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# CLOSED DENSELY DEFINED OPERATORS COMMUTING WITH MULTIPLICATIONS IN A MULTIPLIER PAIR

DON HADWIN, ZHE LIU, AND ERIC NORDGREN

(Communicated by Richard Rochberg)

*Dedicated to the memory of Bill Arveson, an inspiration to us all*

Abstract. For a multiplier pair $(X, Y)$ we study the closed densely defined operators $T$ on $X$ that commute with all of the multiplications by right multipliers in $X$. We apply our general results to special cases involving $H^p$, completions of $L^\infty[0,1]$ with respect to certain norms, and the completion of a $II_1$ factor von Neumann algebra with respect to a unitarily invariant norm, where we show that each such $T$ is a "left multiplication". However, we give an example of a closed densely defined operator on the Bergman space that commutes with all multiplications by $H^\infty$-functions but is not a multiplication operator.

## 1. Introduction

In [6], [7] the first and third authors introduced and studied the notion of a *multiplier pair* $(X, Y)$, where $X$ is a Banach space that is a vector subspace of a Hausdorff topological vector space $Y$ with a separately continuous bilinear map (multiplication) $\cdot : X \times X \to Y$ such that the set $\mathcal{L}_0 = \{x \in X : x \cdot X \subseteq X\}$ of left multipliers and the set $\mathcal{R}_0 = \{x \in X : X \cdot x \subseteq X\}$ of right multipliers are dense in $X$, and such that there are dense subsets $\mathcal{E} \subseteq \mathcal{L}_0$, $\mathcal{F} \subseteq X$, $\mathcal{G} \subseteq \mathcal{R}_0$ such that

$$(a \cdot b) \cdot c = a \cdot (b \cdot c)$$

whenever $a \in \mathcal{E}$, $b \in \mathcal{F}$, $c \in \mathcal{G}$. Moreover, there is an $e \in X$ such that $e \cdot x = x \cdot e = x$ for every $x \in X$. It was shown that, for each $x \in X$, the linear transformations $R_x$ and $L_x$ defined by

$$L_x a = x \cdot a, \ R_x a = a \cdot x$$

are closed densely defined operators on $X$. Moreover, $L_x$ is bounded on $\mathcal{R}_0$ if and only if $x \in \mathcal{L}_0$ and $R_x$ is bounded on $\mathcal{L}_0$ if and only if $x \in \mathcal{R}_0$. Thus $\mathcal{R} = \{R_x : x \in \mathcal{R}_0\}$ and $\mathcal{L} = \{L_x : x \in \mathcal{L}_0\}$ are unital subalgebras of $B(X)$ (the set of bounded linear operators on $X$). It was proved in [6] that $\mathcal{L}$ and $\mathcal{R}$ are each other's commutant, which implies that $\mathcal{L}$ and $\mathcal{R}$ are closed in the weak operator topology. In the setting of multiplier pairs a general notion of a composition operator was defined.

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Suppose $B(W)$ denotes the Banach algebra of all bounded linear transformations on a Banach space $W$, and suppose $\mathcal{A}$, $\mathcal{B}$ are unital Banach subalgebras of $B(W)$ such that $\mathcal{B}' = \mathcal{A}$ and $\mathcal{A}' = \mathcal{B}$, where $\mathcal{S}'$ denotes the *commutant* of a set of operators $\mathcal{S}$, i.e., the set of operators commuting with every element of $\mathcal{S}$. Suppose $T$ is a linear transformation whose domain $\mathcal{D}(T)$ and range $\mathcal{R}(T)$ are linear subspaces of $W$. We say that $T$ *commutes* with $\mathcal{B}$ (or $T$ *is affiliated with* $\mathcal{A}$) if, for every $S \in \mathcal{B}$, we have $S(\mathcal{D}(T)) \subseteq \mathcal{D}(T)$ and, for every $x \in \mathcal{D}(T)$, we have $STx = TSx$. This is equivalent to saying that, for every invertible $S \in \mathcal{B}$, we have $ST = TS$. It is also equivalent to saying that, for every $S \in \mathcal{B}$,

$$(S \oplus S)\,\mathrm{Graph}\,(T) \subseteq \mathrm{Graph}(T),$$

where $\mathrm{Graph}\,(T) = \{(x, y) \in W \times W : y = Tx\}$ is the graph of $T$. It easily follows that if $\mathrm{Graph}\,(T)$ is closed, then the set

$$\mathcal{S} = \{S \in B(W) : S(\mathcal{D}(T)) \subseteq \mathcal{D}(T), \forall x \in \mathcal{D}(T) \;\; STx = TSx\}$$

is a unital algebra that is closed in the weak operator topology. In order to show that $\mathcal{B} \subseteq \mathcal{S}$, it is sufficient to show that $\mathcal{S}$ contains a set of operators $\mathcal{B}_0$ such that the unital weak operator closed algebra generated by $\mathcal{B}_0$ is $\mathcal{B}$.

In this paper we study the problem of determining for a multiplier pair $(X, Y)$ the closed densely defined operators $T$ on $X$ that commute with $\mathcal{R}$ (i.e., are affiliated with $\mathcal{L}$). The symmetry of the situation makes this problem "equivalent" to that of finding the closed densely defined operators on $X$ that commute with $\mathcal{L}$ (i.e., are affiliated with $\mathcal{R}$).

In [6] it was shown that, for every $x \in X$, $L_x$ commutes with every operator in $\mathcal{R}$ and $R_x$ commutes with every operator in $\mathcal{L}$. In general, we expect that the closed densely defined operators commuting with $\mathcal{R}$ should be left multiplications of some sort. We will prove general results that affirm this notion in a large number of cases, but, when $X$ is the Bergman space on the unit disk, we construct a closed densely defined operator commuting with $\mathcal{L} = \mathcal{R} = H^\infty$ that is not a multiplication by any function.

Examples of this problem have been studied by D. Suárez [18] and S. Seubert [17] in the case of the algebra generated by the unilateral shift operator, by D. Sarason [13] in the case of a restricted shift operator [14], [15], and by H. Bercovici [1] in the case of a $C_0$-contraction. A result for von Neumann algebras was proved by Nelson [11]. The problem for $L^\infty[0,1]$ acting on $L^2[0,1]$ was discussed in [9].

