# A double commutant theorem for Murray–von Neumann algebras

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**Murray–von Neumann algebras are algebras of operators affiliated with finite von Neumann algebras. In this article, we study commutativity and affiliation of self-adjoint operators (possibly unbounded). We show that a maximal abelian self-adjoint subalgebra $\mathscr{A}$ of the Murray–von Neumann algebra $\mathscr{A}_f(\mathcal{R})$ associated with a finite von Neumann algebra $\mathcal{R}$ is the Murray–von Neumann algebra $\mathscr{A}_f(\mathscr{A}_0)$, where $\mathscr{A}_0$ is a maximal abelian self-adjoint subalgebra of $\mathcal{R}$ and, in addition, $\mathscr{A}_0$ is $\mathscr{A} \cap \mathcal{R}$. We also prove that the Murray–von Neumann algebra $\mathscr{A}_f(\mathcal{C})$ with $\mathcal{C}$ the center of $\mathcal{R}$ is the center of the Murray–von Neumann algebra $\mathscr{A}_f(\mathcal{R})$. Von Neumann's celebrated double commutant theorem characterizes von Neumann algebras $\mathcal{R}$ as those for which $\mathcal{R}'' = \mathcal{R}$, where $\mathcal{R}'$, the commutant of $\mathcal{R}$, is the set of bounded operators on the Hilbert space that commute with all operators in $\mathcal{R}$. At the end of this article, we present a double commutant theorem for Murray–von Neumann algebras.**

affiliated operators | unbounded operators

In ref. 1, J. von Neumann initiates the subject of von Neumann algebras (which he refers to as "rings of operators") and proves what is the most fundamental theorem of the subject, his celebrated "double commutant theorem." That theorem is, in effect, an infinite-dimensional version of Schur's lemma. The complex-Hilbert-space setting in which this theorem of von Neumann's is proved is the most "classical" venue available for such an infinite-dimensional version. Von Neumann's purpose in proving this result was twofold. On the one hand, Schur's lemma is basic to the study of representation of finite groups; von Neumann was preparing the way for an extension of that study to infinite groups. The "rings of operators" introduced by von Neumann were to (and they do) play the role of a (complex) group algebra of the group. There are other possibilities for such a group algebra that are useful for different purposes. As is often the case when a concept from finite-dimensional algebra is extended to infinite dimensions, that concept "ramifies," splits into several distinct concepts, subtly different from one another, especially when analysis and topology make an important appearance. It was von Neumann's hope that his group algebra would "coalesce" large families of groups (that is, be the same algebra for each group of the family) and, therefore, provide a simpler target for classification than the groups themselves. A sample instance of this coalescing occurs when considering "locally finite" groups, those for which each finite subset generates a finite subgroup. One important example of a locally finite group is given by the group of permutations of the integers each of which moves at most a finite set of integers. This example describes a group that has an additional important feature: Each of its conjugacy classes, with the exception of that of the group identity, is infinite. We refer to such groups as "i.c.c. groups." The von Neumann group algebras of i.c.c. groups have centers consisting just of scalar multiples of the identity operator, the so-called "factors." The locally finite, i.c.c. groups all have the same group algebra (up to isomorphism). This group algebra is a key von Neumann algebra, the "hyperfinite factor of type $\mathrm{II}_1$."

F. J. Murray and J. von Neumann undertook a thorough study of factors (2–5) classifying them into various types. Each of the i.c.c. groups has a von Neumann algebra that is (a factor) of type $\mathrm{II}_1$. The free, noncommutative groups on two or more generators are among these i.c.c. groups. To this day, we do not know whether the $\mathrm{II}_1$-factor group algebras of the free groups on two and three generators are or are not isomorphic to one another.

On the other hand, at the time von Neumann was proving his double commutant theorem and introducing his "rings of operators" on Hilbert spaces, quantum theorists were seeking a workable mathematical model that would encompass the ad hoc quantum assumptions with which they altered the strictly classical (Hamiltonian–Newtonian) analysis of the atomic and subatomic physical systems they were examining. Planck's formula for radiation associated with a full, black-body radiator, Einstein's photoelectric effect, and Bohr's remarkable derivation of the visible (Balmer) energy spectrum from his quantized version of the Rutherford (planetary) model of the hydrogen atom are among the first and most basic instances of these quantum assumptions. During this period, Heisenberg discovered his fundamental commutation relation (see also ref. 6), $QP - PQ = i\hbar I$, where $Q$ and $P$ play the role of position and "conjugate" momentum of a particle in the physical system and $\hbar$ is $h/2\pi$, where $h$ is Planck's experimentally determined constant. Heisenberg's relation carries with it the declaration that the mathematical model being sought must be "noncommutative" (so, $QP = PQ$ has been ruled out) and (finite) matrices would not be suitable (as the "trace" functional on the matrices makes clear). Now, von Neumann was aware of these developments and aware as well that Hilbert spaces and the linear operators on them, especially families of self-adjoint operators, with the algebraic structure such families inherit from the usual product and addition of (everywhere-defined) operators, provided an especially hospitable environment for modeling quantum mechanical systems and supplying them with the necessary degree of noncommutativity.

The simplest examples of such families are the self-adjoint elements in subalgebras of $\mathscr{B}(\mathscr{H})$, the family of all bounded operators on the Hilbert space $\mathscr{H}$, that contain $A^*$, the adjoint of the operator $A$, when they contain $A$. Such subalgebras are said to be self-adjoint ("closed under the adjoint operation"). Among the self-adjoint subalgebras of $\mathscr{B}(\mathscr{H})$, the most useful for the purposes of modeling quantum mechanics are those closed in $\mathscr{B}(\mathscr{H})$ with respect to some of the "natural" topologies on $\mathscr{B}(\mathscr{H})$, the "norm topology," corresponding to the topology on $\mathscr{B}(\mathscr{H})$ arising from the metric $\|A - B\|$ as the distance between $A$ and $B$ in $\mathscr{B}(\mathscr{H})$ and the "strong-operator topology" corresponding to convergence of sequences (nets) on vectors in $\mathscr{H}$ (that is, $A_n \to A$ in this topology when $A_n x \to Ax$, for each $x$ in $\mathscr{H}$). Those self-adjoint algebras closed in the norm topology are called "$C^*$-algebras," and those closed in the strong-operator topology are called "von Neumann algebras." In both cases, we include the condition that $I$, the identity operator on $\mathscr{H}$ is in the algebra.

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In ref. 3, Murray and von Neumann discover that unbounded operators closely associated with (we say "affiliated with" in a technical definition) a certain class of von Neumann algebras, the "finite" von Neumann algebras (again, a technical term), have remarkable domain properties that allow virtually unlimited algebraic manipulation with such operators. We study some aspects of the structure of such families of affiliated operators in this article, notably, commutativity.

In Section 2, we provide a background discussion of unbounded operators, with some of the technical details we shall need, affiliated operators, and some of their related spectral theory, and some of the algebraic properties of families of affiliated operators. Some of the technical results on commutativity, maximal abelian subalgebras, and central elements in the algebra of affiliated operators that serve as the basis for our main results, the commutant theorems in Section 4, appear in Section 3.

