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On some mathematical aspects of the Heisenberg relation

Dedicated to Professor Richard V. Kadison on the Occasion of his 85th Birthday

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Abstract The Heisenberg commutation relation, $QP-PQ = i\hbar I$, is the most fundamental relation of quantum mechanics. Heisenberg's encoding of the ad-hoc quantum rules in this simple relation embodies the characteristic indeterminacy and uncertainty of quantum theory. Representations of the Heisenberg relation in various mathematical structures are discussed. In particular, after a discussion of unbounded operators affiliated with finite von Neumann algebras, especially, factors of Type II_1, we answer the question of whether or not the Heisenberg relation can be realized with unbounded self-adjoint operators in the algebra of operators affiliated with a factor of type II_1.

Keywords Heisenberg relation, quantum mechanics, unbounded operators, affiliated operators, finite von Neumann algebras, factors of type II_1

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0 Introduction

In a paper [14] published in the 1929–1930 Math Ann, von Neumann defines a class of algebras of bounded operators on a Hilbert space that have acquired the name "von Neumann algebras" [1] (von Neumann refers to them as "rings of operators"). Such algebras are *self-adjoint*, *strong-operator closed*, and contain the identity operator. In that article, the celebrated Double Commutant Theorem is proved. It characterizes von Neumann algebras $\mathcal{R}$ as those for which $\mathcal{R}'' = \mathcal{R}$, where $\mathcal{R}'$, the *commutant* of $\mathcal{R}$, is the set of bounded operators on the Hilbert space that commute with all operators in $\mathcal{R}$. Since then, the subject popularly known as "operator algebras" has come upon the mathematical stage. (We use [5] as our basic reference for results in the theory of operator algebras as well as for much of our notation and terminology.)

Five to six years after the appearance of [14], von Neumann, together with Murray, resumes the study of von Neumann algebras. It is one of the most successful mathematical collaborations. The point to that study was largely to supply a (complex) group algebra crucial for working with infinite groups and to provide a rigorous framework for the most natural mathematical model of the early formulation of quantum mechanics (see [2,3,6,9,16]). In their series of papers [10–12] and [15], Murray and von Neumann focus their attention on those von Neumann algebras that are completely *noncommutative*, those whose centers consist of scalar multiples of the identity operator I. They call such algebras "factors". In [10], they construct factors without *minimal projections* in which I is *finite*. Such factors are said to be of "Type II_1", and they are of particular interest to Murray and von Neumann. In [11], they note that the family of *unbounded* operators on a Hilbert space $\mathcal{H}$ "affiliated" with a type II_1 factor $\mathcal{M}$ has unusual properties. We say that a *closed densely defined* operator T on $\mathcal{H}$ is *affiliated* with $\mathcal{M}$ when $U'T = TU'$ for each *unitary* operator U' in $\mathcal{M}'$, the commutant of $\mathcal{M}$. This definition applies to a general von Neumann algebra, not just to factors of type II_1. However, as Murray and von Neumann show, at the end of [11], the family of operators $\mathscr{A}(\mathcal{M})$ affiliated with a factor $\mathcal{M}$ of type II_1 (or, more generally, affiliated with *finite* von Neumann algebras, those in which the identity operator is *finite*) admits surprising operations of addition and multiplication that suit the formal algebraic manipulations used by the founders of quantum mechanics in their mathematical formulation. This is the case because of very special domain properties that are valid for finite families of operators affiliated with a factor of type II_1. These properties are not valid for *infinite* factors; their families of affiliated operators do not admit such algebraic operations. We review the basic theory of factors in Section 1, and later, in Section 3, we shall show that $\mathscr{A}(\mathcal{M})$ is an associative algebra with unit I and an adjoint operation ("involution") relative to these algebraic operations. (This "detail" is largely ignored in the literature. Experience has shown that it is unwise to ignore "details" when dealing with unbounded operators.)

The development of modern quantum mechanics in the mid-1920s, which studies the physical behavior of systems at atomic length scales and smaller, was an important motivation for the great interest in the study of operator algebras in general and von Neumann algebras in particular. In Dirac's treatment of physical systems [2], there are two basic constituents: the family of *observables* and the family of *states* in which the system can be found. In classical (Newtonian-Hamiltonian) mechanics, states in a physical system are described by an assignment of numbers to the observables (the values certain to be found by measuring the observables in the given state), and the observables are represented as *functions*, on the space of states, that form an algebra, necessarily *commutative*. Contrary to the classical formulation, in quantum mechanics, each state is described in terms of an assignment of probability measures to the spectra of the observables (a measurement of the observable with the system in a given state will produce a value in a given portion of the spectrum with a specific probability). A state that assigns a definite value to one observable assigns a dispersed measure to the spectrum of some other observable — the amount of dispersion involving the experimentally reappearing Planck's constant. So, in quantum mechanics, it is not possible to describe states in which a particle has both a definite position and a definite momentum. The more precise the position, the less precise the momentum. This is the celebrated *Heisenberg Uncertainty Principle*. It entails the *non-commutativity* of the algebra of observables.

In the mathematical formulation of quantum mechanics, many "natural" observables are represented as *self-adjoint* operators (possibly unbounded) on a Hilbert space. Heisenberg's encoding of the ad-hoc quantum rules in his famous commutation relation, $QP - PQ = i\hbar I$, where Q and P are the observables corresponding to the position and momentum (say, of a particle in the system) respectively, I is the identity operator and $i\hbar$ is some complex scalar involving Planck's constant, embodies the characteristic *indeterminacy* and *uncertainty* of quantum theory. The very essence of the relation is its introduction of non-commutativity between the particle's position Q and its corresponding conjugate momentum P. This is the basis for the view of quantum physics as employing noncommutative mathematics, while classical (Newtonian-Hamiltonian) physics involves just commutative mathematics. If we look for mathematical structures that can accommodate this non-commutativity and permit the necessary computations, families of matrices come quickly to mind. In the classical case, commutativity leads immediately to algebras of (complex-valued) functions on a topological space. Of course, we, and the early quantum physicists, can hope that the finite matrices will suffice for our computational work in quantum physics. Unhappily, this is not the case, as the trace (functional) on the algebra of complex $n \times n$ matrices makes clear to us. The trace of the left side of the Heisenberg relation is 0 for matrices P and Q, while the trace of the right side is $i\hbar$ ($\neq 0$). That is to say, the Heisenberg relation cannot be satisfied by finite matrices. Of course, the natural extension of this attempt is to wonder if infinite-dimensional Hilbert spaces might not "support" such a representation with bounded operators. Even this is not possible as we shall show in Section 4; the Heisenberg relation is not representable in terms of elements of complex Banach algebras with a unit element. Therefore, in our search for ways to represent the Heisenberg relation in some (algebraic) mathematical structure, we can eliminate finite matrices, bounded operators on an infinite-dimensional Hilbert space, and even elements of more general complex Banach algebras. Is there anything left?

We are not only representing the Heisenberg relation in mathematical terms; we are trying to do so in a way that allows us to calculate with the representing elements. Another possibility, not yet eliminated, might be to represent the relation by unbounded operators on a Hilbert space. The techniques of unbounded operators are somewhat familiar. At the same time, we have become aware that the unbounded operator theory is deep, difficult, and dangerous. Statements we would want to be true, and that are easily shown to be true for bounded operators, often fail for unbounded operators — sometimes for subtle reasons — occasionally, noted only after such statements have been used as crucial parts of a "proof". The fundamentals of the theory of unbounded operators are presented (without proofs) in Section 2. As it turns out, there is a representation of the Heisenberg relation in terms of unbounded operators, and it is just about the best we are going to get. This classic representation is discussed in Section 4. However, to specify a dense domain on which the representing *differentiation* operator (corresponding to the momentum P) is self-adjoint is not so simple. Some problems, elementary but subtle, arise on the way. We manage to bypass them, and we present an elegant way to approach the problem of finding precisely the self-adjoint operator and its domain through the use of "Stone's theorem" (from the very beginning of the theory of unitary representations of infinite groups).

