(iv) Let $q_- > p-1$. Let $\Omega' \supset \Omega$ be a smooth bounded connected domain and ψ the non-negative first eigenfunction of p-Laplacian in Ω' with $\sup_{x\in\Omega} \psi(x) = 1$. Then ψ is a supersolution. When the initial data $u_0(x) \leqslant \psi(x)$, by the comparison principle the solution is global. □

Remark 4.4. If $q_+ = p-1$, in the spirit of [27], the blow-up or the global existence of solutions will be closely related to the size of the area where $q(x) \equiv p-1$. In fact, if Ω contains a domain Ω' where $q(x) \equiv p-1$ and the first eigenvalue of Ω' for the p-Laplacian operator is less than 1, all solutions will blow up. Other cases are rather intricate and we will not treat them here.

5 The case of whole space

In this section we discuss the case when $\Omega = \mathbb{R}^N$. As we have seen in the previous section, the solution is more likely to blow up when the domain Ω is larger. So the solutions in $\mathbb{R}^N$ behave differently from solutions in bounded domains.

Proof of Theorem 1.2. (i) $q_+ \leqslant 1$.

It suffices to observe that

$$w(t) = C\|u_0\|_\infty \mathrm{e}^t$$

with $C \geqslant 1, C\|u_0\|_\infty \geqslant 1$ is a supersolution and $w(0)$ is larger than $u_0(x)$. Moreover, w is strictly positive, thus the comparison principle implies that the maximal solution to the problem is global. Therefore, any solution is global.

(ii) $q_+ > p-1$.

Suppose in a region $q(x) \geqslant \gamma > p-1$ and the initial data is sufficiently large, the solution must blow up. The proof is the same as the proof of Theorem 1.1(i).

(iii) $q_- > p-1+p/N$.

In [14], a global supersolution h less than 1 is constructed when $q(x) \equiv q > p-1+\frac{p}{N}$. So it is also a supersolution to (1.1) since $q(x) \geqslant q_- > p-1+\frac{p}{N}$. It follows that if $u_0(x) \leqslant h(x,0)$, then u is global.

(iv) $1 < q_- \leqslant q_+ \leqslant p-1+p/N$.

We use the method in [15] to derive the nonexistence of global solution. Since the procedure is similar, we only give an outline of the proof and point out the difference.

We prove by contradiction by supposing that u is a global solution and then derive a contradiction.

Step 1. Let $\theta(y,\tau)=(1+t)^{\kappa N}u(x,t)$, where $y=\frac{x}{(1+t)^{\kappa}}$, $\tau=\log(1+t)$ and $\kappa=\frac{1}{(p-2)N+p}$. Then

$$\begin{aligned}\theta_\tau &= \operatorname{div}(|\nabla\theta|^{p-2}\nabla\theta)+\kappa\nabla\theta\cdot y+\kappa N\theta+(1+t)^{\frac{(p-1)N+p-Nq(x)}{(p-2)N+p}}\theta^{q(x)}\\ &\geqslant \operatorname{div}(|\nabla\theta|^{p-2}\nabla\theta)+\kappa\nabla\theta\cdot y+\kappa N\theta+\theta^{q(x)}\\ &\geqslant \operatorname{div}(|\nabla\theta|^{p-2}\nabla\theta)+\kappa\nabla\theta\cdot y+\kappa N\theta+\min(\theta^{q_-},\theta^{q_+}) \end{aligned}\tag{5.1}$$

and

$$\theta(y,0)=u_0(y).$$

Denote $g(y,\tau)$ to be the solution of

$$g_\tau=\operatorname{div}(|\nabla g|^{p-2}\nabla g)+\kappa\nabla g\cdot y+\kappa Ng+\min(g^{q_-},g^{q_+})\tag{5.2}$$

with the initial data

$$g(y,0)=V_S(|y|;a)\leqslant u_0(y).$$

Then $\theta(y,\tau)\geqslant g(y,\tau)$ in $\mathbb{R}^N\times\mathbb{R}^+$, which implies g is a global solution of (5.2).

Step 2. $g(y,\tau)$ is nondecreasing in τ in $\mathbb{R}^N\times\mathbb{R}^+$.