## 2 Murray–von Neumann Algebras

We use Section 2.7 and Section 5.6 in refs. 7–9 as our basic reference for results in the theory of unbounded operators as well as for much of our notation and terminology.

**2.1 Basic Results on Unbounded Operators.** Let $T$ be a linear mapping of the Hilbert space $\mathscr{H}$ into the Hilbert space $\mathscr{K}$. We denote by "$\mathscr{D}(T)$" the domain of $T$. Note that $\mathscr{D}(T)$ is a linear submanifold of $\mathscr{H}$ (not necessarily closed). We associate a graph $\mathscr{G}(T)$ with $T$, where $\mathscr{G}(T) = \{(x, Tx) : x \in \mathscr{D}(T)\}$. We say that $T$ is closed when $\mathscr{G}(T)$ is closed. The closed graph theorem tells us that if $T$ is defined on all of $\mathscr{H}$, then $\mathscr{G}(T)$ is closed if and only if $T$ is bounded. The unbounded operators $T$ we consider will usually be densely defined; that is, $\mathscr{D}(T)$ is dense in $\mathscr{H}$. We say that $T_0$ extends (or is an extension of) $T$ and write $T \subseteq T_0$ when $\mathscr{D}(T) \subseteq \mathscr{D}(T_0)$ and $T_0x = Tx$ for each $x$ in $\mathscr{D}(T)$. If $\mathscr{G}(T)^-$, the closure of the graph of $T$ is the graph of a linear transformation $\bar{T}$, clearly $\bar{T}$ is the "smallest" closed extension of $T$, we say that $T$ is preclosed (or closable) and refer to $\bar{T}$ as the closure of $T$. From the point of view of calculations with an unbounded operator $T$, it is often much easier to study its restriction $T|\mathscr{D}_0$ to a dense linear manifold $\mathscr{D}_0$ in its domain $\mathscr{D}(T)$ than to study $T$ itself. If $T$ is closed and $\mathscr{G}(T|\mathscr{D}_0)^- = \mathscr{G}(T)$, we say that $\mathscr{D}_0$ is a core for $T$. Each dense linear manifold in $\mathscr{G}(T)$ corresponds to a core for $T$.

***Definition 1:*** If $T$ is a linear transformation with $\mathscr{D}(T)$ dense in the Hilbert space $\mathscr{H}$ and range contained in the Hilbert space $\mathscr{K}$, we define a mapping $T^*$, the adjoint of $T$, as follows. Its domain consists of those vectors $y$ in $\mathscr{K}$ such that, for some vector $z$ in $\mathscr{H}$, $\langle x, z\rangle = \langle Tx, y\rangle$ for all $x$ in $\mathscr{D}(T)$. For such $y$, $T^*y$ is $z$. If $T = T^*$, we say that $T$ is self-adjoint. (Note that the formal relation $\langle Tx, y\rangle = \langle x, T^*y\rangle$, familiar from the case of bounded operators, remains valid in the present context only when $x \in \mathscr{D}(T)$ and $y \in \mathscr{D}(T^*)$.)

***Remark 2:*** If $T$ is densely defined, then $T^*$ is closed. If $T_0$ is an extension of $T$, then $T^*$ is an extension of $T_0^*$.

**Theorem 3:** *If $T$ is a densely defined linear transformation from the Hilbert space $\mathscr{H}$ to the Hilbert space $\mathscr{K}$, then*

*i. if $T$ is preclosed, $(\bar{T})^* = T^*$;*
*ii. $T$ is preclosed if and only if $\mathscr{D}(T^*)$ is dense in $\mathscr{K}$;*
*iii. if $T$ is preclosed, $T^{**} = \bar{T}$;*
*iv. if $T$ is closed, $T^*T + I$ is one-to-one with range $\mathscr{H}$ and positive inverse of bound not exceeding 1.*

***Definition 4:*** We say that $T$ is symmetric when $\mathscr{D}(T)$ is dense in $\mathscr{H}$ and $\langle Tx, y\rangle = \langle x, Ty\rangle$ for all $x$ and $y$ in $\mathscr{D}(T)$. Equivalently, $T$ is symmetric when $T \subseteq T^*$. (Since $T^*$ is closed and $\mathscr{G}(T) \subseteq \mathscr{G}(T^*)$, in this case, $T$ is preclosed if it is symmetric. If $T$ is self-adjoint, $T$ is both symmetric and closed.)

***Remark 5:*** If $A \subseteq T$ with $A$ self-adjoint and $T$ symmetric, then $A \subseteq T \subseteq T^*$, so that $T^* \subseteq A^* = A \subseteq T \subseteq T^*$ and $A = T$. It follows that $A$ has no proper symmetric extension. That is, a self-adjoint operator is maximal symmetric.

**Proposition 6.** *If $T$ is a closed symmetric operator on the Hilbert space $\mathscr{H}$, the following assertions are equivalent:*

*i. $T$ is self-adjoint;*
*ii. $T^* \pm iI$ have $(0)$ as null space;*
*iii. $T \pm iI$ have $\mathscr{H}$ as range;*
*iv. $T \pm iI$ have ranges dense in $\mathscr{H}$.*

**Proposition 7.** *If $T$ is a closed linear operator with domain dense in a Hilbert space $\mathscr{H}$ and with range in $\mathscr{H}$, then*

$$R(T) = I - N(T^*), \qquad N(T) = I - R(T^*),$$
$$R(T^*T) = R(T^*), \qquad N(T^*T) = N(T),$$

*where $N(T)$ and $R(T)$ denote the projections whose ranges are, respectively, the null space of $T$ and the closure of the range of $T$.*

**Proposition 8.** *Suppose that $A$ and $B$ are linear operators with their domains dense in a Hilbert space $\mathscr{H}$ and their ranges in $\mathscr{H}$. Then $A^* + B^* \subseteq (A + B)^*$ if $A + B$ is densely defined, and $B^*A^* \subseteq (AB)^*$ if $AB$ is densely defined.*

**Proposition 9.** *If $A$ and $C$ are densely defined preclosed operators and $B$ is a bounded (everywhere-defined) operator such that $A = BC$, then $A^* = C^*B^*$.*

There is an extension of the polar decomposition for bounded operators to the case of a closed densely defined linear operator from one Hilbert space to another.

**Theorem 10.** *If $T$ is a closed densely defined linear transformation from one Hilbert space to another, there is a partial isometry $V$ with initial space the closure of the range of $(T^*T)^{1/2}$ and final space the closure of the range of $T$ such that $T = V(T^*T)^{1/2} = (T^*T)^{1/2}V$. Restricted to the closures of the ranges of $T^*$ and $T$, respectively, $T^*T$ and $TT^*$ are unitarily equivalent (and $V$ implements this equivalence). If $T = WH$, where $H$ is a positive operator and $W$ is a partial isometry with initial space the closure of the range of $H$, then $H = (T^*T)^{1/2}$ and $W = V$.*

**2.2 Affiliated Operators and Some Spectral Theory.**

***Definition 11:*** We say that a closed densely defined operator $T$ is *affiliated* with a von Neumann algebra $\mathscr{R}$ and write $T\eta\mathscr{R}$ when $U^*TU = T$ for each unitary operator $U$ commuting with $\mathscr{R}$. (Note that the equality, $U^*TU = T$, is to be understood in the strict sense that $U^*TU$ and $T$ have the same domain and formal equality holds for the transforms of vectors in that domain. As far as the domains are concerned, the effect is that $U$ transforms $\mathscr{D}(T)$ onto itself.)