In this article, we ask further whether there is a representation of the Heisenberg commutation relation in terms of unbounded operators affiliated with a factor of type II_1. As mentioned earlier, the operators affiliated with a factor $\mathcal{M}$ of type II_1 have special properties and they form an algebra $\mathscr{A}(\mathcal{M})$. von Neumann had great respect for his physicist colleagues and the uncanny accuracy of their results in experiments at the subatomic level. In effect, the physicists worked with unbounded operators, but in a loose way. If taken at face value, many of their mathematical assertions were demonstrably incorrect. When the algebra $\mathscr{A}(\mathcal{M})$ appeared, von Neumann hoped that it would provide a framework for the formal computations the physicists made with the unbounded operators. As it turned out, in more advanced areas of modern physics, factors of type II_1 do not suffice, by themselves, for the mathematical framework needed. It remains a tantalizing question, nonetheless, whether the most fundamental relation of quantum mechanics, the Heisenberg relation, can be realized with self-adjoint operators in some $\mathscr{A}(\mathcal{M})$. It is the answer to this question that we provide in this article.

The work in this article is substantially the author's thesis written under the supervision of Professor Richard Kadison; much of the work is joint with him. The author would like to thank Professor Kadison for his patient guidance and unwavering support during her work on this project.

1 Factors

There are two main classes of examples of von Neumann algebras introduced by Murray and von Neumann in their series of papers. One is obtained from the "group-measure space construction". Let G be a discrete group with unit e, and $\mathscr{A}$ be a maximal abelian self-adjoint subalgebra of $\mathcal{B}(\mathcal{H})$. Let $\mathcal{K}$ be the Hilbert space $\sum_{g\in G}\oplus\mathcal{H}_g$, where each $\mathcal{H}_g$ is $\mathcal{H}$, so that $\mathcal{K}$ consists of all mappings $x : G \to \mathcal{H}$ for which $\sum_{g\in G}\|x(g)\|^2 < \infty$. With S in $\mathcal{B}(\mathcal{H})$, there is a naturally associated operator $\Phi(S)$ in $\mathcal{B}(\mathcal{K})$ defined by $(\Phi(S)x)(g) = S(x(g))$. Let $U : g \to U(g)$ be a unitary representation of G on $\mathcal{H}$. We assume that $U(g)\mathscr{A}U(g)^* = \mathscr{A}$ for each g in G (that is, each $U(g)$ implements a * automorphism of $\mathscr{A}$ and U gives rise to a representation of G by automorphisms of $\mathscr{A}$) and that $\mathscr{A} \cap (U(g)\mathscr{A}) = \{0\}$ for each $g(\neq e)$ in G (that is, G acts *freely*, by automorphisms, on $\mathscr{A}$). We say that G acts *ergodically* on $\mathscr{A}$ (through the representation U) when the only elements A in $\mathscr{A}$ such that $U(g)AU(g)^* = A$, for all g in G, are the scalars. Let $(V(g)x)(g')$ be $U(g)x(g^{-1}g')$. It is easily checked that $V(g)V(h) = V(gh)$ and $V(g)\Phi(S)V(g)^* = \Phi(U(g)SU(g)^*)$ when $S \in \mathcal{B}(\mathcal{H})$ and $g, h \in G$. Thus the representation V gives rise to the "same" representation of G by automorphisms of the copy $\Phi(\mathscr{A})$ of $\mathscr{A}$ as U does. The von Neumann algebra $\mathcal{R}$ generated by $\Phi(\mathscr{A})$ and the group $\{V(g)\}_{g\in G}$ is a factor if and only if G acts ergodically on $\mathscr{A}$. Moreover, $\Phi(\mathscr{A})$ is a maximal abelian * subalgebra of $\mathcal{R}$.

Specific examples of the structures described above are obtained from a measure space $(S, \mathscr{S}, m)$ that is countably separated ($\mathscr{S}$ contains a countable family of $E_1, E_2, \dots$ of non-null sets of finite measure such that if s and t are distinct points of S, then $t \in E_j$ and $s \notin E_j$ for some j) and a group G of one-to-one mappings of S onto S that preserves measurability and measure 0 subsets and acts freely on S (that is, $m(\{s \in S : g(s) = s\}) = 0$ when g is not the unit element of G). In this case, $\mathscr{A}$ is the multiplication algebra of the measure space (acting on $L_2(S, \mathscr{S}, m)$). The Radon-Nikodým theorem yields, for each g, a non-negative, real-valued, measurable function φ_g on S such that $\int x(g(s))dm(s) = \int x(s)\varphi_g(s)dm(s)$ for each x in $L_1(S, \mathscr{S}, m)$. If U_g is defined by $(U_g x)(s) = [\varphi_g(s)]^{\frac{1}{2}} x(g^{-1}(s))$, for each x in $L_2(S, \mathscr{S}, m)$, then $g \to U_g$ is a unitary representation of G that gives rise to automorphisms of $\mathscr{A}$ satisfying $\mathscr{A} \cap (U_g\mathscr{A}) = \{0\}$ for each $g(\neq e)$ in G. We say that G acts ergodically on S when $m(g(S_0)\backslash S_0) > 0$ for some g in G unless $m(S_0) = 0$ or $m(S\backslash S_0) = 0$. The representation $g \to U_g$ acts ergodically on $\mathscr{A}$ if and only if G acts ergodically on S. With U_g in place of $U(g)$, $L_2(S, \mathscr{S}, m)$ for $\mathcal{H}$, and the multiplication algebra of the measure space $(S, \mathscr{S}, m)$ for $\mathscr{A}$, the conditions for the construction of $\mathcal{R}$ described earlier are satisfied.

Theorem 1.1. *If G acts ergodically on S, then $\mathcal{R}$ is a factor and*

(i) *$\mathcal{R}$ is of type I if and only if some point in S has positive measure; in this case, $\mathcal{R}$ is of type I_n where n is the number of points in S.*

(ii) *$\mathcal{R}$ is of type II when S admits a G-invariant measure m_0 such that $m_0(S_0) = 0$ if $m(S_0) = 0$. In this case, $\mathcal{R}$ is of type II_1 when $m_0(S) < \infty$ and of type II_∞ when $m_0(S) = \infty$.*

(iii) *$\mathcal{R}$ is of type III when there is no m_0 as described in* (ii).

The other class is based on regular representations of (countable) discrete groups. In [12], Murray and von Neumann provide one of the possible extensions of the notion of group algebra from finite to infinite discrete groups. Let G be a infinite discrete group with unit e and $\mathcal{H}$ be the Hilbert space $l_2(G)$ (the family of complex-valued functions x on G such that $\sum_{g\in G} |x(g)|^2 < \infty$, provided with the inner product $\langle x, y\rangle = \sum_{g\in G} x(g)\overline{y(g)}$). Let $(L_{g_0}x)(g)$ be $x(g_0^{-1}g)$ for each g in G. Then L_g is a unitary operator on the Hilbert space $\mathcal{H}$ (for $L_{g^{-1}}$ is its inverse and $\langle L_g x, L_g y\rangle = \langle x, y\rangle$ for all x and y in $\mathcal{H}$). Moreover, $L_g L_{g'} = L_{gg'}$; the mapping $g \to L_g$ is a (group) isomorphism of G into the group of unitary operators on $\mathcal{H}$. In the same way, we can define the unitary operators R_{g_0} by $(R_{g_0}x)(g) = x(gg_0)$. Let $\mathscr{L}_G$ and $\mathscr{R}_G$ be the weak-operator closures of the algebras of finite, complex linear combinations of the operators $\{L_g : g \in G\}$ and $\{R_g : g \in G\}$, respectively. Then $\mathscr{L}_G$ and $\mathscr{R}_G$ are von Neumann algebras. In addition, each of $\mathscr{L}_G$ and $\mathscr{R}_G$ is the commutant of the other.