To prove this, we consider for arbitrary $\varepsilon>0$ the solution g_ε of the following problem:

$$\begin{cases}(g_\varepsilon)_\tau=\operatorname{div}((|\nabla g_\varepsilon|^{p-2}+\varepsilon)\nabla g_\varepsilon)+\kappa\nabla g_\varepsilon\cdot y+\kappa Ng_\varepsilon+\min(g_\varepsilon^{q_-},g_\varepsilon^{q_+}),\\ g_\varepsilon(y,0)=V_S(|y|;a).\end{cases}\tag{5.3}$$

Differentiating the equation above with respect to τ, we have that $z=(g_\varepsilon)_\tau$ solves a linear uniformly parabolic equation with initial value

$$z(y,0)=\varepsilon\Delta V_S+\min\{V_S^{q_-},V_S^{q_+}\}.$$

It is easy to verify that when ε is sufficiently small, $z(y,0)\geqslant 0$. Therefore, by the maximum principle [11], $z(y,\tau)\geqslant 0$ for $\tau\geqslant 0$, which means g_ε is non-decreasing in τ.

By the regularity results in [7, 24], as $\varepsilon\to 0$,

$$g_\varepsilon\to g\ \text{ uniformly on compact subsets of }\ \mathbb{R}^N\times[0,\infty).$$

Therefore g is non-decreasing in τ.

Step 3. By construction, $g=g(\eta,\tau)$ with $\eta=|y|$ is a radical symmetric solution of (5.2) satisfying the following equation:

$$g_\tau=\frac{1}{\eta^{N-1}}(\eta^{N-1}|g'|^{p-2}g')'+\kappa g'\eta+\kappa Ng+F(g),\tag{5.4}$$

where

$$F(x)=\begin{cases}x^{q_-}, & 0\leqslant x\leqslant 1,\\ x^{q_+}, & x>1\end{cases}\tag{5.5}$$

is a locally Lipschitz function of x.

Since $g(0,\tau)>0$, the following symmetric boundary condition holds:

$$g'(0,\tau)=0,\quad \text{for}\quad \tau\geqslant 0.$$

Applying the maximum principle [11] and a standard regularisation argument, we have $g(\eta,\tau)$ is nonincreasing in η for all $\tau>0$.

Step 4. We claim that for any $\eta > 0$, there exists a finite limit $g(\eta, \tau) \to f(\tau)$ as $\tau \to \infty$. Indeed, if this claim is not valid, then for a fixed $\eta_0 > 0$,

$$g(\eta_0, \tau) \to \infty.$$

Since $g(\eta, \tau)$ is nonincreasing in η, it follows that

$$g(\eta, \tau) \to \infty \quad \text{as} \quad \tau \to \infty$$

uniformly on $\eta \in [0, \eta_0]$.

We set $M(\tau) = \inf_{\eta \in [0,\eta_0]} g(\eta, \tau)$. Then $M(\tau)$ is nondecreasing and $M(\tau) \to \infty$ as $\tau \to \infty$.

For the original function u, this means when $|x| \leqslant \eta_0(1+t)^\kappa$, $u(x,t) \geqslant \frac{M_1(t)}{(1+t)^{\kappa N}}$ with $M_1(t) \to \infty$ as $t \to \infty$.

Since $M_1(t) \to \infty$ as $t \to \infty$ and $M_1(t)$ is nondecreasing, we can find a t_0 such that when $t \geqslant t_0$, $u(x,t) \geqslant \frac{M_0}{(1+t)^{\kappa N}}$ in the ball $|x| \leqslant \eta_0(1+t)^\kappa$ and

$$\eta_0(1+t)^\kappa \left(\frac{M_0}{(1+t)^{\kappa N}}\right)^{\frac{q_+-p+1}{p}} \geqslant \eta_0 M_0^{\frac{q_+-p+1}{p}} \geqslant \frac{M_2}{C_2}(C_1 M_1)^{\frac{q_+-p+1}{p}}.$$

Next, we take $t_1 \geqslant t_0$ sufficiently large such that

$$\frac{C_1 M_1}{M_0}(1+t_1)^{\kappa N} \geqslant 1$$

and let $\varepsilon = \frac{M_0}{(1+t_1)^{\kappa N}}$, $\delta = \eta_0(1+t_1)^\kappa$. Then $u(x,t_1) \geqslant \varepsilon$ when $|x| \leqslant \delta$, and ε and δ satisfy the condition in Lemma 3.3, which implies u blows up. This leads to a contradiction.