***Remark 12:*** If $T$ is a closed densely defined operator with core $\mathscr{D}_0$ and $U^*TUx = Tx$ for each $x$ in $\mathscr{D}_0$ and each unitary operator $U$ commuting with a von Neumann algebra $\mathscr{R}$, then $T\eta\mathscr{R}$.

If $A$ is a bounded self-adjoint operator acting on a Hilbert space $\mathscr{H}$ and $\mathscr{A}$ is an abelian von Neumann algebra containing $A$, there is a family $\{E_\lambda\}$ of projections in $\mathscr{A}$ (indexed by $\mathbb{R}$), called the spectral resolution of $A$, such that

i. $E_\lambda = 0$ if $\lambda < -\|A\|$, and $E_\lambda = I$ if $\|A\| \le \lambda$;
ii. $E_\lambda \le E_{\lambda'}$ if $\lambda \le \lambda'$;
iii. $E_\lambda = \wedge_{\lambda' > \lambda} E_{\lambda'}$;
iv. $AE_\lambda \le \lambda E_\lambda$ and $\lambda(I - E_\lambda) \le A(I - E_\lambda)$ for each $\lambda$;
v. $A = \int_{-\|A\|}^{\|A\|} \lambda dE_\lambda$ in the sense of norm convergence of approximating Riemann sums; and $A$ is the norm limit of finite linear combinations with coefficients in $\mathrm{sp}(A)$, the spectrum of $A$, of orthogonal projections $E_{\lambda'} - E_\lambda$.

$\{E_\lambda\}$ is said to be a resolution of the identity if $\{E_\lambda\}$ satisfies (ii), (iii), $\wedge_{\lambda \in \mathbb{R}} E_\lambda = 0$, and $\vee_{\lambda \in \mathbb{R}} E_\lambda = I$.

With the abelian von Neumann algebra $\mathscr{A}$ isomorphic to $C(X)$ and $X$ an extremely disconnected compact Hausdorff space, if $f$ and $e_\lambda$ in $C(X)$ correspond to $A$ and $E_\lambda$ in $\mathscr{A}$, then $e_\lambda$ is the characteristic function of the largest clopen subset $X_\lambda$ on which $f$ takes values not exceeding $\lambda$.

The spectral theory described above can be extended to unbounded self-adjoint operators. We associate an unbounded spectral resolution with each of them.

**Theorem 13.** *If $A$ is a self-adjoint operator acting on a Hilbert space $\mathscr{H}$, $A$ is affiliated with some abelian von Neumann algebra $\mathscr{A}$. There is a resolution of the identity $\{E_\lambda\}$ in $\mathscr{A}$ such that $\cup_{n=1}^\infty F_n(\mathscr{H})$ is a core for $A$, where $F_n = E_n - E_{-n}$, and $Ax = \int_{-n}^n \lambda dE_\lambda x$ for each $x$ in $F_n(\mathscr{H})$ and all $n$, in the sense of norm convergence of approximating Riemann sums.*

Since $A$ is self-adjoint, from Proposition 6, $A + iI$ and $A - iI$ have range $\mathscr{H}$ and null space $(0)$; in addition, they have inverses, say, $T_+$ and $T_-$, that are everywhere-defined with bound not exceeding 1. Let $\mathscr{A}$ be an abelian von Neumann algebra containing $I$, $T_+$ and $T_-$. If $U$ is a unitary operator in $\mathscr{A}'$, for each $x$ in $\mathscr{D}(A)$, $Ux = UT_+(A + iI)x = T_+U(A + iI)x$ so that $(A + iI)Ux = U(A + iI)x$; and $U^{-1}(A + iI)U = A + iI$. Thus $U^{-1}AU = A$ and $A\eta\mathscr{A}$. In particular, $A$ is affiliated with the abelian von Neumann algebra generated by $I$, $T_+$ and $T_-$. Because $\mathscr{A}$ is abelian, $\mathscr{A}$ is isomorphic to $C(X)$ with $X$ an extremely disconnected compact Hausdorff space. Let $g_+$ and $g_-$ be the functions in $C(X)$ corresponding to $T_+$ and $T_-$. Let $f_+$ and $f_-$ be the functions defined as the reciprocals of $g_+$ and $g_-$, respectively, at those points where $g_+$ and $g_-$ do not vanish. Then, $f_+$ and $f_-$ are continuous where they are defined on $X$, as is the function $f$ defined by $f = (f_+ + f_-)/2$. In a formal sense, $f$ is the function that corresponds to $A$. Let $X_\lambda$ be the largest clopen set on which $f$ takes values not exceeding $\lambda$. Let $e_\lambda$ be the characteristic function of $X_\lambda$ and $E_\lambda$ be the projection in $\mathscr{A}$ corresponding to $e_\lambda$. In this case, $\{E_\lambda\}$ satisfies $E_\lambda \le E_{\lambda'}$ if $\lambda \le \lambda'$, $E_\lambda = \wedge_{\lambda' > \lambda} E_{\lambda'}$, $\vee_\lambda E_\lambda = I$ and $\wedge_\lambda E_\lambda = 0$. That is, we have constructed a resolution of the identity $\{E_\lambda\}$. This resolution is unbounded if $f \notin C(X)$. Let $F_n = E_n - E_{-n}$, the spectral projection corresponding to the interval $[-n, n]$ for each positive integer $n$. Then $AF_n$ is bounded and self-adjoint. Moreover, $\cup_{n=1}^\infty F_n(\mathscr{H})$ is a core for $A$. From the spectral theory of bounded self-adjoint operators, $Ax = \int_{-n}^n \lambda dE_\lambda x$, for each $x$ in $F_n(\mathscr{H})$ and all $n$. If $x \in \mathscr{D}(A)$,

$$\int_{-n}^n \lambda dE_\lambda x = \int_{-n}^n \lambda dE_\lambda F_n x = AF_n x = F_n A x \to Ax.$$

Interpreted as an improper integral, we write

$$Ax = \int_{-\infty}^\infty \lambda dE_\lambda x \quad (x \in \mathscr{D}(A)).$$

***Remark 14:*** The abelian von Neumann algebra $\mathscr{A}_0$ generated by $T_+$ and $T_-$ in the above discussion is the smallest von Neumann algebra with which the self-adjoint operator $A$ is affiliated. We refer to $\mathscr{A}_0$ as the von Neumann algebra generated by $A$.