Theorem 1.2. *The von Neumann algebra $\mathscr{L}_G$ is a factor if and only if the conjugacy class of each group element* (*other than the group identity*) *is infinite. In this case, $\mathscr{L}_G$ is a factor of type II_1.*

The groups satisfying the infinite conjugacy class condition are called *i.c.c. groups*. Some examples of such groups are $\mathscr{F}_n$, the free (nonabelian) groups on $n\,(\geqslant 2)$ generators, and Π, the group of those permutations of the integers that move at most a finite number of integers.

Theorem 1.3. *$\mathscr{L}_\Pi$ is not isomorphic to $\mathscr{L}_{\mathscr{F}_n}$.*

It is still not known whether $\mathscr{L}_{\mathscr{F}_n}$ and $\mathscr{L}_{\mathscr{F}_m}$ are isomorphic when $n \neq m$.

A factor $\mathcal{M}$ of type I_n, with n finite, is isomorphic to the algebra $\mathcal{M}_n(\mathbb{C})$ of $n \times n$ matrices over the complex numbers. A key element of structure for $\mathcal{M}_n(\mathbb{C})$ (and $\mathcal{M}$) is the linear functional τ with the properties

$$\tau(AB) = \tau(BA), \quad A, B \in \mathcal{M},$$
$$\tau(I) = 1.$$

We refer to τ as the *normalized trace* on $\mathcal{M}$. With the properties noted, τ is unique. In addition, τ takes on non-negative real values at positive matrices. If we denote by $[a_{jk}]$ a matrix in $\mathcal{M}_n(\mathbb{C})$, where a_{jk} is the entry in row j and column k, then $\tau([a_{jk}])$ is $n^{-1}(\sum_{j=1}^n a_{jj})$.

A discovery that intrigued Murray and von Neumann greatly was the existence of a functional on a factor $\mathcal{M}$ of type II_1 with the main properties exhibited by the trace on $\mathcal{M}_n(\mathbb{C})$. They referred to this functional on $\mathcal{M}$ as the (normalized) trace. To define such a trace, Murray and von Neumann proceeded in a measure-theoretic manner. With $\mathcal{M}$ a factor of type II_1, it can be shown that, for each positive integer n and each projection E in $\mathcal{M}$, there are n equivalent mutually orthogonal projections in $\mathcal{M}$ with sum E. If we assign to I the measure (or "normalized dimension") 1 and use I in place of E, then each of the n equivalent projections should be assigned measure n^{-1}. Each projection in $\mathcal{M}$ is a (possibly infinite) sum of such (rational) projections, which provides it with a measure. Murray and von Neumann arrived at a dimension function $\boldsymbol{d}$ that assigns to each projection E in $\mathcal{M}$ a number in $[0,1]$. They noted that the range of $\boldsymbol{d}$ is precisely $[0,1]$, and recognized that they were dealing with "continuous dimensionality". By virtue of the spectral theorem, it was not difficult to determine the value that the trace τ must assume at each element of $\mathcal{M}$. If A is a self-adjoint operator in $\mathcal{B}(\mathcal{H})$, there is a family $\{E_\lambda\}$ of projections in each von Neumann algebra containing A such that

$$A = \int_{-\|A\|}^{\|A\|} \lambda dE_\lambda$$

in the sense of norm convergence of approximating Riemann sums; and A is the norm limit of finite linear combinations with coefficients in $\mathrm{sp}(A)$, the spectrum of A, of orthogonal projections $E_{\lambda'} - E_\lambda$. The family $\{E_\lambda\}$ is referred to as *the resolution* of the identity for A or *the spectral resolution* of A. The value of τ at each self-adjoint operator in $\mathcal{M}$ is defined by

$$\tau(A) = \int_{-\|A\|}^{\|A\|} \lambda d\boldsymbol{d}(E_\lambda).$$

Finally, each operator T in $\mathcal{M}$ is a sum $H + iK$ where H $(= (T+T^*)/2)$ and K $(= (T-T^*)/2i)$ are self-adjoint. If τ is to be linear, we must define $\tau(T)$ as $\tau(H) + i\tau(K)$. This construction of τ was carried out in [10]. It was relatively easy to prove that τ, so determined, is unique. But proving that τ is additive ($\tau(A+B) = \tau(A) + \tau(B)$) was a considerable challenge; it was not established until [11]. The trace has many interesting and important properties. As constructed, it restricts to the dimension function on projections. In addition, $\tau(AB) = \tau(BA)$ for all A and B in $\mathcal{M}$, $\tau(A) \geqslant 0$ when $A \geqslant 0$, and $\tau(A_n) \to \tau(A)$ when $A_n x \to Ax$ for each x in $\mathcal{H}$ (that is, τ is strong-operator continuous on $\mathcal{M}$). It is the unique *tracial state* of $\mathcal{M}$ (positive linear functional taking value 1 at I such that $\tau(AB) = \tau(BA)$ for all A and B in $\mathcal{M}$).

2 Unbounded operators

2.1 Definitions and facts

Let T be a linear mapping, with domain $\mathscr{D}(T)$ a linear submanifold (not necessarily closed), of the Hilbert space $\mathcal{H}$ into the Hilbert space $\mathcal{K}$. We associate a *graph* $\mathscr{G}(T)$ with T, where $\mathscr{G}(T) = \{(x, Tx) : x \in \mathscr{D}(T)\}$. We say that T is *closed* when $\mathscr{G}(T)$ is closed. The *closed graph theorem* tells us that if T is defined on all of $\mathcal{H}$, then $\mathscr{G}(T)$ is closed if and only if T is bounded. The unbounded operators T we consider will usually be *densely defined*, that is, $\mathscr{D}(T)$ is dense in $\mathcal{H}$. Whatever T we consider, it has a graph $\mathscr{G}(T)$, and the closure $\mathscr{G}(T)^-$ of $\mathscr{G}(T)$ will be a linear subspace of $\mathcal{H} \bigoplus \mathcal{K}$. It may be the case that $\mathscr{G}(T)^-$ is the graph of a linear transformation $\bar{T}$, but it need not be. If it is, $\bar{T}$ "extends" T and is closed. We say that T_0 *extends* (or is *an extension of*) T, and write $T \subseteq T_0$, when $\mathscr{D}(T) \subseteq \mathscr{D}(T_0)$ and $T_0 x = Tx$ for each x in $\mathscr{D}(T)$. If $\mathscr{G}(T)^-$ is the graph of a linear transformation $\bar{T}$, clearly $\bar{T}$ is the "smallest" closed extension of T, we say that T is *preclosed* (or *closable*) and refer to $\bar{T}$ as the *closure* of T.

From the point of view of calculations with an unbounded operator T, it is often much easier to study its restriction $T|\mathscr{D}_0$ to a dense linear manifold $\mathscr{D}_0$ in its domain $\mathscr{D}(T)$ than to study T itself. If T is closed and $\mathscr{G}(T|\mathscr{D}_0)^- = \mathscr{G}(T)$, the information obtained in this way is much more applicable to T. In this case, we say that $\mathscr{D}_0$ is a *core* for T. Each dense linear manifold in $\mathscr{G}(T)$ corresponds to a core for T.