Step 5. $f(\eta) \geqslant 0$ is a weak symmetric stationary solution satisfying

$$\frac{1}{\eta^{N-1}}(\eta^{N-1}|f'|^{p-2}|f')' + \kappa f'\eta + \kappa N f + F(f) = 0, \tag{5.6}$$

$$f'(0) = 0, \quad 0 < f(0) < \infty. \tag{5.7}$$

Step 6. It is easy to verify that (5.5) and (5.6) do not have a non-trivial non-negative solution $f(\eta)$. Thus we derive a contradiction again and complete the proof. $\square$

Remark 5.1. For the case $1 < q_+ < p-1$, we have found an interesting phenomenon that all the solutions with compactly supported initial data will blow up or be global, depending on the detailed property of $q(x)$.

To prove this, first we claim that for any compact set $D \subset \mathbb{R}^N$ and $M > 0$, there exists a time t_0 such that $u(x,t) \geqslant M$, for $x \in D, t \geqslant t_0$. By this claim, there cannot exist a global solution and blow-up solution at the same time. Let us suppose that there are two solutions u and v, where u is global and v blows up. Then by the claim above, after some time, $u(x,t_0) \geqslant v_0(x)$ since $v_0(x)$ is compactly supported, thus u must blow up, which leads to a contradiction.

Now we prove the claim.

Let $q(x) \leqslant \sigma < p-1$.

By Corollary 4.2, for any $B_R \supset B_{\frac{R}{2}} \supset D$, there exists a time $t_1, \delta > 0$, such that $u(x,t) \geqslant \delta$ in B_R for $t \geqslant t_1$. By the method in the proof of Theorem 1.1(iii), for t large, $u(x,t) \geqslant cR^{\frac{p}{p-1-\sigma}}$ in $B_{\frac{R}{2}}$. Then if R is sufficiently large, $u \geqslant M$.

We have proved above that if $1 < q_- \leqslant q_+ < p-1$, all the solutions will blow up. However, if $q_- < 1$, the comparison method cannot be applied in the whole $\mathbb{R}^N$ and we need a new argument to judge which case it belongs to. This is still an open case.

6 Conclusion and discussion

The central topic of this article is to study the existence/nonexistence of global solutions under various conditions of $q(x)$ and Ω. By making use of the comparison argument, especially the one designated for the variable exponent case (see Lemma 3.2), we are able to analyse in depth the rich dynamics of the solutions.

When Ω is a bounded domain, the exponent $p-1$ plays a crucial role. If $q_+ < p-1$, all the solutions are global, while if $q_+ > p-1$, there exist blow-up solutions. If $q_- > p-1$, there exist global solutions, while if $q_- < p-1 < q_+$, as we have shown, the Fujita phenomenon will occur for some $q(x)$ and Ω, which is in strong contrast to the constant exponent case. In the bounded domain case, we use the first eigenfunction of p-Laplacian operator in Ω, and self-similar subsolution to prove suitable lower bound. The proof process indicates that in larger domain the solution is more likely to blow up, and in sufficiently small domain there always exist global solutions for a given functon $q(x)$. From the eigenfunction argument, we also know the critical case $q_+ = p-1$ will be highly complicated.

When $\Omega = \mathbb{R}^N$, the problem exhibits very different dynamics of solutions, because of the unboundedness of the domain. In this case, the Fujita exponent $p-1+p/N$ plays a crucial role. Our result shows that if $1 < q_- \leqslant q_+ \leqslant p-1+p/N$, all the solutions will blow up, while if $q_- > p-1+p/N$, there exist global solutions. Unlike the case of bounded domains, here the condition $q_+ < p-1$ fails to guarantee the solution to be global, and the strict sublinear condition $q_+ \leqslant 1$ seems to be necessary. There is a gap between 1 and $p-1$, which does not appear in the semilinear case ($p=2$). As in Remark 5.1, we have found that if $1 < q_+ < p-1$, all the solutions with compactly supported initial data will blow up or be global, depending on the detailed property of $q(x)$. This is a significant difference from the semilinear case and need further investigation. Since if $q_- < 1$, the comparison argument fails to apply, some new techniques are needed to study this open case.