In [6] many examples of multiplier pairs were constructed. Many of them satisfy more conditions than assumed in the definition. We define a multiplier pair $(X, Y)$ to be a *special multiplier pair* if the multiplication $\cdot : Y \times Y \to Y$ is defined and separately continuous and $(Y, +, \cdot)$ is a ring with identity $e$.

## 2. General results

We begin with algebraic results that will apply to special multiplier pairs. If $\mathcal{R}$ is a ring and $\mathcal{S} \subseteq \mathcal{R}$, we say that $\mathcal{S}$ is *left-separating* if, for every $x \in \mathcal{R}$, $x \cdot \mathcal{S} = \{0\}$ implies $x = 0$.

**Theorem 1.** *Suppose $1 \in Y$ is a ring, $1 \in \mathcal{R}$ is a subring of $Y$ and $\mathcal{R} \subset X \subset Y$, where $X$ is a right $\mathcal{R}$-module. Suppose $G \subset X \times X$ is a graph and a right $\mathcal{R}$-module*

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*containing an element* $(g,h)$ *such that*

(1) $g$ *has a left-inverse* $g^{-1}$ *in* $Y$,
(2) *for every* $x \in X$, $\{u \in \mathcal{R} : \exists v \in \mathcal{R}$ *such that* $xu = gv\}$ *is left-separating for* $Y$.

*Then* $G \subseteq \text{Graph}\left(L_{hg^{-1}}\right)$.

*Proof.* Suppose $(x,y) \in G$, $u$,$v \in \mathcal{R}$ and $xu = gv$. Then $(x,y)\,u - (g,h)\,v = (0, yu - hv) \in G$. Since $(0,0) \in G$ and $G$ is a graph, we know $yu - hv = 0$. Thus $yu = hv = hg^{-1}(gv) = hg^{-1}(xu)$. Hence, for every $u \in \mathcal{R}$ for which there is a $v \in \mathcal{R}$ such that $xu = gv$, we have

$$\left[y - L_{hg^{-1}}x\right] u = 0.$$

It follows from (2) that $y = L_{hg^{-1}}x$ for every $(x,y) \in G$. □

If $R$ is an integral domain, the field of fractions $\hat{R}$ of $R$ is the field of formal quotients $\frac{a}{b}$ with $a$,$b \in R$ and the natural addition and multiplication. We think of $R \subseteq \hat{R}$ by identifying $x \in R$ with $\frac{x}{1} \in \hat{R}$.

**Corollary 1.** *Suppose* $F$ *is the field of fractions of an integral domain* $R$, *and* $R \subset X \subset F$ *and* $X$ *is an* $R$*-module. Suppose* $G \subset X \times X$ *is a graph and an* $R$*-module. Then there is a* $\varphi \in F$ *such that* $G$ *is contained in the graph of* $y = \varphi x$ *in* $F \times F$.

**Corollary 2.** *Suppose* $(X,Y)$ *is a special multiplier pair such that* $Y$ *is an integral domain and let* $\hat{Y}$ *denote the field of quotients over* $Y$. *Suppose also that* $X \subseteq \hat{\mathcal{R}}_0$. *If* $G \subseteq X \times X$ *is a graph that is an* $\mathcal{R}_0$*-module, then there is a* $\varphi \in \hat{\mathcal{R}}_0$ *such that* $G \subseteq \text{Graph}\,(L_\varphi)$. *Moreover, if the domain of* $G$ *is dense in* $X$ *and* $\tau : Y \to \mathbb{C}$ *is a unital algebra homomorphism whose restriction to* $X$ *is continuous, then there are* $f$, $g \in \mathcal{R}_0$ *such that* $\varphi = g/f$ *and* $\tau(f) \neq 0$.

*Proof.* The first part follows immediately from Theorem 1. Next, suppose the domain of $G$ is dense and $\tau : Y \to \mathbb{C}$ is a multiplicative linear functional whose restriction to $X$ is continuous. It follows that the domain of $G$ cannot be contained in $\ker \tau$. Hence there is an $(f,g) \in G$ such that $\tau(f) \neq 0$. Since $g = \varphi f$, we have that $\varphi = g/f$ and $\tau(f) \neq 0$. □

Throughout this paper we use (SOT) to denote the strong operator topology.

**Theorem 2.** *Suppose* $(X,Y)$ *is a multiplier pair and* $T$ *is a closed densely defined operator on* $X$ *commuting with every operator in* $\mathcal{R}$ *and such that* $T \subseteq L_y$ *for some* $y \in Y$. *Suppose also that, for some* $g \in \mathcal{D}(T)$ *and for every* $x \in X$, *there exist* $\{u_n\}$, $\{v_n\}$ *in* $\mathcal{R}_0$ *such that*

(1) $xu_n = gv_n \in \mathcal{D}(T) \subseteq \mathcal{D}(L_y)$,
(2) $R_{u_n} \to 1$ *(SOT)*.

*Then* $T = L_y$.

*Proof.* Suppose $x \in \mathcal{D}(L_y)$, so $yx \in X$, and choose $\{u_n\}$ and $\{v_n\}$ as above. Then $xu_n \to x$ and $(yx)\,u_n \to yx$, but $xu_n = gv_n \in D(T)$, so $yxu_n = Txu_n$. Hence $(x, yx) = \lim_{n\to\infty}(xu_n, Txu_n) \in \mathcal{G}(T)^- = \mathcal{G}(T)$. Hence $x \in \mathcal{D}(T)$. Thus $\mathcal{D}(T) = \mathcal{D}(L_y)$, so $T = L_y$. □

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**Theorem 3.** *Suppose $(X, Y)$ is a special multiplier pair such that, for each $y \in Y$, there is a sequence $\{q_n\}$ in $\mathcal{R}_0$ such that*

(1) $R_{q_n} \to 1$ *in the (SOT) on* $X$,
(2) $yq_n \in \mathcal{R}_0$ *for every* $n$.