We define the operations of addition and multiplication for unbounded operators so that the domains of the resulting operators consist precisely of those vectors on which the indicated operations can be performed. Thus $\mathscr{D}(A+B)=\mathscr{D}(A)\cap\mathscr{D}(B)$ and $(A+B)x=Ax+Bx$ for x in $\mathscr{D}(A+B)$. Assuming that $\mathscr{D}(B)\subseteq\mathcal{H}$ and $\mathscr{D}(A)\subseteq\mathcal{K}$, where B has its range in $\mathcal{K}$, AB is defined as the linear transformation, with $\{x : x\in\mathscr{D}(B) \text{ and } Bx\in\mathscr{D}(A)\}$ as its domain, assigning $A(Bx)$ to x. Of course $\mathscr{D}(aA)=\mathscr{D}(A)$ and $(aA)x=a(Ax)$ for a complex scalar a.

Definition 2.1. *If T is a linear transformation with $\mathscr{D}(T)$ dense in the Hilbert space $\mathcal{H}$ and range contained in the Hilbert space $\mathcal{K}$, we define a mapping T^*, the adjoint of T, as follows. Its domain consists of those vectors y in $\mathcal{K}$ such that, for some vector z in $\mathcal{H}$, $\langle x,z\rangle=\langle Tx,y\rangle$ for all x in $\mathscr{D}(T)$. For such y, T^*y is z. If $T=T^*$, we say that T is self-adjoint.*

The formal relation $\langle Tx,y\rangle=\langle x,T^*y\rangle$, familiar from the case of bounded operators, remains valid in the present context only when $x\in\mathscr{D}(T)$ and $y\in\mathscr{D}(T^*)$.

Remark 2.2. If T_0 is densely defined and T is an extension of T_0, then T_0^* is an extension of T^*.

Remark 2.3. If T is densely defined, then T^* is a closed linear operator.

Theorem 2.4. *If T is a densely defined transformation from the Hilbert space $\mathcal{H}$ to the Hilbert space $\mathcal{K}$, then*

(i) *if T is preclosed, $(\bar{T})^*=T^*$;*
(ii) *T is preclosed if and only if $\mathscr{D}(T^*)$ is dense in $\mathcal{K}$;*
(iii) *if T is preclosed, $T^{**}=\bar{T}$;*
(iv) *if T is closed, T^*T+I is one-to-one with range $\mathcal{H}$ and positive inverse of bound not exceeding* 1;
(v) *T^*T is self-adjoint when T is closed, and $\mathscr{D}(T^*T)$ is a core for T.*

The statement that T is self-adjoint ($T=T^*$) contains information about the domain of T as well as the formal information that $\langle Tx,y\rangle=\langle x,Ty\rangle$ for all x and y in $\mathscr{D}(T)$. When $\mathscr{D}(T)$ is dense in $\mathcal{H}$ and $\langle Tx,y\rangle=\langle x,Ty\rangle$ for all x and y in $\mathscr{D}(T)$, we say that T is *symmetric*. Equivalently, T is symmetric when $T\subseteq T^*$. Since T^* is closed and $\mathscr{G}(T)\subseteq\mathscr{G}(T^*)$, in this case, T is preclosed if it is symmetric. If T is self-adjoint, T is both symmetric and closed. The operation of differentiation on an appropriate domain provides an example of a closed symmetric operator that is not self-adjoint (see Example 4.2).

Remark 2.5. If $A\subseteq T$ with A self-adjoint and T symmetric, then $A\subseteq T\subseteq T^*$, so that $T^*\subseteq A^*=A\subseteq T\subseteq T^*$ and $A=T$. It follows that A has no proper symmetric extension. That is, a self-adjoint operator is *maximal symmetric*.

Proposition 2.6. *If T is a closed symmetric operator on the Hilbert space $\mathcal{H}$, the following assertions are equivalent*:

(i) *T is self-adjoint*;
(ii) *$T^*\pm iI$ have* (0) *as null space*;
(iii) *$T\pm iI$ have $\mathcal{H}$ as range*;
(iv) *$T\pm iI$ have ranges dense in $\mathcal{H}$.*

Proposition 2.7. *If T is a closed linear operator with domain dense in a Hilbert space $\mathcal{H}$ and with range in $\mathcal{H}$, then*

$$R(T)=I-N(T^*),\quad N(T)=I-R(T^*),\quad R(T^*T)=R(T^*),\quad N(T^*T)=N(T),$$

where $N(T)$ and $R(T)$ denote the projections whose ranges are, respectively, the null space of T and the closure of the range of T.

2.2 Spectral theory

If A is a bounded self-adjoint operator acting on a Hilbert space $\mathcal{H}$ and $\mathscr{A}$ is an abelian von Neumann algebra containing A, there is a family $\{E_\lambda\}$ of projections in $\mathscr{A}$ (indexed by $\mathbb{R}$), called the spectral resolution of A, such that

(i) $E_\lambda=0$ if $\lambda<-\|A\|$, and $E_\lambda=I$ if $\|A\|\leqslant\lambda$;

(ii) $E_\lambda \leqslant E_{\lambda'}$ if $\lambda \leqslant \lambda'$;

(iii) $E_\lambda = \wedge_{\lambda' > \lambda} E_{\lambda'}$;

(iv) $AE_\lambda \leqslant \lambda E_\lambda$ and $\lambda(I - E_\lambda) \leqslant A(I - E_\lambda)$ for each λ;

(v) $A = \int_{-\|A\|}^{\|A\|} \lambda dE_\lambda$ in the sense of norm convergence of approximating Riemann sums; and A is the norm limit of finite linear combinations with coefficients in sp(A), the spectrum of A, of orthogonal projections $E_{\lambda'} - E_\lambda$.

With the abelian von Neumann algebra $\mathscr{A}$ isomorphic to $C(X)$ and X an extremely disconnected compact Hausdorff space, if f and e_λ in $C(X)$ correspond to A and E_λ in $\mathscr{A}$, then e_λ is the characteristic function of the largest clopen subset X_λ on which f takes values not exceeding λ.

The spectral theory described above can be extended to unbounded self-adjoint operators. (We associate an unbounded spectral resolution with each of them.) We begin with a discussion that details the relation between unbounded self-adjoint operators and the multiplication algebra of a measure space.