**Lemma 15.** *If $\{E_\lambda\}$ is a resolution of the identity on a Hilbert space $\mathscr{H}$ and $\mathscr{A}$ is an abelian von Neumann algebra containing $\{E_\lambda\}$, there is a self-adjoint operator $A$ affiliated with $\mathscr{A}$ such that*

$$Ax = \int_n^n \lambda dE_\lambda x,$$

*for each $x$ in $F_n(\mathscr{H})$ and all $n$, where $F_n = E_n - E_{-n}$; and $\{E_\lambda\}$ is the spectral resolution of $A$.*

**Lemma 16.** *If $A$ is a closed operator on the Hilbert space $\mathscr{H}$, $\{E_\lambda\}$ is a resolution of the identity on $\mathscr{H}$, $\cup_{n=1}^\infty F_n(\mathscr{H})$ is a core for $A$, where $F_n = E_n - E_{-n}$, and*

$$Ax = \int_n^n \lambda dE_\lambda x,$$

*for each $x$ in $F_n(\mathscr{H})$ and all $n$, then $A$ is self-adjoint and $\{E_\lambda\}$ is the spectral resolution of $A$.*

**Lemma 17.** *If $A$ is a closed operator acting on the Hilbert space $\mathscr{H}$ and $CA \subseteq AC$ for each $C$ in a self-adjoint subset $\mathscr{F}$ of $\mathscr{B}(\mathscr{H})$, then $TA \subseteq AT$ for each $T$ in the von Neumann algebra generated by $\mathscr{F}$.*

**Lemma 18.** *If $BA \subseteq AB$ and $\mathscr{D}(A) \subseteq \mathscr{D}(B)$, where $A$ is a self-adjoint operator and $B$ is a closed operator on the Hilbert space $\mathscr{H}$, then $E_\lambda B \subseteq BE_\lambda$ for each $E_\lambda$ in the spectral resolution $\{E_\lambda\}$ of $A$.*

***Definition 19:*** We say that a closed densely defined operator $A$ is *normal* when the two self-adjoint operators $A^*A$ and $AA^*$ are equal.

**Theorem 20:** *An operator $A$ is normal if and only if it is affiliated with an abelian von Neumann algebra. If $A$ is normal, there is a smallest von Neumann algebra $\mathscr{A}_0$ with which $A$ is affiliated. The algebra $\mathscr{A}_0$ is abelian.*

**2.3 The Algebra of Affiliated Operators.** Let $\mathscr{H}$ be a Hilbert space. Two projections $E$ and $F$ acting on $\mathscr{H}$ are said to be orthogonal if $EF = 0$. If the range of $F$ is contained in the range of $E$ (equivalently, $EF = F$), we say that $F$ is a subprojection of $E$ and write $F \le E$. Let $\mathscr{R}$ be a von Neumann algebra acting on $\mathscr{H}$. Suppose that $E$ and $F$ are nonzero projections in $\mathscr{R}$. We say that $E$ is a minimal projection in $\mathscr{R}$ if $F \le E$ implies $F = E$. Murray and von Neumann conceived the idea of comparing the "sizes" of projections in a von Neumann algebra in the following way: $E$ and $F$ are said to be equivalent (modulo or relative to $\mathscr{R}$), written $E \sim F$, when $V^*V = E$ and $VV^* = F$ for some $V$ in $\mathscr{R}$. (Such an operator $V$ is called a partial isometry with initial projection $E$ and final projection $F$.) We write $E \precsim F$ when $E \sim F_0$ and $F_0 \le F$ and $E \prec F$ when $E$ is, in addition, not equivalent to $F$. It is apparent that $\sim$ is an equivalence relation on the projections in $\mathscr{R}$. In addition, $\precsim$ is a partial ordering of the equivalence classes of projections in $\mathscr{R}$, and it is a nontrivial and crucially important fact that this partial ordering is a total ordering when $\mathscr{R}$ is a factor. (Factors are von Neumann algebras whose centers consist of scalar multiples of the identity operator.) Murray and von Neumann also define infinite and finite projections in this framework modeled on the set-theoretic approach. The projection $E$ in $\mathscr{R}$ is infinite (relative to $\mathscr{R}$) when $E \sim F < E$, and finite otherwise. We say that the von Neumann algebra $\mathscr{R}$ is finite when the identity operator $I$ is finite.

Throughout the rest of this subsection, $\mathscr{R}$ denotes a finite von Neumann algebra acting on a Hilbert space $\mathscr{H}$, and $\mathscr{A}(\mathscr{R})$ denotes the family of operators affiliated with $\mathscr{R}$.

In ref. 10, the following are proved.

**Proposition 21.** *Suppose that operators $S$ and $T$ are affiliated with $\mathscr{R}$, then:*

i. *$S+T$ is densely defined, preclosed and its closure, denoted by $S\hat{+}T$, is affiliated with $\mathscr{R}$;*
ii. *$ST$ is densely defined, preclosed and its closure, denoted by $S\hat{\cdot}T$, is affiliated with $\mathscr{R}$.*

**Proposition 22.** *Suppose that operators $A$, $B$ and $C$ are affiliated with $\mathscr{R}$, then*

$$(A\hat{\cdot}B)\hat{\cdot}C = A\hat{\cdot}(B\hat{\cdot}C),$$

*that is, the associative law holds for the multiplication $\hat{\cdot}$.*

**Proposition 23.** *Suppose that operators $A$, $B$ and $C$ are affiliated with $\mathscr{R}$, then*

$$(A\hat{+}B)\hat{\cdot}C = (A\hat{\cdot}C)\hat{+}(B\hat{\cdot}C) \quad \text{and } C\hat{\cdot}(A\hat{+}B) = (C\hat{\cdot}A)\hat{+}(C\hat{\cdot}B),$$

*that is, the distributive laws hold for the multiplication $\hat{\cdot}$ relative to the addition $\hat{+}$.*

**Proposition 24.** *Suppose that operators $A$ and $B$ are affiliated with $\mathscr{R}$, then*

$$(aA\hat{+}bB)^* = \bar{a}A^*\hat{+}\bar{b}\,B^* \quad \text{and } (A\hat{\cdot}B)^* = B^*\hat{\cdot}A^*,$$
$$(a, b \in \mathbb{C})$$

*where * is the usual adjoint operation on operators (possibly unbounded).*

Therefore, $\mathscr{A}(\mathscr{R})$, provided with the operations $\hat{+}$ (addition) and $\hat{\cdot}$ (multiplication), is a * algebra (with unit $I$). Recall, $\mathscr{R}$ is finite (and must be) as a von Neumann algebra for this to be valid.

***Definition 25:*** We use "$\mathscr{A}_f(\mathscr{R})$" to denote the * algebra $(\mathscr{A}(\mathscr{R}), \hat{+}, \hat{\cdot})$. We call $\mathscr{A}_f(\mathscr{R})$ *the Murray–von Neumann algebra* associated with $\mathscr{R}$.