*If $T$ is a closed densely defined operator on $X$ commuting with $\mathcal{R}$ and there is a $g \in \mathcal{D}(T)$ that is invertible in $Y$, then $T = L_{T(g)g^{-1}}$.*

*Proof.* It follows from condition (1) that $\|e - eq_n\| \to 0$, which, from the separate continuity of multiplication in $Y$, implies $w - wq_n = w \cdot (e - eq_n) \to 0$ and $w - q_n w = (e - eq_n) \cdot w \to 0$ in $Y$. Hence $\{q_1, q_2, \ldots\}$ is both left-separating and right-separating in $Y$. Suppose $x \in X$ and let $y = g^{-1}x \in Y$. Choose $\{q_n\}$ as above and note that

$$xq_n = g(yq_n),$$

so, by Theorem 1, we see that $T \subseteq L_{T(g)g^{-1}}$. If $x \in \mathcal{D}\left(L_{T(g)g^{-1}}\right)$, then $xq_n = R_{yq_n}(g) \in \mathcal{D}(T)$ (since $T$ commutes with $R_{T(g)g^{-1}}$), $\|xq_n - x\| \to 0$ and $T(xq_n) = L_{T(g)g^{-1}}(xq_n) = \left(L_{T(g)g^{-1}}x\right)q_n \to L_{T(g)g^{-1}}x$. Since the operator $T$ is closed, $\left(x, L_{T(g)g^{-1}}x\right)$ is in the graph of $T$. Hence $T = L_{T(g)g^{-1}}$. $\square$

We now consider a special case in which $X$ is a Hilbert space and $\mathcal{L}$ and $\mathcal{R}$ are von Neumann algebras.

**Lemma 1.** *Suppose $(X, Y)$ is a multiplier pair, $X$ is a Hilbert space and $\mathcal{L}$ is a von Neumann algebra. Suppose $T$ is a closed densely defined operator affiliated with $\mathcal{L}$. Then there are $u$,$v \in \mathcal{L}$ with $0 \leq u$ and $\ker u = 0$ such that $T \subseteq L_{u^{-1}v}$.*

*Proof.* Since $\mathrm{Graph}(T)$ is closed and invariant for the von Neumann algebra $\mathcal{M} = \{A \oplus A : A \in \mathcal{R}\}$, the projection $P$ from $X \oplus X$ onto $\mathrm{Graph}(T)$ is in the commutant of $\mathcal{M}$, which is $\mathcal{M}_2(\mathcal{R}') = \mathcal{M}_2(\mathcal{L})$. Hence, we can write $P = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ with $A$,$B$,$C$,$D \in \mathcal{L}$. If $\begin{pmatrix} x \\ Tx \end{pmatrix} \in \mathrm{Graph}(T)$, then

$$\begin{pmatrix} x \\ Tx \end{pmatrix} = P\begin{pmatrix} x \\ Tx \end{pmatrix} = \begin{pmatrix} Ax + BTx \\ Cx + DTx \end{pmatrix},$$

which implies

$$Tx = Cx + DTx$$

or

$$(1 - D)\,Tx = Cx.$$

Suppose $x \in X$, $\|x\| = 1$ and $Dx = x$.

Then

$$\left\|P\begin{pmatrix} 0 \\ x \end{pmatrix}\right\|^2 = \left(P\begin{pmatrix} 0 \\ x \end{pmatrix}, \begin{pmatrix} 0 \\ x \end{pmatrix}\right) = \left(\begin{pmatrix} Bx \\ Dx \end{pmatrix}, \begin{pmatrix} 0 \\ x \end{pmatrix}\right)$$
$$= (Dx, x) = \left\|\begin{pmatrix} 0 \\ x \end{pmatrix}\right\|^2.$$

Thus $P\begin{pmatrix} 0 \\ x \end{pmatrix} = \begin{pmatrix} 0 \\ x \end{pmatrix}$, which means $\begin{pmatrix} 0 \\ x \end{pmatrix} \in \mathrm{Graph}(T)$, which, in turn, implies $x = T0 = 0$. Thus $\ker(1 - D) = 0$. We have from $(1 - D)\,Tx = Cx$ that $Tx = (1 - D)^{-1}\,Cx$ for every $x \in \mathcal{D}(T)$. $\square$

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We now look at a case that will apply to operator algebras with a *strictly cyclic separating vector*. Suppose $X$ is a Banach space and $\mathcal{A}$ is a norm closed unital subalgebra of $B(X)$ and suppose $0 \neq e \in X$ satisfies

(1) $\mathcal{A}e = X$ (i.e., $e$ is a strictly cyclic vector for $\mathcal{A}$),
(2) $e$ is a separating vector for $\mathcal{A}$, i.e., for every $A \in \mathcal{A}$,

$$Ae = 0 \Longrightarrow A = 0.$$

We can define a multiplication $\cdot$ on $X$ by

$$(Ae) \cdot (Be) = (AB)\, e.$$

In this case if we let $X = Y$, we have $(X, Y)$ as a multiplier pair with $\mathcal{L}_0 = \mathcal{R}_0 = X$.

Conversely, suppose $(X, Y)$ is a multiplier pair with $\mathcal{L}_0 = \mathcal{R}_0 = X$. Then $e$ is a strictly cyclic separating vector for $\mathcal{L} = \{L_x : x \in \mathcal{L}_0\}$, since $\mathcal{L}e = \mathcal{L}_0 = X$ and since $L_x e = 0$ implies $xe = x = 0$.

**Proposition 1.** *Suppose $(X, Y)$ is a multiplier pair and $\mathcal{L}_0 = \mathcal{R}_0 = X$. Suppose $T$ is a closed densely defined operator on $X$ affiliated with $\mathcal{L}$. Then $T \in \mathcal{L}$.*

*Proof.* The map $\gamma : \mathcal{L} \to X$ defined by $\gamma(L_x) = x$ is continuous and bijective. It follows from the open mapping theorem that $\gamma^{-1}$ is continuous. Since $\mathcal{D}(T)$ is dense in $X$, there is a $g \in \mathcal{D}(T)$ with $\|e - g\| < 1/\left\|\gamma^{-1}\right\|$, which means that $\|1 - L_g\| = \left\|\gamma^{-1}(e - g)\right\| < 1$. Hence $L_g$ is invertible in $B(X)$ and in the commutant of $\mathcal{R}$. Hence there is an $h \in \mathcal{L}_0$ such that $\eta = hg = e$. It follows from the fact that $\mathcal{D}(T) \subseteq \mathcal{R}(\mathcal{D}(T))$ that $\mathcal{D}(T) = X$, which implies $T \in \mathcal{R}' = \mathcal{L}$. □

We conclude with some results in a different direction.