In addition to the problem raised above, there are many other interesting topics which can be pursued. For example, what happens if Ω is a half space or other cone-shaped domain? In these cases, the Fujita phenomenon can develop with different exponents. The interaction among different values of $q(x)$ in different areas could present highly complex situations.

Also, the results in this article can be generalized to the equation

$$u_t = \operatorname{div}(|\nabla u^m|^{p-2}\nabla u^m) + u^{q(x)},$$

where $m > 1$. Here a new comparison technique should be used.

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Appendix A: Proof of the comparison principle

In this appendix, we give a detailed proof of Lemma 2.4. First we list the properties satisfied in the class $\mathfrak{R}$,

$$\sup_{x\in\mathbb{R}^N}\int_{B_1(x)}|u(y,t)|dy \leqslant C, \tag{A.1}$$

$$\sup_{x\in\mathbb{R}^N}|u(x,t)| \leqslant Ct^{-\delta}, \tag{A.2}$$

$$\sup_{x\in\mathbb{R}^N}\frac{|Du(x,t)|}{(1+|x|)^{\frac{2}{p-2}}} \leqslant Ct^{-\delta_1}, \tag{A.3}$$

for $t\in(0,T)$. Here C,δ,δ_1 are positive constants, and

$$\delta<\frac{1}{\lambda-1},\quad \delta_1<\frac{1}{p-2},\quad \lambda=\max\{q,p-1\},\quad q=\max\{1,q_+\}.$$

We rewrite our equation as

$$u_t(x,t)=\mathrm{div}(|\nabla u|^{p-2}\nabla u)+|u|^{q(x)-1}u \tag{A.4}$$

to include the potential case of variable sign. To prove Lemma 2.4, it is sufficient to prove the following lemma.

Lemma A.1. *Suppose $q_-\geqslant 1$ and u is a subsolution of* (A.4) *with initial value u_0. If v is a solution of* (A.4) *with initial value v_0, $u_0(x)\leqslant v_0(x)$ in $\mathbb{R}^N$ and u and v belong to the class $\mathfrak{R}$, then $u\leqslant v$ in S_T.*

Proof. Letting $w=u-v$, then w satisfies

$$w_t\leqslant (a^{ij}(x,t)w_{x_i})_{x_j}+b(x,t)w\quad\text{in}\quad S_T \tag{A.5}$$

and

$$w_+(x,t)\to 0\quad\text{in}\quad L^1_{\mathrm{loc}}(\mathbb{R}^N)\quad\text{as}\quad t\to 0,$$

where

$$\begin{aligned} a^{ij}(x,t)&=\int_0^1|D(su+(1-s)v)|^{p-2}ds\cdot\delta_{ij}\\ &\quad+(p-2)\int_0^1|D(su+(1-s)v)|^{p-4}\cdot(su+(1-s)v)_{x_i}(su+(1-s)v)_{x_j}ds, \\ b(x,t)&=q(x)\int_0^1|su+(1-s)v|^{q(x)-1}ds. \end{aligned}\tag{A.6}$$

The matrix a^{ij} is positive semi-definite and for all $\xi\in\mathbb{R}^N,(x,t)\in S_T$,

$$a_0(x,t)|\xi|^2\leqslant a^{ij}(x,t)\xi_i\xi_j\leqslant(p-1)a_0(x,t)|\xi|^2, \tag{A.7}$$

where

$$a_0(x,t)=\int_0^1|D(su+(1-s)v)|^{p-2}ds.$$

For $\beta>0$, set

$$\begin{aligned} A_\beta(x)&=(1+|x|^p)^{-\beta},\\ h_\beta(t)&=\sup_{0<\tau<t}\int_{\mathbb{R}^N}u(x,\tau)A_\beta(x)dx. \end{aligned}$$

If

$$\beta\geqslant\frac{\kappa}{p(p-2)}, \tag{A.8}$$

where $\kappa=N(p-2)+p$, by (A.1), we have

$$h_\beta(t)\leqslant C(\beta)\quad\text{for}\quad t\in(0,T). \tag{A.9}$$

To prove the lemma, we need the following lemmas.