If g is a complex measurable function (finite almost everywhere) on a measure space $(S, \mathscr{S}, m)$, without the restriction that it be essentially bounded — multiplication by g will not yield an everywhere-defined operator on $L_2(S)$, for many of the products will not lie in $L_2(S)$. Enough functions f will have product gf in $L_2(S)$, however, to form a dense linear submanifold $\mathscr{D}$ of $L_2(S)$ and constitute a (dense) domain for an (unbounded) multiplication operator M_g. To see this, let E_n be the (bounded) multiplication operator corresponding to the characteristic function of the (measurable) set on which $|g| \leqslant n$. Since g is finite almost everywhere, $\{E_n\}$ is an increasing sequence of projections with union I. The union $\mathscr{D}_0$ of the ranges of the E_n is a dense linear manifold of $L_2(S)$ contained in $\mathscr{D}$. A measure-theoretic argument shows that M_g is closed with $\mathscr{D}_0$ as a core. In fact, if $\{f_n\}$ is a sequence in $\mathscr{D}$ converging in $L_2(S)$ to f and $\{gf_n\}$ converges in $L_2(S)$ to h, then, passing to subsequences, we may assume that $\{f_n\}$ and $\{gf_n\}$ converge almost everywhere to f and h, respectively. But, then, $\{gf_n\}$ converges almost everywhere to gf, so that gf and h are equal almost everywhere. Thus $gf \in L_2(S)$, $f \in \mathscr{D}$, $h = M_g(f)$, and M_g is closed. With f_0 in $\mathscr{D}$, $E_n f_0$ converges to f_0 and $\{M_g E_n f_0\} = \{E_n M_g f_0\}$ converges to $M_g f_0$. Now $E_n f_0 \in \mathscr{D}_0$, so that $\mathscr{D}_0$ is a core for M_g. Note that $M_g E_n$ is bounded with norm not exceeding n. One can show that M_g is an (unbounded) self-adjoint operator when g is real-valued. If M_g is unbounded, we cannot expect it to belong to the multiplication algebra $\mathscr{A}$ of the measure space $(S, \mathscr{S}, m)$. Nonetheless, there are various ways in which M_g behaves as if it were in $\mathscr{A}$ — for example, M_g is unchanged when it is "transformed" by a unitary operator U commuting with $\mathscr{A}$. In this case, $U \in \mathscr{A}$, so that $U = M_u$ where u is a bounded measurable function on S with modulus 1 almost everywhere. With f in $\mathscr{D}(M_g)$, $guf \in L_2(S)$; while, if $guh \in L_2(S)$, then $gh \in L_2(S)$ and $h \in \mathscr{D}(M_g)$. Thus U transforms $\mathscr{D}(M_g)$ onto itself. Moreover

$$(U^* M_g U)(f) = \overline{u} guf = |u|^2 gf = gf.$$

Thus $U^* M_g U = M_g$. The fact that M_g "commutes" with all unitary operators commuting with $\mathscr{A}$ in conjunction with the fact that each element of a C*-algebra is a finite linear combination of unitary elements in the algebra and the double commutant theorem (from which it follows that a bounded operator that commutes with all unitary operators commuting with $\mathscr{A}$ lies in $\mathscr{A}$) provides us with an indication of the extent to which M_g "belongs" to $\mathscr{A}$. We formalize this property in the definition that follows.

Definition 2.8. *We say that a closed densely defined operator T is affiliated with a von Neumann algebra $\mathcal{R}$ and write $T\eta\mathcal{R}$ when $U^*TU = T$ for each unitary operator U commuting with $\mathcal{R}$.*

Note that the equality, $U^*TU = T$, of the preceding definition is to be understood in the strict sense that U^*TU and T have the same domain and (formal) equality holds for the transforms of vectors in that domain. As far as the domains are concerned, the effect is that U transforms $\mathscr{D}(T)$ onto itself.

Remark 2.9. If T is a closed densely defined operator with core $\mathscr{D}_0$ and $U^*TUx = Tx$ for each x in $\mathscr{D}_0$ and each unitary operator U commuting with a von Neumann algebra $\mathcal{R}$, then $T\eta\mathcal{R}$.

To see this, note that, with y in $\mathscr{D}(T)$, there is a sequence $\{y_n\}$ in $\mathscr{D}_0$ such that $y_n \to y$ and $Ty_n \to Ty$ (since $\mathscr{D}_0$ is a core for T). Now, $Uy_n \to Uy$ and $TUy_n = UTy_n \to UTy$. Since T is closed, $Uy \in \mathscr{D}(T)$

and $TUy = UTy$. Thus $\mathscr{D}(T) \subseteq U^*(\mathscr{D}(T))$. Applied to U^*, we have $\mathscr{D}(T) \subseteq U(\mathscr{D}(T))$, so that $U(\mathscr{D}(T)) = \mathscr{D}(T)$. Hence $\mathscr{D}(U^*TU) = \mathscr{D}(T)$ and $U^*TUy = Ty$ for each y in $\mathscr{D}(T)$.

Theorem 2.10. *If A is a self-adjoint operator acting on a Hilbert space $\mathcal{H}$, A is affiliated with some abelian von Neumann algebra $\mathscr{A}$. There is a resolution of the identity $\{E_\lambda\}$ in $\mathscr{A}$ such that $\bigcup_{n=1}^{\infty} F_n(\mathcal{H})$ is a core for A, where $F_n = E_n - E_{-n}$, and $Ax = \int_{-n}^{n} \lambda dE_\lambda x$ for each x in $F_n(\mathcal{H})$ and all n, in the sense of norm convergence of approximating Riemann sums.*

Since A is self-adjoint, from Proposition 2.6, $A + iI$ and $A - iI$ have range $\mathcal{H}$ and null space (0); in addition, they have inverses, say, T_+ and T_-, that are everywhere defined with bound not exceeding 1. Let $\mathscr{A}$ be an abelian von Neumann algebra containing I, T_+ and T_-. If U is a unitary operator in $\mathscr{A}'$, for each x in $\mathscr{D}(A)$, $Ux = UT_+(A + iI)x = T_+U(A + iI)x$ so that $(A + iI)Ux = U(A + iI)x$; and $U^{-1}(A + iI)U = A + iI$. Thus $U^{-1}AU = A$ and $A\eta\mathscr{A}$. In particular, A is affiliated with the abelian von Neumann algebra generated by I, T_+ and T_-. Since $\mathscr{A}$ is abelian, $\mathscr{A}$ is isomorphic to $C(X)$ with X an extremely disconnected compact Hausdorff space. Let g_+ and g_- be the functions in $C(X)$ corresponding to T_+ and T_-. Let f_+ and f_- be the functions defined as the reciprocals of g_+ and g_-, respectively, at those points where g_+ and g_- do not vanish. Then, f_+ and f_- are continuous where they are defined on X, as is the function f defined by

$$f = (f_+ + f_-)/2.$$

In a formal sense, f is the function that corresponds to A. Let X_λ be the largest clopen set on which f takes values not exceeding λ. Let e_λ be the characteristic function of X_λ and E_λ be the projection in $\mathscr{A}$ corresponding to e_λ. In this case, $\{E_\lambda\}$ satisfies $E_\lambda \leqslant E_{\lambda'}$ if $\lambda \leqslant \lambda'$, $E_\lambda = \wedge_{\lambda'>\lambda} E_{\lambda'}$, $\vee_\lambda E_\lambda = I$ and $\wedge_\lambda E_\lambda = 0$. That is, we have constructed a resolution of the identity $\{E_\lambda\}$. This resolution is unbounded if $f \notin C(X)$. Let $F_n = E_n - E_{-n}$, the *spectral projection* corresponding to the interval $[-n, n]$ for each positive integer n. AF_n is bounded and self-adjoint. Moreover, $\bigcup_{n=1}^{\infty} F_n(\mathcal{H})$ is a core for A. From the spectral theory of bounded self-adjoint operators, $Ax = \int_{-n}^{n} \lambda dE_\lambda x$, for each x in $F_n(\mathcal{H})$ and all n. If $x \in \mathscr{D}(A)$,

$$\int_{-n}^{n} \lambda dE_\lambda x = \int_{-n}^{n} \lambda dE_\lambda F_n x = AF_n x \to Ax.$$

Interpreted as an improper integral, we write

$$Ax = \int_{-\infty}^{\infty} \lambda dE_\lambda x, \quad x \in \mathscr{D}(A).$$

2.3 Polar decomposition

Each T in $\mathcal{B}(\mathcal{H})$ has a unique decomposition as VH, the *polar decomposition* of T, where $H = (TT^*)^{1/2}$ and V maps the closure of the range of H, denoted by $r(H)$, isometrically onto $r(T)$ and maps the orthogonal complement of $r(H)$ to 0. We say that V is a *partial isometry* with *initial space* $r(H)$ and *final space* $r(T)$. If $R(H)$ is the projection with range $r(H)$ (the *range projection* of H), then $V^*V = R(H)$ and $VV^* = R(T)$. We note that the components V and H of this polar decomposition lie in the von Neumann algebra $\mathcal{R}$ when T does, from which we conclude that $R(H) \sim R(T)$. In fact, $R(T^*) = R(T^*T) = R((T^*T)^{1/2})$ for any $T \in \mathcal{B}(\mathcal{H})$.