**Proposition 2.** *Suppose $(X, Y)$ is a multiplier pair. Then:*

(1) *If $a \in X$, then $L_a$ is a closed densely defined operator on $X$ that commutes with every operator in $\mathcal{R}$.*
(2) *If $(X, Y)$ is a special multiplier pair and $y \in Y$, then $L_y$ is a closed operator on $X$ that commutes with every operator in $\mathcal{R}$.*
(3) *If $(X, Y)$ is a special multiplier pair, $y \in Y$ and, for each $x \in \mathcal{R}_0$, there is a sequence $\{e_n\}$ in $\mathcal{L}_0$ such that*
  (a) *$\|e_n x - x\| \to 0$,*
  (b) *$ye_n \in X$ for every $n \geq 1$,*
  *then $\mathcal{D}(L_y)$ is dense in $X$.*

*Proof.* (1) This was proved in [6].

(2) If $\{(x_n, yx_n)\}$ is a sequence in Graph$(L_y)$ and $\|(x_n, yx_n) - (x, w)\| \to 0$, then $\|x_n - x\| \to 0$ and $\|yx_n - w\| \to 0$. Thus $x_n \to x$ and $yx_n \to w$ in $Y$. Since $(X, Y)$ is a special multiplier pair, multiplication is separately continuous, so $yx_n \to yx$ in $Y$. This implies $w = yx$. Thus $L_y$ is closed. Since $(Y, +, \cdot)$ is a ring, $L_y$ is in the commutant of $\mathcal{R}$.

(3) It is clear from conditions (a) and (b) that the closure of $\mathcal{D}(L_y)$ contains $\mathcal{R}_0$, so $\mathcal{D}(L_y)$ is dense in $X$. □

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## 3. Applications

We now apply the results of the preceding section to some special cases of multiplier pairs.

### 3.1. Analytic functions.

Suppose $1 \leq p < \infty$ and $X = H^p$ (on the unit disk) and that $Y = N$ is the set of meromorphic functions in the Nevanlinna class, i.e., functions of the form $f/g$ with $f,g \in H^\infty$ and $g$ not identically 0. Then $(X,Y)$ is a special multiplier pair with $\mathcal{L}_0 = \mathcal{R}_0 = H^\infty$. The Smirnov class $N^+$ consists of all members of $N$ having a denominator that is an outer function. Sarason [13] has observed that the closed densely defined operators that commute with the unilateral shift on $H^2$ are multiplications induced by members of the Smirnov class, and we prove it here more generally in Theorem 4 as a consequence of Corollary 1. Sarason also observed [14] that members of $N^+$ have a canonical form related to $H^2$, and the following lemma shows that the analogous result related to $H^p$ holds for $1 \leq p < \infty$ and is established by the same proof.

**Lemma 2.** *If $1 \leq p < \infty$, $\phi \in N$, and $\phi \neq 0$, then there exist relatively prime inner functions $u$ and $v$ and outer functions $a$ and $b$ satisfying $|a|^p + |b|^p = 1$ a.e. on the unit circle such that*

$$\phi = \frac{vb}{ua}.$$

*Proof.* Recall that an outer function is positive at zero and is uniquely determined by its absolute boundary values, which are necessarily absolutely log integrable. Suppose $\phi$ is a nonzero function in $N$ and the inner-outer factorization is applied to each of the numerator and denominator of $\phi$, so

$$\phi = \frac{uf_1}{vf_2},$$

where $u$ and $v$ are relatively prime inner functions and $f_1$ and $f_2$ are outer functions in $H^\infty$.

Observe that on the unit circle $\mathbb{T}$,

$$\max\{|f_1|, |f_2|\} \leq (|f_1|^p + |f_2|^p)^{1/p} \leq |f_1| + |f_2|,$$

and therefore $(|f_1|^p + |f_2|^p)^{1/p}$ is log integrable. Thus there exists an outer function $\psi$ in $H^\infty$ such that $|\psi| = (|f_1|^p + |f_2|^p)^{1/p}$ a.e. on $\mathbb{T}$. Put $a = f_2/\psi$ and $b = f_1/\psi$ and observe that the definition of $\psi$ implies that $|a|^p + |b|^p = 1$ a.e. on $\mathbb{T}$. The asserted representation of $\phi$ follows. □

**Corollary 3.** *If $\phi \in N$, where $\phi = (vb)/(ua)$ as in Lemma 2, then the graph* $\text{Graph}(M_\phi)$ *of $M_\phi$ is the closed subset $\{(uag) \oplus (vbg) : g \in H^p\}$ of $H^p \oplus H^p$.*

*Proof.* If $g \in H^p$, then $uag \in H^p$ and $M_\phi uag = vbg \in H^p$, and it follows that the set asserted to be the graph of $M_\phi$ is a subset of the graph. For the opposite inclusion suppose both $f$ and $\phi f$ belong to $H^p$. Then both $|f|^p$ and $|\phi|^p\,|f|^p$ are integrable on $\mathbb{T}$, and because

$$\frac{|f|^p}{|a|^p} = \frac{|a|^p + |b|^p}{|a|^p}|f|^p = |f|^p + |\phi|^p|f|^p$$

on $\mathbb{T}$, it follows that if $g_1 = f/a$, then $g_1 \in H^p$. (If $f \in H^p$, $a$ is outer, and $f/a \in L^p$, then $f/a \in H^p$. See Nikolskii [12, Theorem 3.9.6].) Thus $f = ag_1$ and $u\phi f = vbg_1$. Since $u$ and $v$ are relatively prime and $b$ is outer, the last equation

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shows that $u$ is a factor of $g_1$, and thus $g_1 = ug$ for some $g \in H^p$. We have shown that $f = aug$ and $\phi f = vbg$, and thus the required inclusion is established. □