Lemma A.2. *There exists a constant $C = C(N, p, q)$ such that*

$$\int_0^t \int_{\mathbb{R}^N} |Du|^{p-1} A_{\beta+\frac{1}{p}}(x)dxd\tau \leqslant C \cdot t^{\frac{1-\delta(\lambda-1)}{p}} \quad \textit{for} \quad t \in (0, T).$$

Proof. Since u is a subsolution, in the inequality that u satisfies, take the test function

$$\phi = (t-\varepsilon)_+^{\frac{1}{p}} |u|^{-\frac{2}{p}} u (A_{\beta+\frac{1}{p}}^{\frac{1}{p}} \xi)^p,$$

where ξ is the usual cutoff function in B_ρ. After a Steklov averaging process and standard calculations, we obtain

$$\begin{aligned}
&\int_\varepsilon^t \int_{B_\rho} (\tau-\varepsilon)^{\frac{1}{p}} \frac{|Du|^p}{|u|^{\frac{2}{p}}} A_{\beta+\frac{1}{p}} \xi^p dxd\tau \\
&\quad \leqslant C \int_\varepsilon^t \int_{B_\rho} (\tau-\varepsilon)^{\frac{1}{p}} |u|^{p-\frac{2}{p}} |D(A_{\beta+\frac{1}{p}}^{\frac{1}{p}} \xi)|^p dxd\tau \\
&\qquad + C \int_\varepsilon^t \int_{B_\rho} (\tau-\varepsilon)^{\frac{1}{p}-1} |u|^{2-\frac{2}{p}} A_{\beta+\frac{1}{p}} dxd\tau \\
&\qquad + \int_\varepsilon^t \int_{B_\rho} (\tau-\varepsilon)^{\frac{1}{p}} |u|^{q(x)+1-\frac{2}{p}} A_{\beta+\frac{1}{p}} \xi^p dxd\tau \\
&\quad =: J_1 + J_2 + J_3.
\end{aligned}$$

For J_2, we have

$$J_2 \leqslant C \int_\varepsilon^t (\tau-\varepsilon)^{\frac{1-p-(p-2)\delta}{p}} \cdot \int_{B_\rho} \tau^{\frac{(p-2)\delta}{p}} \frac{|u|^{1-\frac{2}{p}}}{1+|x|} |u| A_\beta(x) dxd\tau.$$

By (A.2) and (A.9), we have

$$J_2 \leqslant C(t-\varepsilon)^{\frac{1-(p-2)\delta}{p}} \quad \text{for} \quad \rho \geqslant 1.$$

We estimate J_1,

$$\begin{aligned}
J_1 &\leqslant C \int_\varepsilon^t \int_{B_\rho} (\tau-\varepsilon)^{\frac{1}{p}} |u|^{p-\frac{2}{p}} A_{\beta+\frac{1}{p}} |D\xi|^p dxd\tau \\
&\quad + C \int_\varepsilon^t \int_{B_\rho} (\tau-\varepsilon)^{\frac{1}{p}} |u|^{p-\frac{2}{p}} \xi^p |D(A_{\beta+\frac{1}{p}}^{\frac{1}{p}})|^p dxd\tau \\
&=: J_{1,1} + J_{1,2}.
\end{aligned}$$

Since $|D(A_{\beta+\frac{1}{p}}^{\frac{1}{p}})|^p \leqslant C A_\beta A_{1+\frac{1}{p}}$, by (A.2) and (A.9), we have

$$J_{1,2} \leqslant \int_\varepsilon^t \int_{B_\rho} (\tau-\varepsilon)^{\frac{1}{p}} |u|^{p-1-\frac{2}{p}} A_{1+\frac{1}{p}} |u| A_\beta dxd\tau \leqslant C(t-\varepsilon)^{(1+\frac{1}{p})(1-\delta(p-2))}.$$