There is an extension of the polar decomposition to the case of a closed densely defined linear operator from one Hilbert space to another.

Lemma 2.11. *If A and C are densely defined preclosed operators and B is a bounded operator such that $A = BC$, then $A^* = C^*B^*$.*

Proof. If $y \in \mathscr{D}(A^*)$, then, for each x in $\mathscr{D}(A)$ ($=\mathscr{D}(C)$),

$$\langle x, A^*y\rangle = \langle Ax, y\rangle = \langle BCx, y\rangle = \langle Cx, B^*y\rangle;$$

so that $B^*y \in \mathscr{D}(C^*)$ and $C^*B^*y = A^*y$.

If $y \in \mathscr{D}(C^*B^*)$, then $B^*y \in \mathscr{D}(C^*)$ and, for each x in $\mathscr{D}(C)$ $(=\mathscr{D}(A))$,

$$\langle x, C^*B^*y\rangle = \langle Cx, B^*y\rangle = \langle BCx, y\rangle = \langle Ax, y\rangle;$$

so that $y \in \mathscr{D}(A^*)$ and $A^*y = C^*B^*y$. □

Theorem 2.12. *If T is a closed densely defined linear transformation from one Hilbert space to another, there is a partial isometry V with initial space the closure of the range of $(T^*T)^{1/2}$ and final space the closure of the range of T such that $T = V(T^*T)^{1/2} = (T^*T)^{1/2}V$. Restricted to the closures of the ranges of T^* and T, respectively, T^*T and TT^* are unitarily equivalent (and V implements this equivalence). If $T = WH$, where H is a positive operator and W is a partial isometry with initial space the closure of the range of H, then $H = (T^*T)^{1/2}$ and $W = V$. If $\mathcal{R}$ is a von Neumann algebra, $T\eta\mathcal{R}$ if and only if $V \in \mathcal{R}$ and $(T^*T)^{1/2}\eta\mathcal{R}$.*

Proof. From Theorem 2.4, T^*T is self-adjoint. If $x \in \mathscr{D}(T^*T)$, then $x \in \mathscr{D}(T)$, $Tx \in \mathscr{D}(T^*)$, and

$$0 \leqslant \langle Tx, Tx\rangle = \langle T^*Tx, x\rangle.$$

Thus T^*T is positive and has a unique positive square root $(T^*T)^{1/2}$. Again, from Theorem 2.4, $\mathscr{D}(T^*T)$ is a core for $(T^*T)^{1/2}$ and for T. Thus $(T^*T)^{1/2}$ and T map $\mathscr{D}(T^*T)$ onto dense subsets of their ranges. Defining $V_0(T^*T)^{1/2}x$ to be Tx, for x in $\mathscr{D}(T^*T)$, V_0 extends to a partial isometry V with initial space the closure of the range of $(T^*T)^{1/2}$ and final space the closure of the range of T, since

$$\langle (T^*T)^{1/2}x, (T^*T)^{1/2}x\rangle = \langle T^*Tx, x\rangle = \langle Tx, Tx\rangle.$$

Moreover, $Tx = V(T^*T)^{1/2}x$ for each x in $\mathscr{D}(T^*T)$.

With x in $\mathscr{D}(V(T^*T)^{1/2})$, choose x_n in $\mathscr{D}(T^*T)$ such that $x_n \to x$ and $(T^*T)^{1/2}x_n \to (T^*T)^{1/2}x$. Then $Tx_n = V(T^*T)^{1/2}x_n \to V(T^*T)^{1/2}x$. Since T is closed, $x \in \mathscr{D}(T)$ and $Tx = V(T^*T)^{1/2}x$. Thus $V(T^*T)^{1/2} \subseteq T$.

Conversely, if $x \in \mathscr{D}(T)$ and x_n is chosen in $\mathscr{D}(T^*T)$ such that $x_n \to x$ and $Tx_n \to Tx$, then $(T^*T)^{1/2}x_n = V^*V(T^*T)^{1/2}x_n = V^*Tx_n \to V^*Tx$. Since $(T^*T)^{1/2}$ is closed, $x \in \mathscr{D}((T^*T)^{1/2})$. It follows that $T = V(T^*T)^{1/2}$.

From Lemma 2.11, $T^* = (T^*T)^{1/2}V^*$, so that $TT^* = VT^*TV^*$. Thus the restriction of TT^* to the closure of the range of T is unitarily equivalent to the restriction of T^*T to the closure of the range of T^*, and V implements this equivalence. It follows that $(TT^*)^{1/2} = V(T^*T)^{1/2}V^*$; so that

$$T = V(T^*T)^{1/2} = V(T^*T)^{1/2}V^*V = (TT^*)^{1/2}V.$$

If $T = WH$ with H positive and W a partial isometry with initial space the closure of the range of H, then, from Lemma 2.11, $T^* = HW^*$ and $T^*T = H^2$. Thus $H = (T^*T)^{1/2}$, so that $W = V$.

Let $\mathcal{R}$ be a von Neumann algebra and U be a unitary operator in $\mathcal{R}'$. Then $UVU^*U(T^*T)^{1/2}U^*$ is the polar decomposition of UTU^*. From uniqueness of the polar decomposition, $T = UTU^*$ if and only if $V = UVU^*$ and $(T^*T)^{1/2} = U(T^*T)^{1/2}U^*$. Thus $T\eta\mathcal{R}$ if and only if $V \in \mathcal{R}$ and $(T^*T)^{1/2}\eta\mathcal{R}$. □

Proposition 2.13. *If T is affiliated with a von Neumann algebra $\mathcal{R}$, then*

(i) *$R(T)$ and $N(T)$ are in $\mathcal{R}$;*

(ii) *$R(T^*) = R(T^*T) = R((T^*T)^{1/2})$;*

(iii) *$R(T) \sim R(T^*)$ relative to $\mathcal{R}$.*

Proof. (i) Note that $x \in N(T)(\mathcal{H})$ if and only if $x \in \mathscr{D}(T)$ and $Tx = 0$. If U' is a unitary operator in $\mathcal{R}'$, then $U'x \in \mathscr{D}(T)$ when $x \in \mathscr{D}(T)$ and $TU'x = U'Tx$. Thus $TU'x = 0$ when $x \in N(T)(\mathcal{H})$, and $N(T)(\mathcal{H})$ is stable under each unitary operator in $\mathcal{R}'$. Hence, $N(T) \in \mathcal{R}$. From Proposition 2.7, $R(T) \in \mathcal{R}$.

(ii) We show that $N((T^*T)^{1/2}) = N(T^*T)$. If $x \in N((T^*T)^{1/2})(\mathcal{H})$, then $x \in \mathscr{D}((T^*T)^{1/2})$ and $(T^*T)^{1/2}x = 0$. Thus $x \in \mathscr{D}(T^*T)$,

$$T^*Tx = (T^*T)^{1/2}(T^*T)^{1/2}x = 0,$$

and $x \in N(T^*T)(\mathcal{H})$.