## 3 Commutativity and Affiliation

**Proposition 26.** *If $\mathscr{R}$ is a finite von Neumann algebra acting on a Hilbert space $\mathscr{H}$, $H$ and $K$ are self-adjoint operators in $\mathscr{A}_f(\mathscr{R})$, and $\{E_\lambda\}_{\lambda\in\mathbb{R}}$, $\{F_\lambda\}_{\lambda\in\mathbb{R}}$ are the spectral resolutions of $H$, $K$, respectively, then $H\hat{\cdot}K = K\hat{\cdot}H$ if and only if $K\hat{\cdot}E_\lambda = E_\lambda\hat{\cdot}K$ for each $\lambda$ in $\mathbb{R}$, and if and only if $E_\lambda F_{\lambda'} = F_{\lambda'}E_\lambda$ for all $\lambda$ and $\lambda'$ in $\mathbb{R}$.*

***Proof:*** Suppose, first, that $H\hat{\cdot}K = K\hat{\cdot}H$. We show that $H$ and $K$ are affiliated with some abelian von Neumann subalgebra $\mathscr{A}$ of $\mathscr{R}$. For this, we prove that $H\hat{+}iK$ is a normal operator affiliated with $\mathscr{R}$. To see this, we observe that

$$\begin{aligned}(H\hat{+}iK)(H\hat{+}iK)^* &= (H\hat{+}iK)(H\hat{-}iK) = H\hat{\cdot}H\hat{+}K\hat{\cdot}K \\ &= (H\hat{-}iK)\hat{\cdot}(H\hat{+}iK) = (H\hat{+}iK)^*(H\hat{+}iK),\end{aligned}$$

from the properties of $\mathscr{A}_f(\mathscr{R})$ as an associative * algebra. (See Section 3.2 in ref. 10.) From Theorem 20, $H\hat{+}iK$ "generates" an abelian von Neumann algebra $\mathscr{A}$ with which it is affiliated as is $H\hat{-}iK(=(H\hat{+}iK)^*)$. Thus $H$ and $K$ are affiliated with $\mathscr{A}$. The spectral resolutions $\{E_\lambda\}_{\lambda\in\mathbb{R}}$ and $\{F_\lambda\}_{\lambda\in\mathbb{R}}$ lie in $\mathscr{A}$. Since $\mathscr{A}$ is abelian, $E_\lambda F_{\lambda'} = F_{\lambda'}E_\lambda$, $H\hat{\cdot}F_\lambda = F_\lambda\hat{\cdot}H$, and $K\hat{\cdot}E_\lambda = E_\lambda\hat{\cdot}K$, for all $\lambda$ and $\lambda'$ in $\mathbb{R}$.

Suppose, now, that $E_\lambda F_{\lambda'} = F_{\lambda'}E_\lambda$ for all $\lambda$ and $\lambda'$ in $\mathbb{R}$. Then $\{E_\lambda, F_\lambda\}_{\lambda\in\mathbb{R}}$ generates an abelian von Neumann algebra $\mathscr{A}$ with which each of $H$ and $K$ are affiliated (Theorem 13, Lemma 15, Lemma 16). From Theorem 5.6.12 and Theorem 5.6.15 in refs. 7–9, $\mathscr{A}_f(\mathscr{A})$ is abelian. Hence $H\hat{\cdot}F_\lambda = F_\lambda\hat{\cdot}H$, $K\hat{\cdot}E_\lambda = E_\lambda\hat{\cdot}K$, and $H\hat{\cdot}K = K\hat{\cdot}H$.

Finally, if $K\hat{\cdot}E_\lambda = E_\lambda\hat{\cdot}K$ for all $\lambda$ in $\mathbb{R}$, then $E_\lambda F_{\lambda'} = F_{\lambda'}E_\lambda$ for all $\lambda$ and $\lambda'$ in $\mathbb{R}$, from what we have just proved, with $E_\lambda$ in place of $H$. Thus, again from what we have just proved, $H\hat{\cdot}K = K\hat{\cdot}H$.

***Remark 27:*** The first part of the preceding proof requires us to find a way to move from a family of commuting elements in $\mathscr{A}_f(\mathscr{R})$ to a closely associated family of operators in $\mathscr{R}$. The natural associated family is the set of projections in the various spectral resolutions. However, our problem is precisely that of showing that the spectral resolutions, for self-adjoint elements of $\mathscr{A}_f(\mathscr{R})$ that happen to commute (algebraically), commute with one another. Of course, the process for connecting a self-adjoint operator to its spectral projections is a vital part of what we must use. That process is "analytic" in nature and calls for a certain amount of "backing and filling" if conclusions are to be drawn from it. We have seemingly avoided that process—but we haven't really done that. What we have done is to take advantage of Theorem 20, which gives us special information about a normal operator, that we have constructed from two commuting, self-adjoint operators $H$ and $K$ in $\mathscr{A}_f(\mathscr{R})$ with the well-functioning algebraic equipment Murray and von Neumann have left us. (See Section 3 in ref. 10.) At the same time, Theorem 20 makes use of the special circumstances a normal operator provides to apply Lemma 18, which moves us from $H$ and $K$ to their respective resolutions, but all in an abelian framework (supplied by Theorem 20). Still, the "analysis," effecting the shift from $H$ and $K$ to their resolutions, is hidden. It has been shifted in Lemma 18 to Remark 14 and from there to Theorem 13 where it appears in full force through the introduction of the operators $T_+$ and $T_-$, bounded, everywhere-defined, and inverse to $H+iI$ and $H-iI$, respectively. Once again, the analysis is partially hidden, because we pass to the von Neumann algebra generated by $T_+$ and $T_-$, with which $H$ is affiliated (on simple set-theoretic and mapping grounds). Now, that algebra contains the spectral resolution $\{E_\lambda\}_{\lambda\in\mathbb{R}}$ of $H$, essentially by virtue of von Neumann's double commutant theorem, a large and very powerful approximation theorem, the guiding theorem of this article.

From this discussion of the pieces of our proof of Proposition 26, we can see a direct route to the proof that $K\hat{\cdot}E_\lambda = E_\lambda\hat{\cdot}K$ if we are willing to enter the proof of Theorem 13 and use $T_+$ and $T_-$, now inverses to $H+iI$ and $H-iI$, respectively. In $\mathscr{A}_f(\mathscr{R})$, we have $(H+iI)\hat{\cdot}K = K\hat{\cdot}(H+iI)$ from our assumption that $H\hat{\cdot}K = K\hat{\cdot}H$. Thus $K = T_+\hat{\cdot}(H+iI)\hat{\cdot}K = (T_+\hat{\cdot}K)\hat{\cdot}(H+iI)$, and $K\hat{\cdot}T_+ = (T_+\hat{\cdot}K)\hat{\cdot}(H+iI)\hat{\cdot}T_+ = T_+\hat{\cdot}K$. Similarly, $K\hat{\cdot}T_- = T_-\hat{\cdot}K$ and $K$ commutes with each element of the von Neumann algebra generated by $T_+$ and $T_-$ (Lemma 17). In particular, $K\hat{\cdot}E_\lambda = E_\lambda\hat{\cdot}K$, for each $\lambda$ in $\mathscr{R}$. From this same conclusion, with $E_\lambda$ in place of $H$, we have that $E_\lambda F_{\lambda'} = F_{\lambda'}E_\lambda$, for all $\lambda$ and $\lambda'$ in $\mathbb{R}$.