As for $J_{1,1}$, since $|D\xi| \leqslant \dfrac{2}{p}$, by (A.2) and (A.9), we have

$$J_{1,1} \leqslant C(t-\varepsilon)^{(1+\frac{1}{p})(1-\delta(p-2))}.$$

We now estimate J_3,

$$\begin{aligned}
J_3 &\leqslant C \int_\varepsilon^t \int_{B_\rho} (\tau-\varepsilon)^{\frac{1}{p}} (|u|^{q+1-\frac{2}{p}} + |u|) A_{\beta+\frac{1}{p}} \xi^p dxd\tau \\
&\leqslant C \int_\varepsilon^t \int_{B_\rho} (\tau-\varepsilon)^{\frac{1}{p}} (\tau-\varepsilon)^{-\delta(q-\frac{2}{p})} h_\beta(\tau) dxd\tau \\
&\leqslant C(t-\varepsilon)^{(1+\frac{1}{p})(1-\delta\frac{pq-2}{p+1})} \\
&\leqslant C(t-\varepsilon)^{(1+\frac{1}{p})(1-\delta(\lambda-1))}.
\end{aligned}$$

Combining these estimates, we have for $\rho \geqslant 1$,

$$\int_\varepsilon^t \int_{B_\rho} (\tau-\varepsilon)^{\frac{1}{p}} \frac{|Du|^p}{|u|^{\frac{2}{p}}} A_{\beta+\frac{1}{p}} dx d\tau \leqslant C(t-\varepsilon)^{\frac{1}{p}(1-\delta(\lambda-1))}.$$

Therefore,

$$\begin{aligned}
&\int_\varepsilon^t \int_{B_\rho} |Du|^{p-1} A_{\beta+\frac{1}{p}} dx d\tau \\
&\quad\leqslant \left(\int_\varepsilon^t \int_{B_\rho} (\tau-\varepsilon)^{\frac{1}{p}} \frac{|Du|^p}{|u|^{\frac{2}{p}}} A_{\beta+\frac{1}{p}} dx d\tau\right)^{\frac{p-1}{p}} \left(\int_\varepsilon^t \int_{B_\rho} (\tau-\varepsilon)^{-\frac{p-1}{p}} |u|^{2-\frac{2}{p}} A_{\beta+\frac{1}{p}} dx d\tau\right)^{\frac{1}{p}} \\
&\quad\leqslant C(t-\varepsilon)^{\frac{1}{p}(1-\delta(\lambda-1))}.
\end{aligned}$$

Letting $\varepsilon \to 0$, then the conclusion holds. □

Since v is a solution of (A.4), the conclusion above also holds for v.

Lemma A.3. *There exists a constant C such that w_+ satisfies*

$$\int_{\mathbb{R}^N} w_+(x,t) A_\beta(x) dx \leqslant Ct^{\frac{1}{p}(1-\delta(\lambda-1))}.$$

Proof. By definition of w, we already have

$$w_+(x,t) \to 0 \quad \text{in} \quad L^1_{\text{loc}}(\mathbb{R}^N) \quad \text{as} \quad t \to 0.$$

It is clear that w_+ is a weak subsolution of (A.5). Take the test function $A_\beta(x)\xi(x)$, where ξ is the usual cutoff function in B_ρ. We have

$$\begin{aligned}
\int_{B_\rho} w_+(x,t) A_\beta(x)\xi(x) dx \leqslant& \int_0^t \int_{B_\rho} (|Du|+|Dv|)^{p-1} |D(A_\beta \xi)| dx d\tau \\
&+ C\int_0^t \int_{B_\rho} (|u|^{q-1}+|v|^{q-1}+1) A_\beta (|u|+|v|)\xi dx d\tau.
\end{aligned} \tag{A.10}$$

Noticing that

$$|DA_\beta| \leqslant CA_{\beta+\frac{1}{p}}, \quad |D\xi| = 0 \quad \text{on} \quad |x| \leqslant \frac{\rho}{2}, \quad A_\beta |D\xi| \leqslant CA_{\beta+\frac{1}{p}},$$

by (A.2) and (A.9), and we have

$$\begin{aligned}
\int_0^t \int_{B_\rho} (|u|^{q-1}+|v|^{q-1}+1) A_\beta (|u|+|v|)\xi dx d\tau \leqslant& C\int_0^t (\tau^{-\delta(q-1)}+1) h_\beta(\tau) d\tau \\
\leqslant& Ct^{1-\delta(q-1)}.
\end{aligned}$$