If $x \in N(T^*T)(\mathcal{H})$, then $x \in \mathscr{D}(T^*T)$ and $T^*Tx = 0$. Thus $x \in \mathscr{D}((T^*T)^{1/2})$,

$$0 = \langle T^*Tx, x\rangle = \langle (T^*T)^{1/2}(T^*T)^{1/2}x, x\rangle = \|(T^*T)^{1/2}x\|^2,$$

and $x \in N((T^*T)^{1/2})(\mathcal{H})$. It follows that $N((T^*T)^{1/2}) = N(T^*T)$. From Proposition 2.7, $R(T^*) = R(T^*T) = R((T^*T)^{1/2})$.

(iii) From Theorem 2.12, $T = V(T^*T)^{1/2}$, where V is a partial isometry in $\mathcal{R}$ with initial projection $R((T^*T)^{1/2})$ and final projection $R(T)$. From (ii), $R(T^*) = R((T^*T)^{1/2})$. Thus $R(T)$ and $R(T^*)$ are equivalent in $\mathcal{R}$. □

3 Operators affiliated with finite von Neumann algebras

3.1 Finite von Neumann algebras

We say that a von Neumann algebra $\mathcal{R}$ is finite when the identity operator I is finite (that is, I is not equivalent, relative to $\mathcal{R}$, to any proper subprojection). Note that factors of type I_n (n finite) and type II_1 are finite von Neumann algebras. We first review some properties of finite von Neumann algebras. They are useful to us in the proof of the main theorem of the section.

Proposition 3.1. *Suppose that E and F are projections in a finite von Neumann algebra $\mathcal{R}$. If $E \sim F$, then $I - E \sim I - F$.*

Proof. Suppose $I - E$ and $I - F$ are not equivalent. Then there is a central projection P such that either

$$P(I - E) \prec P(I - F) \quad \text{or} \quad P(I - F) \prec P(I - E).$$

Suppose $P(I - E) \sim G < P(I - F)$. Then, since $PE \sim PF$,

$$P = P(I - E) + PE \sim G + PF < P(I - F) + PF = P,$$

contrary to the assumption that $\mathcal{R}$ is finite. The symmetric argument applies if $P(I - F) \prec P(I - E)$. Thus $I - E \sim I - F$. □

Proposition 3.2. *For any projections E and F in a finite von Neumann algebra $\mathcal{R}$,*

$$\Delta(E \vee F) + \Delta(E \wedge F) = \Delta(E) + \Delta(F),$$

where Δ is the center-valued dimension function on $\mathcal{R}$.

Proof. Since $E \vee F - F \sim E - E \wedge F$ (Kaplansky formula), we have

$$\Delta(E \vee F) - \Delta(F) = \Delta(E \vee F - F) = \Delta(E - E \wedge F) = \Delta(E) - \Delta(E \wedge F).$$

Thus $\Delta(E \vee F) + \Delta(E \wedge F) = \Delta(E) + \Delta(F)$. □

Proposition 3.3. *Suppose that E, F, and G are projections in a finite von Neumann algebra $\mathcal{R}$, and E and F are the* (*strong-operator*) *limits of increasing nets $\{E_a\}$ and $\{F_a\}$, respectively, of projections in $\mathcal{R}$* (*the index set being the same*). *Then*

(i) *$\{E_a \vee G\}$ is strong-operator convergent to $E \vee G$;*

(ii) *$\{E_a \wedge G\}$ is strong-operator convergent to $E \wedge G$;*

(iii) *$\{E_a \wedge F_a\}$ is strong-operator convergent to $E \wedge F$.*

Proof. (i) Since the net $\{E_a \vee G\}$ is increasing and bounded above by $E \vee G$, it converges to a projection P in $\mathcal{R}$, and $P \leqslant E \vee G$. For each index a, $E_a \leqslant E_a \vee G \leqslant P$, so $\bigvee E_a \leqslant P$; that is $E \leqslant P$. Also, $G \leqslant E_a \vee G \leqslant P$; so $E \vee G \leqslant P$. Thus $P = E \vee G$.

(ii) Since the net $\{E_a \wedge G\}$ is increasing and bounded above by $E \wedge G$, it converges to a projection P in $\mathcal{R}$, and $P \leqslant E \wedge G$. Recall that the center-valued dimension function Δ on $\mathcal{R}$ is weak-operator continuous on the set of all projections on $\mathcal{R}$; together with Proposition 3.2,

$$\begin{aligned}\Delta(P) &= \lim \Delta(E_a \wedge G)\\ &= \lim[\Delta(E_a) + \Delta(G) - \Delta(E_a \vee G)]\\ &= \Delta(E) + \Delta(G) - \Delta(E \vee G) = \Delta(E \wedge G).\end{aligned}$$

Since $E \wedge G - P$ is a projection in $\mathcal{R}$ and $\Delta(E \wedge G - P) = 0$, it follows that $P = E \wedge G$.

(iii) The net $\{E_a \wedge F_a\}$ is increasing and therefore has a projection P as a strong-operator limit and least upper bound. Since $E_a \wedge F_a \leqslant E \wedge F$ for each a, $P \leqslant E \wedge F$. With a' fixed the net $\{E_a \wedge F_{a'}\}$ has strong-operator limit $E \wedge F_{a'}$ from (ii). Since $E_a \wedge F_{a'} \leqslant E_a \wedge F_a$ when $a' \leqslant a$, $E \wedge F_{a'} \leqslant P$ for each a'. Again, from (ii), $\{E \wedge F_a\}$ has $E \wedge F$ as its strong-operator limit. Thus $E \wedge F \leqslant P$. Hence $P = E \wedge F$. □

Proposition 3.4. *Let E be a projection in a finite von Neumann algebra $\mathcal{R}$ acting on a Hilbert space $\mathcal{H}$. With T in $\mathcal{R}$, let F be the projection with range $\{x : Tx \in E(\mathcal{H})\}$. Then $F \in \mathcal{R}$ and $E \precsim F$.*

Proof. With A' in $\mathcal{R}'$ and Tx in $E(\mathcal{H})$, $TA'x = A'Tx \in E(\mathcal{H})$ since $A'E = EA'$. Thus $F(\mathcal{H})$ is stable under $\mathcal{R}'$, and $F \in \mathcal{R}''(= \mathcal{R})$.

Note that $Tx \in E(\mathcal{H})$ if and only if $(I - E)Tx = 0$. Thus $F(\mathcal{H})$ is the null space of $(I - E)T$ (that is, $F = N[(I - E)T]$). Then

$$I - F = I - N[(I - E)T] = R[T^*(I - E)] \sim R[(I - E)T] \leqslant I - E. \tag{3.1}$$

If $E \not\precsim F$, then there is a central projection P in $\mathcal{R}$ such that $PF \prec PE$. Now $P(I - F) \precsim P(I - E)$ from (3.1) so that $P(I - F) \sim E_0 \leqslant P(I - E)$. Thus

$$P = PF + P(I - F) \prec PE + E_0 \leqslant PE + P(I - E) = P$$

since $P(I - F)$ and E_0 are finite. This is contrary to the assumption that $\mathcal{R}$ is finite. It follows that $E \precsim F$. □

3.2 The algebra of affiliated operators

Throughout this subsection, $\mathcal{R}$ denotes a finite von Neumann algebra acting on a Hilbert space $\mathcal{H}$, and $\mathscr{A}(\mathcal{R})$ denotes the family of operators affiliated with $\mathcal{R}$.