**Theorem 4.** *Suppose $1 \leq p < \infty$ and $\mathcal{G} \subseteq H^p \oplus H^p$ is a graph that is invariant under $M_z \oplus M_z$. Then there is a meromorphic $\phi \in N$ such that $\mathcal{G} \subseteq \mathrm{Graph}(M_\phi)$. If the domain of $\mathcal{G}$ is dense in $H^p$, then $\phi$ is in the Smirnov class. If, in addition, $\mathcal{G}$ is closed, then $\phi$ is in the Smirnov class and $\mathcal{G} = \mathrm{Graph}\,(M_\phi)$.*

*Proof.* The first assertion follows from Corollary 1. Suppose the domain $\mathcal{D}(\mathcal{G})$ of $\mathcal{G}$ is dense. Then for every $z$ with $|z| < 1$, there is a $g \in \mathcal{D}(\mathcal{G})$ such that $g(z) \neq 0$, so $\phi g \in H^p$ implies that $\phi$ has a removable singularity at $z$. Hence $\phi$ is analytic on the open unit disk. By Corollary 3, the domain of $M_\phi$, which includes that of $\mathcal{G}$, is $uaH^p$. Thus $uaH^p$ is dense, and it follows that $u = 1$. Thus $\phi$ is in $N^+$.

Suppose that $\mathcal{G}$ is closed. If $H^p \oplus H^p$ is given the norm defined by $\|f \oplus g\| = (\|f\|_p^p + \|g\|_p^p)^{1/p}$, then the proof of Corollary 3 shows that the map $V : H^p \to H^p \oplus H^p$ defined by

$$Vg = uag \oplus vbg$$

is an isometry of $H^p$ onto $\mathrm{Graph}(M_\phi)$. Let $\mathcal{M}$ be the inverse image of $\mathcal{G}$ under $V$. Then $\mathcal{G}$ is a closed subspace of $H^p$, and for $g \in \mathcal{M}$ we have

$$VM_z g = uazg \oplus vbzg = M_z \oplus M_z V g \in \mathcal{G}.$$

Hence $\mathcal{M}$ is invariant under $M_z$, and, by Duren [4, Theorem 7.4] or Helson [10, page 25], $\mathcal{M} = \omega H^p$ for some inner function $\omega$, and thus

$$\mathcal{G} = \{\omega uag \oplus \omega vbg : g \in H^p\} = (M_\omega \oplus M_\omega)\mathrm{Graph}(M_\phi).$$

It follows that if the domain of $\mathcal{G}$ is dense, then $\omega$, as well as $u$, must be 1, and hence the asserted equality holds. □

As a corollary to the proof we have the following.

**Corollary 4.** *If $1 \leq p < \infty$ and $\mathcal{G} \subseteq H^p \oplus H^p$ is a closed graph that is invariant under $M_z \oplus M_z$, then there is a meromorphic function $\phi$ in the Nevanlinna class and an inner function $\omega$ such that*

$$\mathcal{G} = (M_\omega \oplus M_\omega)\mathrm{Graph}(M_\phi).$$

The following is a direct consequence of the preceding corollary (i.e., $\omega$ is constant).

**Corollary 5.** *Suppose $1 \leq p < \infty$ and $\mathcal{G} \subseteq H^p \oplus H^p$ is a closed graph that is invariant under $M_z \oplus M_z$ and the sum of its domain and range is dense in $H^p$. Then it is the graph of some meromorphic function in the Nevanlinna class.*

3.2. **Measure theory.** A *symmetric norm* on $L^\infty[0,1]$, with respect to Lebesgue measure $\mu$, is a norm $\alpha$ such that

1. $\alpha(f) = \alpha(|f|)$ for every $f \in L^\infty[0,1]$,
2. $\alpha(1) = 1$,
3. $\alpha(f \circ \tau) = \alpha(f)$ for every $f \in L^\infty[0,1]$ and every invertible measure-preserving $\tau : [0,1] \to [0,1]$.

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We say that a symmetric norm $\alpha$ on $L^{\infty}[0,1]$ is *continuous* if

$$\lim_{t\to 0^{+}} \alpha\left(\chi_{[0,t]}\right) = 0.$$

We define $L^{\alpha}[0,1]$ to be the completion of $L^{\infty}[0,1]$ with respect to the norm $\alpha$. We can realize the elements of $L^{\alpha}[0,1]$ as (equivalence classes of) measurable functions in $L^{1}[0,1]$. Moreover, if $X = L^{\alpha}[0,1]$ and $Y$ is the set of all measurable functions on $[0,1]$ with the topology of convergence in measure, then $(X,Y)$ is a special multiplier pair with pointwise multiplication, and $\mathcal{R}_0 = \mathcal{L}_0 = L^{\infty}[0,1]$ (see [7]). Moreover, if $\alpha$ is continuous, we have

$$\lim_{\mu(F)\to 0} \alpha\left(\chi_F h\right) = 0$$

for every $h \in L^{\alpha}[0,1]$.

**Proposition 3.** *Suppose $\alpha$ is a continuous symmetric norm on $L^{\infty}[0,1]$ and $\mathcal{G} \subseteq L^{\alpha}[0,1] \oplus L^{\alpha}[0,1]$ is a closed linear subspace and a graph that is invariant under $M_f \oplus M_f$ for every $f \in L^{\infty}[0,1]$. Then there is a measurable function $\varphi$ (with possibly infinite values) such that $\mathcal{G} \subseteq \mathcal{G}(M_{\varphi})$.*

*Proof.* Since $\mathcal{D}(\mathcal{G})$ is an $L^{\infty}[0,1]$-module, we have $|g| \in \mathcal{D}(\mathcal{G})$ whenever $g \in \mathcal{D}(\mathcal{G})$. Moreover, if $\{g_n\}$ is a sequence in $\mathcal{D}(\mathcal{G})$ and $(|g_n|, h_n) \in \mathcal{G}$ for $n \geq 1$, then $g = \sum_{n=1}^{\infty} t_n |g_n| \in \mathcal{D}(\mathcal{G})$, where $0 < t_n < \frac{1}{2^n[\alpha(g_n)+\alpha(h_n)+1]}$ for $n \geq 1$. Since $\{x : g(x) > 0\} = \bigcup_{n\geq 1} \{x : |g_n(x)| > 0\}$, if we choose $g_n$ so that

$$\mu\left(\{x : g_n(x) \neq 0\}\right) > \sup\left\{\mu\left(\{x : h(x) \neq 0\}\right) : h \in \mathcal{D}(\mathcal{G})\right\} - \frac{1}{n},$$

then we see that

$$\mu\left(\{x : g(x) \neq 0\}\right) = \sup\{\mu\left(\{x : h(x) \neq 0\}\right) : h \in \mathcal{D}(\mathcal{G})\}.$$