Letting $\rho \to \infty$ in (A.10), we have

$$\int_{\mathbb{R}^N} w_+(x,t) A_\beta(x)\xi(x) dx \leqslant C\int_0^t \int_{\mathbb{R}^N} (|Du|+|Dv|)^{p-1} |D(A_\beta\xi)| dx d\tau + Ct^{1-\delta(q-1)}.$$

By Lemma A.2, Lemma A.3 is proved. □

Lemma A.4. *For any $\varepsilon \in (0, \frac{1}{p\delta}(1-\delta(\lambda-1)))$,*

$$w_+(x,t) \to 0 \quad \textit{in} \quad L^{1+\varepsilon}_{\text{loc}}(\mathbb{R}^N) \quad \textit{as} \quad t \to 0.$$

Proof. Let $\varepsilon \in (0, \frac{1}{p\delta}(1-\delta(\lambda-1)))$ be fixed. Then by (A.2), for $t \in (0,T)$,

$$\int_{\mathbb{R}^N} w_+(x,t)^{1+\varepsilon} A_{\beta+\frac{\varepsilon}{p-2}}(x) dx \leqslant Ct^{-\varepsilon\delta} \int_{\mathbb{R}^N} w_+(x,t) A_\beta(x) dx \leqslant Ct^{\frac{1}{p}(1-\delta(\lambda-1))-\varepsilon\delta}.$$

Then for $\rho \geqslant 1$,

$$\int_{B_\rho} w_+(x,t)^{1+\varepsilon}dx \leqslant C\rho^{(\beta+\frac{\varepsilon}{p-2})p}\int_{\mathbb{R}^N} w_+(x,t)^{1+\varepsilon}A_{\beta+\frac{\varepsilon}{p-2}}(x)dx$$
$$\leqslant C(N,p,q,\rho)t^{\frac{1}{p}(1-\delta(\lambda-1))-\varepsilon\delta}. \qquad \square$$

Now we return to the proof of Lemma A.1. We use the test function

$$(w_+ + \eta)^\varepsilon (A_\beta^{\frac{1}{2}}\xi)^2, \quad \varepsilon \in \left(0, \frac{1}{p\delta}(1-\delta(\lambda-1))\right), \quad \eta \in (0,1).$$

Since $w_+(\cdot,t) \to 0$ in $L^1_{\rm loc}(\mathbb{R}^N)$, a standard Steklov averaging process gives that this is an admissible test function. Therefore we can deduce

$$\begin{aligned}
&\frac{1}{1+\varepsilon}\int_{B_\rho(t)}(w_+ + \eta)^{1+\varepsilon}A_\beta\xi^2dx + \varepsilon\int_\eta^t\int_{B_\rho} a_0(x,\tau)\frac{|Dw_+|^2}{(w_+ + \eta)^{1-\varepsilon}}A_\beta\xi^2dxd\tau\\
&\quad\leqslant \frac{1}{1+\varepsilon}\int_{B_\rho(\eta)}(w_+ + \eta)^{1+\varepsilon}A_\beta\xi^2dx\\
&\qquad + \int_\eta^t\int_{B_\rho} b(x,\tau)w_+(w_+ + \eta)^\varepsilon A_\beta\xi^2dxd\tau\\
&\qquad + C\int_\eta^t\int_{B_\rho} a_0(x,\tau)\frac{|Dw_+|}{(w_+ + \eta)^{\frac{1-\varepsilon}{2}}}(w_+ + \eta)^{\frac{1+\varepsilon}{2}}A_\beta^{\frac{1}{2}}\xi|D(A_\beta^{\frac{1}{2}}\xi)|dxd\tau.
\end{aligned}$$