**Corollary 28:** *Let $\mathscr{R}$ be a finite von Neumann algebra acting on a Hilbert space $\mathscr{H}$. Suppose that $\mathscr{F}$ is a self-adjoint, abelian subset of $\mathscr{A}_f(\mathscr{R})$. Then there is an abelian von Neumann subalgebra of $\mathscr{R}$ with which every element in $\mathscr{F}$ is affiliated.*

***Proof:*** Since $\mathscr{F}$ is a self-adjoint family, it will suffice to show that the "real" and "imaginary" parts of each element in $\mathscr{F}$ are affiliated with some one abelian algebra. This follows immediately from the preceding proposition because that proposition assures us that all the spectral resolutions of this family of self-adjoint operators commute with one another. Thus the family of projections in all the spectral resolutions is a commuting self-adjoint family in $\mathscr{R}$ and generates an abelian subalgebra of $\mathscr{R}$ with which all the operators in $\mathscr{F}$ are affiliated.

**Proposition 29.** *Let $\mathscr{R}$ be a finite von Neumann algebra acting on a Hilbert space $\mathscr{H}$ and $\mathscr{A}$ be a maximal abelian self-adjoint (masa) subalgebra of $\mathscr{A}_{\mathrm{f}}(\mathscr{R})$. Let $\mathscr{A}_0$ be $\mathscr{A} \cap \mathscr{R}$. Then $\mathscr{A}_0$ is a maximal abelian self-adjoint subalgebra of $\mathscr{R}$. In addition, $\mathscr{A} = \mathscr{A}_{\mathrm{f}}(\mathscr{A}_0)$.*

***Proof:*** Suppose $H$ is a self-adjoint operator in $\mathscr{A}$ with spectral resolution $\{E_\lambda\}$. If $K$ is a self-adjoint operator in $\mathscr{A}$, $K$ commutes with $H$. From Proposition 26, $K$ commutes with $\{E_\lambda\}$. Since $\mathscr{A}$ is a masa, $E_\lambda \in \mathscr{A}$ for every $\lambda$. Hence all spectral resolutions of self-adjoint operators in $\mathscr{A}$ lie in $\mathscr{A}_0$. Suppose $A$ is a self-adjoint operator in $\mathscr{R}$ commuting with $\mathscr{A}_0$. Then, again, from Proposition 26, $A$ commutes with each self-adjoint operator $H$ in $\mathscr{A}$ (since $A$ commutes with the spectral resolution of $H$). By maximality (and self-adjointness), $A$ is in $\mathscr{A}$, hence in $\mathscr{A}_0$. Hence $\mathscr{A}_0$ is a masa in $\mathscr{R}$. [To see that $\mathscr{A}_0$ is a strong-operator-closed algebra, let $A$ be an operator in $\mathscr{R}$ that is a strong-operator limit of a net of operators $\{A_j\}$ in $\mathscr{A}_0$. Then any $B$ in $\mathscr{A}_0$ commutes with the operators in the net. Since $A_j \to A$ in the strong-operator topology, $A_jBx \to ABx$ for every $x \in \mathscr{H}$. At the same time, $A_jBx = BA_jx \to BAx$, from the continuity (boundedness) of $B$. Thus $A$ commutes with every $B$ in $\mathscr{A}_0$ and hence $A$ commutes with $\mathscr{A}_0$. From maximality of $\mathscr{A}_0$, $A$ is in $\mathscr{A}_0$. Hence, $\mathscr{A}_0$ is a strong-operator-closed subalgebra of $\mathscr{R}$.]

We show, next, that $\mathscr{A}$ is $\mathscr{A}_{\mathrm{f}}(\mathscr{A}_0)$. Suppose $H$ is a self-adjoint operator in $\mathscr{A}_{\mathrm{f}}(\mathscr{A}_0)$. Then the spectral resolution of $H$ lies in $\mathscr{A}_0$ hence in $\mathscr{A}$. Thus every self-adjoint operator in $\mathscr{A}$ commutes with this spectral resolution, hence with $H$ (Proposition 26). By maximality of $\mathscr{A}$, $H$ is in $\mathscr{A}$. Thus $\mathscr{A}_{\mathrm{f}}(\mathscr{A}_0) \subseteq \mathscr{A}$.

Suppose $H$ is a self-adjoint operator in $\mathscr{A}$. Its spectral resolution is in $\mathscr{A}_0$, and hence $H \in \mathscr{A}_{\mathrm{f}}(\mathscr{A}_0)$ since $H$ is affiliated with the von Neumann algebra generated by its spectral resolution. Thus $\mathscr{A} \subseteq \mathscr{A}_{\mathrm{f}}(\mathscr{A}_0)$.

**Proposition 30.** *If $\mathscr{R}$ is a finite von Neumann algebra acting on a Hilbert space $\mathscr{H}$ with center $\mathcal{C}$, then $\mathscr{A}_{\mathrm{f}}(\mathcal{C})$ is the center, $\mathscr{C}$, of $\mathscr{A}_{\mathrm{f}}(\mathscr{R})$.*

***Proof:*** If $C$ is in $\mathcal{C}$, then $C$ commutes with every projection in $\mathscr{R}$, hence, with every spectral resolution in $\mathscr{R}$, and therefore, with every self-adjoint element in $\mathscr{A}_{\mathrm{f}}(\mathscr{R})$. Thus $C$ lies in $\mathscr{C}$ and $\mathcal{C} \subseteq \mathscr{C}$.

From the properties of the adjoint operation, with $C$, now, in $\mathscr{C}$ and $A$ in $\mathscr{A}_{\mathrm{f}}(\mathscr{R})$,

$$C^* \hat{\cdot} A^* = (A \hat{\cdot} C)^* = (C \hat{\cdot} A)^* = A^* \hat{\cdot} C^*.$$

Since $\mathscr{A}_{\mathrm{f}}(\mathscr{R})$ is a self-adjoint algebra, $C^*$ is in $\mathscr{C}$. Thus $\mathscr{C}$ is a self-adjoint algebra. Hence it suffices to show that each self-adjoint element in $\mathscr{C}$ is in $\mathscr{A}_{\mathrm{f}}(\mathcal{C})$ in order to show $\mathscr{C} \subseteq \mathscr{A}_{\mathrm{f}}(\mathcal{C})$. For this, if $A$ is a self-adjoint operator in $\mathscr{C}$, it commutes with all elements of $\mathscr{A}_{\mathrm{f}}(\mathscr{R})$. From Proposition 26, the spectral resolution of $A$ commutes with all elements of $\mathscr{A}_{\mathrm{f}}(\mathscr{R})$. Therefore, the spectral resolution of $A$ lies in $\mathcal{C}$ and $A$ is in $\mathscr{A}_{\mathrm{f}}(\mathcal{C})$. Thus $\mathscr{C} \subseteq A_{\mathrm{f}}(\mathcal{C})$.

To show that $\mathscr{A}_{\mathrm{f}}(\mathcal{C}) \subseteq \mathscr{C}$, note first, that $\mathscr{A}_{\mathrm{f}}(\mathcal{C})$ is a self-adjoint family. Hence it suffices to show that each self-adjoint $A$ in $\mathscr{A}_{\mathrm{f}}(\mathcal{C})$ is in $\mathscr{C}$. In this case, the spectral resolution of $A$ lies in $\mathcal{C}$, hence in $\mathscr{C}$. Since the spectral resolution of $A$ commutes with every self-adjoint operator in $\mathscr{A}_{\mathrm{f}}(\mathscr{R})$, $A$ commutes with every self-adjoint operator in $\mathscr{A}_{\mathrm{f}}(\mathscr{R})$ and hence $A \in \mathscr{C}$. Thus $\mathscr{A}_{\mathrm{f}}(\mathcal{C}) = \mathscr{C}$.