If $E = \{x : g(x) \neq 0\}$, then $\chi_E h = h$ a.e. $(\mu)$ for every $h \in \mathcal{D}(\mathcal{G})$, and, since $\mathcal{G}$ is a graph and an $L^{\infty}[0,1]$-module, we see that $\chi_E h = h$ a.e. $(\mu)$ for every $h$ in the range of $\mathcal{G}$. We can now replace $L^{\alpha}[0,1]$ with $\chi_E L^{\alpha}[0,1] = X$ and let $Y$ be the set of measurable functions that vanish on $[0,1]\setminus E$. Now $g$ has a multiplicative inverse, and, for each $h \in X$ and each positive integer $n \geq 1$, we have $h\chi_{E_n} = g\left(\chi_{E_n} h/g\right)$, where $E_n = \{x : g(x) \geq 1/n \text{ and } |f(x)| \leq n\}$. Since $\bigcup_{n\geq 1} E_n = E$ (a.e.), the set $\{\chi_{E_n} : n \geq 1\}$ is left separating for $Y$. Applying Theorem 1 to $\chi_E \mathcal{G}$, we see that

$$\mathcal{G} = \chi_E \mathcal{G} \subseteq L_{f/g},$$

where $(g,f) \in \mathcal{G}$. Moreover, the fact that $\mu(E\setminus E_n) \to 0$ and $\alpha$ is continuous implies

$$R_{\chi_{E_n}} \to 1$$

in the strong operator topology. It follows from Theorem 2 that $\chi_E \mathcal{G}$ is the graph of $L_{f/g}$ on $\chi_E L^{\alpha}[0,1]$. If we let $\varphi = \frac{f}{g}\chi_E + \infty\chi_{([0,1]\setminus E)}$, we see that $\mathcal{G} = \mathcal{G}(L_{\varphi})$. ☐

**Corollary 6.** *Suppose $\alpha$ is a continuous symmetric norm on $L^{\infty}[0,1]$ and $T$ is a closed densely defined linear transformation on $L^{\alpha}[0,1]$ that commutes with $L_f$ for every $f \in L^{\infty}[0,1]$. Then there is a measurable function $\varphi : [0,1] \to \mathbb{C}$ such that $T = L_{\varphi}$.*

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**Theorem 5.** *Suppose $\alpha$ is a symmetric norm on $L^{\infty}[0,1]$ and $T$ is a closed densely defined operator on $L^{\alpha}[0,1]$ that commutes with $M_x$. Then there is a measurable function $\varphi : [0,1] \to \mathbb{C}$ such that $T = M_{\varphi}$.*

*Proof.* If $\alpha$ is continuous, the desired conclusion follows from Corollary 6. If $\alpha$ is not continuous, then $\alpha$ is equivalent to $\|\cdot\|_{\infty}$ on $L^{\infty}[0,1]$ (see [7]), so $L^{\alpha}[0,1] = L^{\infty}[0,1]$, and the desired conclusion follows from Proposition 1. □

3.3. **Finite von Neumann algebras.** Suppose $\mathcal{M}$ is a $II_1$ factor von Neumann algebra with a faithful normal tracial state $\tau$. A *unitarily invariant norm* on $\mathcal{M}$ is a norm $\beta$ such that

(1) $\beta(UTV) = \beta(T)$ for all $U,T,V \in \mathcal{M}$ with $U,V$ unitary,
(2) $\beta(1) = 1$.

Since $\mathcal{M}$ is a $II_1$ factor, there is a chain $\{P_t : t \in [0,1]\}$ of projections in $\mathcal{M}$ such that $\tau(P_t) = t$ for every $t \in [0,1]$. The map $\chi_{[0,t]} \mapsto P_t$ extends to a unital $*$-homomorphism $\pi : L^{\infty}[0,1] \to \mathcal{M}$ such that

$$\tau(\pi(f)) = \int_0^1 f(x)\,dx$$

for every $f \in L^{\infty}[0,1]$. If we define $\alpha : L^{\infty}[0,1] \to [0,\infty)$ by

$$\alpha(f) = \beta(\pi(f)),$$

we obtain a symmetric norm $\alpha$. It turns out that $\alpha$ is independent of the chain $\{P_t : t \in [0,1]\}$. Moreover, every element $A \in \mathcal{M}$ has a polar decomposition $A = U(A^*A)^{1/2}$ and $\beta(A) = \beta\left((A^*A)^{1/2}\right)$. There is a chain $\{P_t : t \in [0,1]\}$ whose generated von Neumann algebra contains $(A^*A)^{1/2}$, and there is a $\varphi \in L^{\infty}[0,1]$ such that $\pi(\varphi) = (A^*A)^{1/2}$, and we get $\beta(A) = \alpha(\varphi)$. In this way we can recapture $\beta$ from $\alpha$. It is shown in [5] that every symmetric norm $\alpha$ on $L^{\infty}[0,1]$ determines a unitarily invariant norm $\beta$ on $\mathcal{M}$, and we will call this norm $\|\cdot\|_{\alpha}$. We let $L^{\alpha}(\mathcal{M},\tau)$ denote the completion of $\mathcal{M}$ with respect to the norm $\|\cdot\|_{\alpha}$. If the symmetric norm $\alpha$ is continuous, then

$$\lim_{\tau(P)\to 0, P=P^*=P^2\in\mathcal{M}} \|P\|_{\alpha} = 0.$$

If $\alpha$ is not continuous, then $\alpha$ is equivalent to $\|\cdot\|_{\infty}$ on $L^{\infty}[0,1]$ and $\|\cdot\|_{\alpha}$ is equivalent to the operator norm on $\mathcal{M}$, so $L^{\beta}(\mathcal{M},\tau) = \mathcal{M}$.