Notice that

$$A_\beta|D\xi|^2 + |DA_\beta^{\frac{1}{2}}|^2 \leqslant CA_\beta(x)A_{\frac{2}{p}}(x)$$

and by Schwarz inequality,

$$\begin{aligned}
&\int_\eta^t\int_{B_\rho} a_0(x,\tau)\frac{|Dw_+|}{(w_+ + \eta)^{\frac{1-\varepsilon}{2}}}(w_+ + \eta)^{\frac{1+\varepsilon}{2}}A_\beta^{\frac{1}{2}}\xi|D(A_\beta^{\frac{1}{2}}\xi)|dxd\tau\\
&\quad\leqslant \frac{\varepsilon}{2}\int_\eta^t\int_{B_\rho} a_0(x,\tau)\frac{|Dw_+|^2}{(w_+ + \eta)^{1-\varepsilon}}A_\beta\xi^2dxd\tau\\
&\qquad + C(\varepsilon)\int_\eta^t\int_{B_\rho} a_0(x,\tau)(w_+ + \eta)^{1+\varepsilon}(A_\beta|D\xi|^2 + |DA_\beta^{\frac{1}{2}}|^2)dxd\tau.
\end{aligned}$$

We get

$$\begin{aligned}
&\int_{B_\rho(t)}(w_+ + \eta)^{1+\varepsilon}A_\beta\xi^2dx\\
&\quad\leqslant \int_{B_\rho(\eta)}(w_+ + \eta)^{1+\varepsilon}A_\beta\xi^2dx\\
&\qquad + C\int_\eta^t\int_{B_\rho} a_0(x,\tau)A_{\frac{2}{p}}(x)(w_+ + \eta)^{1+\varepsilon}A_\beta(x)dxd\tau\\
&\qquad + C\int_\eta^t\int_{B_\rho} b(x,\tau)(w_+ + \eta)^{1+\varepsilon}A_\beta(x)dxd\tau.
\end{aligned}$$

By (A.2),

$$b(x,\tau) \leqslant C(|u|^{q-1} + |v|^{q-1} + 1) \leqslant C\tau^{-\delta(q-1)}.$$

By (A.3),

$$a_0(x,\tau) \leqslant C\tau^{-\delta_1(p-2)}.$$

Therefore,

$$\int_{B_{\frac{\rho}{2}}(t)} (w_+ + \eta)^{1+\varepsilon} A_\beta(x)dx \leqslant C\int_\eta^t \tau^{-\sigma}\int_{\mathbb{R}^N} (w_+ + \eta)^{1+\varepsilon} A_\beta(x)dxd\tau \\ + \int_{B_\rho(\eta)} (w_+ + \eta)^{1+\varepsilon} A_\beta(x)dx,$$

where $\sigma = \max\{\delta_1(p-2), \delta(\lambda-1)\}$. Let $\eta \to 0, \rho \to \infty$. By Lemma A.4, we have

$$\int_{\mathbb{R}^N} w_+(x,t)^{1+\varepsilon} A_\beta(x)dx \leqslant C\int_\eta^t \tau^{-\sigma}\int_{\mathbb{R}^N} w_+(x,\tau)^{1+\varepsilon} A_\beta(x)dxd\tau.$$

Since $\tau^{-\sigma}$ is integrable, this implies

$$\int_{\mathbb{R}^N} w_+(x,t)^{1+\varepsilon} A_\beta(x)dx = 0 \quad \text{for} \quad t \in (0,T)$$

by Gronwall's inequality, provided

$$\int_{\mathbb{R}^N} w_+(x,t)^{1+\varepsilon} A_\beta(x)dx \in L^\infty(0,T). \tag{A.11}$$

Notice that the parameter β in the calculation above is only restricted by (A.8), thus by Lemma A.3, if we choose $\beta > \frac{\kappa}{p(p-2)} + \frac{\varepsilon}{p-2}$, we have

$$\begin{aligned} &\int_{\mathbb{R}^N} w_+(x,t)^{1+\varepsilon} A_\beta(x)dx \\ &\quad \leqslant \int_{\mathbb{R}^N} (w_+(x,t)A_{\frac{1}{p-2}})^\varepsilon w_+(x,t)A_{\frac{\kappa}{p(p-2)}}(x)dx \\ &\quad \leqslant C\int_{\mathbb{R}^N} t^{-\delta\varepsilon} w_+(x,t)A_{\frac{\kappa}{p(p-2)}}(x)dx \\ &\quad \leqslant Ct^{-\delta\varepsilon+\frac{1}{p}(1-\delta(\lambda-1))} \leqslant C. \end{aligned}$$

Lemma A.1 is proved. $\square$