***Definition 31:*** If $A$ is a closed, densely defined operator on a Hilbert space $\mathscr{H}$ and $B$ is a bounded, everywhere-defined operator on $\mathscr{H}$, we say that $A$ and $B$ commute when $BA \subseteq AB$.

Toward understanding what we should mean by "the commutant of $\mathscr{A}_{\mathrm{f}}(\mathscr{R})$," we prove the proposition that follows.

**Proposition 32.** *Suppose $H$ and $K$ are self-adjoint operators (possibly unbounded) acting on a Hilbert space $\mathscr{H}$ with spectral resolutions $\{E_\lambda\}_{\lambda\in\mathbb{R}}$ and $\{F_\lambda\}_{\lambda\in\mathbb{R}}$, respectively. Then the following conditions are equivalent:*

*i. $E_\lambda K \subseteq KE_\lambda$, for all $\lambda$ in $\mathbb{R}$; that is $E_\lambda$ and $K$ commute;*
*ii. $E_\lambda F_{\lambda'} = F_{\lambda'}E_\lambda$, for all $\lambda$ and $\lambda'$ in $\mathbb{R}$;*
*iii. $H$ and $K$ are affiliated with the (abelian) von Neumann algebra generated by $\{E_\lambda\}_{\lambda\in\mathbb{R}}$ and $\{F_\lambda\}_{\lambda\in\mathbb{R}}$;*
*iv. $F_\lambda H \subseteq HF_\lambda$, for all $\lambda$ in $\mathbb{R}$.*

***Proof:*** (i) → (ii) From Lemma 18, with $E_\lambda$ in place of $B$ and $K$ in place of $A$ in that lemma. Since $F_{\lambda'}E_\lambda$ is defined on all of $\mathscr{H}$, $F_{\lambda'}E_\lambda = E_\lambda F_{\lambda'}$, for all $\lambda$ and $\lambda'$ in $\mathbb{R}$.

(ii) → (iii) Since $E_\lambda F_{\lambda'} = F_{\lambda'}E_\lambda$ for all $\lambda$ and $\lambda'$ in $\mathbb{R}$, $\{E_\lambda, F_\lambda\}_{\lambda\in\mathbb{R}}$ generates an abelian von Neumann algebra $\mathscr{A}$ on $\mathscr{H}$ with which $H$ and $K$ are affiliated.

(iii) → (iv) Since $\mathscr{A}$ is abelian, $\mathscr{A}_{\mathrm{f}}(\mathscr{A})$ is abelian from Theorem 5.6.12 and Theorem 5.6.15 in refs. 7–9. As $H$ and $F_\lambda$ are in $\mathscr{A}_{\mathrm{f}}(\mathscr{A})$, $F_\lambda \hat{\cdot} H = H \hat{\cdot} F_\lambda$. Now,

$$F_\lambda H \subseteq F_\lambda \hat{\cdot} H = H \hat{\cdot} F_\lambda = HF_\lambda,$$

since $H$ is self-adjoint (hence, closed) and $F_\lambda$ is bounded.

By symmetry, (i) and (iv) are the same condition, so that we have proved the equivalence of (i), (ii), (iii) and (iv).

**Proposition 33:** *Suppose $T$ is a self-adjoint operator acting on a Hilbert space $\mathscr{H}$ and $B$ is a self-adjoint, everywhere-defined and bounded operator acting on $\mathscr{H}$ with spectral resolution $\{E_\lambda\}_{\lambda\in\mathbb{R}}$. Then the following conditions are equivalent:*

*i. $E_\lambda T \subseteq TE_\lambda$ for all $\lambda$ in $\mathscr{R}$;*
*ii. $BT \subseteq TB$.*

***Proof:*** (i) → (ii) It follows from Lemma 17 with $A$, $C$, $T$ and $\mathscr{F}$ of that lemma replaced by $T$, $E_\lambda$, $B$ and $\{E_\lambda\}_{\lambda\in\mathbb{R}}$, respectively.

(ii) → (i) From Lemma 18, $F_\lambda B \subseteq BF_\lambda$ for all $\lambda$ in $\mathscr{R}$, where $\{F_\lambda\}_{\lambda\in\mathbb{R}}$ is the spectral resolution of $T$. From the preceding proposition, $E_\lambda T \subseteq TE_\lambda$.

***Definition 34:*** Suppose $S$ and $T$ are self-adjoint operators (possibly unbounded) acting on a Hilbert space $\mathscr{H}$. We say that $S$ commutes with $T$ if $S$ commutes with the spectral resolution of $T$.

***Remark 35:*** From Proposition 32, $S$ commutes with $T$ if and only if $T$ commutes with $S$ (if and only if their spectral resolutions commute), in which case we also say that $S$ and $T$ commute.

## 4 Main Theorem

***Definition 36:*** If $\mathscr{F}$ is a family of self-adjoint operators (possibly unbounded) acting on a Hilbert space $\mathscr{H}$, we call the set $\mathscr{F}^{\mathrm{sa}\,\prime}$ of all self-adjoint operators that commute with all the operators in $\mathscr{F}$ the *self-adjoint commutant* of $\mathscr{F}$ (written, "sa-commutant").

**Theorem 37.** *Let $\mathscr{R}$ be a von Neumann algebra. Suppose $\mathscr{S}$ is the family of all self-adjoint operators affiliated with $\mathscr{R}$. Then the double sa-commutant of $\mathscr{S}$ coincides with $\mathscr{S}$.*

***Proof:*** We prove first that $\mathscr{S}^{\mathrm{sa}\,\prime}$, the sa-commutant of $\mathscr{S}$, is the set of all self-adjoint operators affiliated with $\mathscr{R}'$. (So, $\mathscr{S}^{\mathrm{sa}\,\prime}$ is the set of self-adjoint elements in $\mathscr{A}_{\mathrm{f}}(\mathscr{R}')$ when $\mathscr{R}'$ is finite.) To see this, choose a self-adjoint operator $H'$ affiliated with $\mathscr{R}'$. Then the spectral resolution of $H'$ is in $\mathscr{R}'$. If $H$ is in $\mathscr{S}$, its spectral resolution is in $\mathscr{R}$, and hence, commutes with the spectral resolution of $H'$. By denition, $H'$ commutes with $H$. Since $H$ is an arbitrary element of $S$, $H'$ is in $\mathscr{S}^{\mathrm{sa}\,\prime}$.