There is a topology of *convergence in measure* on $\mathcal{M}$ (see [11]), and $L^{\alpha}(\mathcal{M},\tau)$ can be viewed as a subset of the completion $\mathcal{Y}$ of $\mathcal{M}$ with respect to this topology. It was shown in [7] that $(L^{\alpha}(\mathcal{M},\tau), \mathcal{Y})$ is a multiplier pair and $\mathcal{L}_0 = \mathcal{R}_0 = \mathcal{M}$, and it follows from [11, Theorem 1] that $(L^{\alpha}(\mathcal{M},\tau), \mathcal{Y})$ is a special multiplier pair.

**Theorem 6.** *Suppose $\mathcal{M}$ is a $II_1$ factor von Neumann algebra with faithful normal tracial state $\tau$, and suppose $\alpha$ is a unitarily invariant norm on $\mathcal{M}$. Suppose $T$ is a closed densely defined operator on $L^{\alpha}(\mathcal{M},\tau)$ that commutes with $R_S$ for every $S \in \mathcal{M}$. Then:*

(1) *If $\alpha$ is not continuous, then $T \in \mathcal{L}$.*
(2) *If $\alpha$ is continuous, then $T = L_y$ for some $y \in \mathcal{Y}$.*
(3) *If $\alpha$ is continuous and $y \in \mathcal{Y}$, then $L_y$ is a closed densely defined operator on $L^{\alpha}(\mathcal{M},\tau)$ that commutes with $R_S$ for every $S \in \mathcal{M}$.*

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*Proof.* (1) If $\alpha$ is not continuous, then $\alpha$ is equivalent to the operator norm on $\mathcal{M}$, so $L^{\alpha}(\mathcal{M},\tau) = \mathcal{M}$, which implies $\mathcal{L}_0 = \mathcal{R}_0 = \mathcal{M}$. The desired conclusion now follows from Proposition 1.

(2) Suppose $x \in \mathcal{D}(T)$. It follows from [11, Theorem 2] that there is a sequence $\{p_n\}$ of projections in $\mathcal{M}$ such that $\tau(p_n) \to 0$ and $x(1-p_n) \in \mathcal{M}$ for each $n \geq 1$, and, since $\mathcal{D}(T)(1-p_n) \subseteq \mathcal{D}(T)$, we have $x(1-p_n) \in \mathcal{M} \cap \mathcal{D}(T)$ for each $n \geq 1$. However, $\alpha$ is continuous, so

$$\alpha(x-(1-p_n)x) = \alpha(p_n x) \leq \alpha(p_n)\|x\| \to 0.$$

Thus the closure of $\mathcal{M}\cap\mathcal{D}(T)$ contains $\mathcal{D}(T)$, and since $\mathcal{D}(T)$ is dense in $L^{\alpha}(\mathcal{M},\tau)$, it follows that $\mathcal{M}\cap\mathcal{D}(T)$ is dense in $L^{\alpha}(\mathcal{M},\tau)$. For each $x \in \mathcal{M}\cap\mathcal{D}(T)$, let $P_x$ denote the projection onto the weak operator topology closure of the range of $x$, and choose a sequence $\{x_n\}$ in $\mathcal{M}\cap\mathcal{D}(T)$ such that

$$\lim \tau(P_{x_n}) = \sup\{\tau(P_x) : x \in \mathcal{M}\cap\mathcal{D}(T)\}.$$

For each $n \geq 1$ we can write $x_n = (x_n x_n^*)^{\frac{1}{2}} u_n$ with $u_n$ unitary, and so $(x_n x_n^*)^{\frac{1}{2}} = R_{u_n^*} x_n \in \mathcal{M}\cap\mathcal{D}(T)$. If we choose $t_n$ so that

$$t_n\left[\alpha\left((x_n x_n^*)^{\frac{1}{2}}\right) + \alpha\left(T(x_n x_n^*)^{\frac{1}{2}}\right)\right] < \frac{1}{2^n},$$

then $g = \sum_{n=1}^{\infty} t_n (x_n x_n^*)^{\frac{1}{2}} \in \mathcal{M}\cap\mathcal{D}(T)$. Moreover, it follows that $P_g \geq P_{(x_n x_n^*)^{\frac{1}{2}}}$ for each $n$. Hence

$$\tau(P_g) = \sup\{\tau(P_x) : x \in \mathcal{M}\cap\mathcal{D}(T)\}.$$

If $x \in \mathcal{D}(T)$, then $(xx^*)^{\frac{1}{2}}$, $g + (xx^*)^{\frac{1}{2}} \in \mathcal{D}(T)$, which means the weak operator topology closure of the range of $x$ (which equals the range of $(xx^*)^{\frac{1}{2}}$) is contained in $P_g$. Hence,

$$\mathcal{M}\cap\mathcal{D}(T) \subseteq P_g\mathcal{M} \subseteq P_g L^{\alpha}(\mathcal{M},\tau).$$

Since $\mathcal{M}\cap\mathcal{D}(T)$ is dense in $L^{\alpha}(\mathcal{M},\tau)$, we must have $P_g = 1$, which implies $g$ is invertible in $\mathcal{Y}$.

Now it follows from [11, Theorem 2] that for each $y \in \mathcal{Y}$ there is a sequence $\{q_n\}$ of projections in $\mathcal{M}$ such that $\tau(q_n) \to 1$ and $yq_n \in \mathcal{M}$ for every $n \geq 1$. Since $\alpha$ is continuous, we know from [5] that $R_{q_n} \to 1$ (SOT) on $L^{\alpha}(\mathcal{M},\tau)$. Hence, Theorem 3 implies that $T = L_{T(g)g^{-1}}$.