If $K'$ is in $\mathscr{S}^{\mathrm{sa}\prime}$, then $K'$ commutes, in particular, with each self-adjoint operator in $\mathscr{R}$ and hence with every operator in $\mathscr{R}$. To see this, with $B$ in $\mathscr{R}$, $B = B_1 + iB_2$ where $B_1$ and $B_2$ are self-adjoint operators in $\mathscr{R}$. Since $B_1K' \subseteq K'B_1$, and $B_2K' \subseteq K'B_2$,

$$(B_1 + iB_2)K' = B_1K' + iB_2K' \subseteq K'B_1 + iK'B_2 \subseteq K'(B_1 + iB_2)$$

Since $\mathscr{R}$ is a von Neumann algebra, from von Neumann's double commutant theorem, $\mathscr{R} = \mathscr{R}'' = (\mathscr{R}')'$. It follow that $K'$ commutes with every operator in the commutant of $\mathscr{R}'$ and hence, by denition, $K'$ is affiliated with $\mathscr{R}'$.

We, now, apply what we have just proved to $\mathscr{S}^{\mathrm{sa}\prime}$ (the family of all self-adjoint operators affiliated with $\mathscr{R}'$). The sa-commutant of $\mathscr{S}^{\mathrm{sa}\prime}$ is the set of all self-adjoint operators affiliated with $(\mathscr{R}')'(=\mathscr{R})$. Therefore, the double sa-commutant of $\mathscr{S}$ coincides with $\mathscr{S}$.

***Definition 38:*** With $\mathscr{R}$ a finite von Neumann algebra acting on a Hilbert space $\mathscr{H}$, the *commutant* $\mathscr{A}_{\mathrm{f}}(\mathscr{R})'$ of $\mathscr{A}_{\mathrm{f}}(\mathscr{R})$ is the set of closed, densely defined operators $C'$ on $\mathscr{H}$ that commute with each self-adjoint operator $H$ in $\mathscr{R}$, that is, $E_\lambda C' \subseteq C'E_\lambda$, for each $E_\lambda$ in the spectral resolution of $H$.

**Proposition 39.** *With $\mathscr{R}$ a finite von Neumann algebra acting on a Hilbert space $\mathscr{H}$, $C' \in \mathscr{A}_{\mathrm{f}}(\mathscr{R})'$ if and only if $BC' \subseteq C'B$ for each $B$ in $\mathscr{R}$.*

***Proof:*** Suppose, first, that $C' \in \mathscr{A}_{\mathrm{f}}(\mathscr{R})'$. Then $EC' \subseteq C'E$ for each projection $E$ in $\mathscr{R}$. Since the set of projections in $\mathscr{R}$ is a self-adjoint family that generates the von Neumann algebra $\mathscr{R}$, $BC' \subseteq C'B$ for each $B$ in $\mathscr{R}$ from Lemma 17.

If $BC' \subseteq C'B$ for each $B$ in $\mathscr{R}$, then $EC' \subseteq C'E$ for each projection $E$ in the spectral resolution of a self-adjoint operator. Thus $C' \in \mathscr{A}_{\mathrm{f}}(\mathscr{R})'$ in this case.

**Theorem 40:** *If $\mathscr{R}$ is a finite von Neumann algebra acting on a Hilbert space $\mathscr{H}$ and $\mathscr{R}'$, the commutant of $\mathscr{R}$, is also finite (as a von Neumann algebra), then $\mathscr{A}_{\mathrm{f}}(\mathscr{R})' = \mathscr{A}_{\mathrm{f}}(\mathscr{R}')$, and*

$$\mathscr{A}_{\mathrm{f}}(\mathscr{R})'' = \mathscr{A}_{\mathrm{f}}(\mathscr{R}')' = \mathscr{A}_{\mathrm{f}}(\mathscr{R}'') = \mathscr{A}_{\mathrm{f}}(\mathscr{R}).$$

***Proof:*** Suppose $C' \in \mathscr{A}_{\mathrm{f}}(\mathscr{R})'$. From Proposition 39, $BC' \subseteq C'B$, for each $B$ in $\mathscr{R}$. In particular, then, $UC' \subseteq C'U$, for each unitary operator $U$ in $\mathscr{R}$. From 5.6(13) in refs. 7–9, $C' = U^{-1}UC' \subseteq U^{-1}C'U$, and $UC'U^{-1} \subseteq C'UU^{-1} = C'$. As this is true for each unitary operator $U$ in $\mathscr{R}$, replacing $U$ by $U^{-1}$ in the second inclusion, we have that $U^{-1}C'U \subseteq C'$, as well as, $C' \subseteq U^{-1}C'U$. Thus $C' = U^{-1}C'U$ for each unitary operator $U$ in $\mathscr{R}$. Hence $C'\eta\mathscr{R}'$ (since $\mathscr{R} = \mathscr{R}''$, from the von Neumann double commutant theorem). It follows that $C' \in \mathscr{A}_{\mathrm{f}}(\mathscr{R}')$ and that $\mathscr{A}_{\mathrm{f}}(\mathscr{R})' \subseteq \mathscr{A}_{\mathrm{f}}(\mathscr{R}')$.

Suppose, next, that $C' \in \mathscr{A}_{\mathrm{f}}(\mathscr{R}')$. Then $C'U = UC'$ for each unitary operator $U$ in $\mathscr{R}''(=\mathscr{R})$. Let $E$ be a projection in $\mathscr{R}$. Then $E + i(I - E)$ $(=U_+)$ and $E - i(I - E)$ $(=U_-)$ are unitary operators in $\mathscr{R}$ $(=\mathscr{R}'')$. Thus

$$2EC' = (U_+ + U_-)C' = U_+C' + U_-C' = C'U_+ + C'U_- \subseteq C'(U_+ + U_-) = 2C'E.$$

Hence $EC' \subseteq C'E$, for each projection $E$ in $\mathscr{R}$, and $C' \in \mathscr{A}_{\mathrm{f}}(\mathscr{R})'$. Therefore $\mathscr{A}_{\mathrm{f}}(\mathscr{R}') \subseteq \mathscr{A}_{\mathrm{f}}(\mathscr{R})'$ and $\mathscr{A}_{\mathrm{f}}(\mathscr{R})' = \mathscr{A}_{\mathrm{f}}(\mathscr{R})'$. The last line of the statement of this theorem now becomes the completion of this proof.


1. von Neumann J (1929) Zur Algebra der Funktionaloperationen und Theorie der normalen Operatoren. *Math Ann* 102:370–427.
2. Murray FJ, von Neumann J (1936) On rings of operators. *Ann of Math* 37:116–229.
3. Murray FJ, von Neumann J (1937) On rings of operators, II. *Trans Amer Math Soc* 41:208–248.
4. Murray FJ, von Neumann J (1943) On rings of operators, IV. *Ann of Math* 44:716–808.
5. von Neumann J (1940) On rings of operators, III. *Ann of Math* 41:94–161.
6. Dirac PAM (1947) *The Principles of Quantum Mechanics* (Clarendon Press, Oxford), 3rd ed.
7. Kadison RV, Ringrose JR (1983) *Fundamentals of the Theory of Operator Algebras*, (Academic, New York), Vol. I.
8. Kadison RV, Ringrose JR (1986) *Fundamentals of the Theory of Operator Algebras*, (Academic, Orlando, FL), Vol. II.
9. Kadison RV, Ringrose JR (1991) *Fundamentals of the Theory of Operator Algebras*, (Academic, Boston), Vol. III.
10. Liu Zhe (2011) On some mathematical aspects of the Heisenberg relation. *Science China Math* 54:2427–2452.