(3) It follows from [6] that $L_y$ is a closed operator on $L^{\alpha}(\mathcal{M},\tau)$. It follows from [11, Theorem 2] that there is a sequence $\{p_n\}$ of projections in $\mathcal{M}$ such that $\tau(p_n) \to 0$ and $y(1-p_n) \in \mathcal{M} \subseteq L^{\alpha}(\mathcal{M},\tau)$, which implies

$$\bigcup_{n=1}^{\infty}(1-p_n)\mathcal{M} \subseteq \mathcal{D}(L_y).$$

Moreover, since $\alpha$ is continuous,

$$\alpha(x-(1-p_n)x) = \alpha(p_n x) \leq \alpha(p_n)\|x\| \to 0.$$

Hence, $\mathcal{D}(L_y)$ is dense in $L^{\alpha}(\mathcal{M},\tau)$. ∎

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## 4. An example on the Bergman space

The closed densely defined operators commuting with the unilateral shift, $M_z$ on $H^2$, are multiplications by functions in the Smirnov class. Is there a similar result for the Bergman shift, multiplication by $z$ on the Bergman space $A^2$? What follows is a counterexample.

**Theorem 7.** *There exists a closed densely defined operator that commutes with the Bergman shift but is not a multiplication operator.*

*Proof.* By a result of Seip (see [16], [3, Theorem 9, p. 186], [8]) there exist two disjoint interpolation sets $\Gamma_1$,$\Gamma_2$ for $A^2$ whose union $\Gamma$ is a sampling set. A sampling set for $A^2$ is defined as a subset $\Gamma$ of $\mathbb{D}$ with the property that there exist positive constants $c$ and $C$ such that, for each $f \in A^2$,

$$c\|f\|^2 \leqslant \sum_{z\in\Gamma} |f(z)|^2(1-|z|^2)^2 \leqslant C\|f\|^2.$$

Thus, with $\mu$ the discrete measure having mass $(1-|z|^2)^2$ at each point $z$ of $\Gamma$, the restriction map $\Phi : f \mapsto f \mid \Gamma$ is an invertible map of $A^2$ onto a subspace $A^2|\Gamma$ of $L^2(\mu)$. An interpolation set for $A^2$ is a subset $\Gamma$ of $\mathbb{D}$ with the property that for each element $w$ in $L^2(\mu)$ there exists $f \in A^2$ such that $w = \Phi(f)$, i.e., the restriction map $f \mapsto f \mid \Gamma$ on $A^2$ is onto $L^2(\mu)$.

Since interpolation sets are also zero sets for $A^2$, we can let $f_1$ be a function in $A^2$ having $\Gamma_1$ as its zero set. Define $\mathcal{G}_1$, $\mathcal{G}_2$ by

$$\mathcal{G}_1 = \{(u, uf_1) : u, uf_1 \in A^2\}$$

and

$$\mathcal{G}_2 = \{(g, g) : g \in A^2,\ g \mid \Gamma_2 = 0\}.$$

Then $\mathcal{G}_1 = \mathcal{G}(M_{f_1})$ and $\mathcal{G}_2 \subseteq \mathcal{G}(I)$ are graphs of closed operators that commute with the Bergman shift $M_z$. Moreover, $\mathcal{G} = \mathcal{G}_1 + \mathcal{G}_2$ is also a graph. To see that $\mathcal{G}$ is a graph, we need only check that if $(0,h) \in \mathcal{G}$, then $h = 0$. Suppose $(u, uf_1) \in \mathcal{G}_1$, $(g, g) \in \mathcal{G}_2$ and $(0, h) = (u, uf_1) + (g, g)$. Then $u = -g$ vanishes on $\Gamma_2$, so $uf_1$ vanishes on the sampling set $\Gamma = \Gamma_1 \cup \Gamma_2$, which implies $uf_1 = 0$. But $f_1 \neq 0$, so $0 = u = -g$, whence $h = 0$.

Finally, it will be shown that the angle between $\mathcal{G}_1$ and $\mathcal{G}_2$ is greater than zero, and thus the graph $\mathcal{G} = \mathcal{G}_1 + \mathcal{G}_2$ is itself a closed direct sum. It is to be shown that there is a $\gamma$, $0 \leq \gamma < 1$, such that, if $F_k \in \mathcal{G}_k$ for $k = 1$,2, then

$$|\langle F_1, F_2\rangle| \leq \gamma \, \|F_1\| \, \|F_2\|.$$

Let $\Phi^{(2)}(f,g) = (\Phi(f),\Phi(g))$. Since $\Phi$ is an isomorphism between $A^2$ and its image $\mathcal{A}$ in $L^2(\mu)$, it suffices to show that the preceding inequality holds for the images of $F_1$ and $F_2$ under $\Phi^{(2)}$. Since an element of $L^2(\mu)$ in the image $\mathcal{A}$ of $\Phi$ is just the restriction of an element of $A^2$, we will simply write $f = \Phi(f)$ and indicate the norm and inner product in $L^2(\mu)$ with a subscript $\mu$. We have $F_1 = (u,uf_1)$ and $F_2 = (g,g)$ with $g,u,uf_1 \in A^2$ and $g \mid \Gamma_2 = 0$. Thus

$$\langle F_1, F_2\rangle_\mu = \langle u, g\rangle_\mu + \langle uf_1, g\rangle_\mu = \langle u, g\rangle_\mu,$$

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where the second equality follows from the fact that $f_1 \mid \Gamma_1 = 0$ as well as $g \mid \Gamma_2 = 0$, so $uf_1\bar{g}$ vanishes a.e.-$\mu$. Also,

$$\|F_1\|_\mu^2 = \|u\|_\mu^2 + \|uf_1\|_\mu^2$$

and

$$\|F_2\|_\mu^2 = 2\|g\|_\mu^2.$$

We have

$$|\langle F_1, F_2\rangle_\mu| = |\langle u, g\rangle_\mu|^2 \leqslant \|u\|_\mu \|g\|_\mu \leq \|F_1\|_\mu \left(\sqrt{2}/2\right) \|F_2\|_\mu,$$

as required with $\gamma = \sqrt{2}/2$. It follows that the subspaces are at a positive angle and the sum of their closures is closed. The operator whose graph is $\mathcal{G}$ has the required properties. □

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Department of Mathematics, Kingsbury Hall, University of New Hampshire, Durham, New Hampshire 03824-3591
*E-mail address*: don@math.unh.edu

Department of Mathematics, University of Pennsylvania, 209 South 33rd Street, Philadelphia, Pennsylvania 19104
*E-mail address*: zheliu@sas.upenn.edu

Department of Mathematics, Kingsbury Hall, University of New Hampshire, Durham, New Hampshire 03824-3591
*E-mail address*: ean@math.unh.edu

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