Proposition 3.5. *If S is a symmetric operator affiliated with $\mathcal{R}$, then S is self-adjoint.*

Proof. Since $S\eta\mathcal{R}$, $(S + iI)\eta\mathcal{R}$. It follows that

$$\begin{aligned}&R(S + iI) \sim R((S + iI)^*) \quad \text{(Proposition 2.13)},\\ &I - R(S + iI) \sim I - R((S + iI)^*) \quad \text{(Proposition 3.1)},\\ &I - R(S + iI) = N((S + iI)^*) \sim N(S + iI) = I - R((S + iI)^*) \quad \text{(Proposition 2.7)}.\end{aligned}$$

If x is in the range of $N(S + iI)$, then $x \in \mathscr{D}(S + iI)(= \mathscr{D}(S))$ and $Sx + ix = 0$. Since $S \subseteq S^*$, $x \in \mathscr{D}(S^*)$ and $Sx = S^*x$, so that

$$\langle Sx, x\rangle = \langle x, S^*x\rangle = \langle x, Sx\rangle = \overline{\langle Sx, x\rangle}$$

and

$$0 = \langle Sx + ix, x\rangle = \langle Sx, x\rangle + i\langle x, x\rangle.$$

Thus $\langle x, x\rangle = 0$ and $x = 0$. Hence $N(S + iI) = 0$ and $N((S + iI)^*) = 0$. Similarly, $N((S - iI)^*) = 0$. From Proposition 2.6, S is self-adjoint (for $(S \pm iI)^* = S^* \mp iI$). □

Proposition 3.6. *Suppose that operators A and B are affiliated with $\mathcal{R}$ and $A \subseteq B$, then $A = B$.*

Proof. Let VH be the polar decomposition of B. Since $A \subseteq B$,

$$V^*A \subseteq V^*B = V^*VH = H = H^* \subseteq (V^*A)^*.$$

Thus V^*A is symmetric. If fact, V^*A is affiliated with $\mathcal{R}$. To see this, first, V^*A is densely defined since $\mathscr{D}(V^*A) = \mathscr{D}(A)$. Now, suppose $\{x_n\}$ is a sequence of vectors in $\mathscr{D}(V^*A)$ such that $x_n \to x$ and $V^*Ax_n \to y$. As V^* is isometric on the range of A,

$$\|Ax_n - Ax_m\| = \|V^*Ax_n - V^*Ax_m\| \to 0 \quad \text{as } m, n \to 0,$$

so that $\{Ax_n\}$ converges to some vector z and $V^*Ax_n \to V^*z = y$. But since A is closed, $x \in \mathscr{D}(A)$ and $Ax = z$. Thus $y = V^*z = V^*Ax$, and V^*A is closed. If U' is a unitary operator in $\mathcal{R}'$, then $U'^*AU' = A$ so that

$$U'^*V^*AU' = V^*U'^*AU' = V^*A$$

since $V^* \in \mathcal{R}$. Thus $V^*A \; \eta \; \mathcal{R}$.

From Proposition 3.5, V^*A is self-adjoint. Since V^*A is contained in H and note, from Remark 2.5, that self-adjoint operators are maximal symmetric, $V^*A = H$. Hence

$$A = R(B)A = VV^*A = VH = B. \qquad \square$$

Proposition 3.7. *Suppose that operators S and T are affiliated with $\mathcal{R}$, then*

(i) *$S + T$ is densely defined, preclosed and has a unique closed extension $S \hat{+} T$ affiliated with $\mathcal{R}$;*

(ii) *ST is densely defined, preclosed and has a unique closed extension $S \hat{\cdot} T$ affiliated with $\mathcal{R}$.*

Proof. Let VH and WK be the polar decompositions of S and T, respectively, and let E_n and F_n be the spectral projections for H and K, respectively, corresponding to the interval $[-n, n]$ for each positive integer n.

(i) From the spectral theorem, $\{E_n\}$ and $\{F_n\}$ are increasing sequences of projections with strong-operator limit I. From Proposition 3.3, $\{E_n \wedge F_n\}$ is an increasing sequence with strong-operator limit I. Thus $\bigcup_{n=1}^{\infty}(E_n \wedge F_n)(\mathcal{H})$ is dense in $\mathcal{H}$. If $x \in (E_n \wedge F_n)(\mathcal{H})$, then $x \in \mathscr{D}(H) \cap \mathscr{D}(K)$. Hence $x \in \mathscr{D}(S+T)$. It follows that $S + T$ is densely defined.

Since S and T are affiliated with $\mathcal{R}$, S^* and T^* are affiliated with $\mathcal{R}$. From the preceding conclusion, $\mathscr{D}(S^* + T^*)$ is dense in $\mathcal{H}$. Since $S^* + T^* \subseteq (S + T)^*$, $\mathscr{D}((S + T)^*)$ is dense in $\mathcal{H}$. From Theorem 2.4, $S + T$ is preclosed. The closure $S \hat{+} T$ of $S + T$ is the smallest closed extension of $S + T$. If U' is a unitary operator in $\mathcal{R}'$ and $x \in \mathscr{D}(S + T)$, then $x \in \mathscr{D}(S)$, $x \in \mathscr{D}(T)$, $U'x \in \mathscr{D}(S)$, $U'x \in \mathscr{D}(T)$ (Recall that a unitary operator transforms the domain of each affiliated operator onto the domain itself.), and

$$(S + T)U'x = SU'x + TU'x = U'Sx + U'Tx = U'(S + T)x.$$

From Remark 2.9, $S \hat{+} T \; \eta \; \mathcal{R}$ since $\mathscr{D}(S + T)$ is a core for $S \hat{+} T$. If A is a closed extension of $(S + T)$ and $A \; \eta \; \mathcal{R}$, then $S \hat{+} T \subseteq A$. From Proposition 3.6, $S \hat{+} T = A$. Therefore, $S \hat{+} T$ is the only closed extension of $S + T$ affiliated with $\mathcal{R}$.

(ii) By choice of F_n, KF_n is a bounded, everywhere-defined, self-adjoint operator in $\mathcal{R}$. Let $T_n = TF_n$. Then $T_n (= TF_n = WKF_n)$ is a bounded, everywhere-defined, operator in $\mathcal{R}$. Let G_n be the projection on the range $F_n(\mathcal{H}) \cap T_n^{-1}(E_n(\mathcal{H}))$. From Proposition 3.4, the projection M_n with range $\{x : T_n x \in E_n(\mathcal{H})\}$ is in $\mathcal{R}$ and $E_n \precsim M_n$. Since $\{E_n\}$ is an increasing sequence of projections with strong-operator limit I, $\Delta(E_n) = \tau(E_n) \uparrow \tau(I) = I$ in the strong-operator topology, where Δ is the center-valued dimension function and τ is the center-valued trace on $\mathcal{R}$. Since $\{M_n\}$ is an increasing sequence and $\tau(E_n) \leqslant \tau(M_n)$, $\tau(M_n) \uparrow I$. Hence $\{M_n\}$ has strong-operator limit I. From Proposition 3.3, $\{G_n\} = \{F_n \wedge M_n\}$ is an increasing sequence with strong-operator limit I. It follows that $\bigcup_{n=1}^{\infty} G_n(\mathcal{H})$ is dense in $\mathcal{H}$. If $x \in G_n(\mathcal{H})$, then $T_n x \in E_n(\mathcal{H})$ so that $T_n x \in \mathscr{D}(H) = \mathscr{D}(S)$. At the same time, $x \in F_n(\mathcal{H})$ so that $x \in \mathscr{D}(K) = \mathscr{D}(T)$ and $Tx = TF_n x = T_n x$. Thus $x \in \mathscr{D}(ST)$. It follows that ST is densely defined.

Now, T^*S^* is densely defined since $S^* \; \eta \; \mathcal{R}$ and $T^* \; \eta \; \mathcal{R}$. Note that $T^*S^* \subseteq (ST)^*$, thus $(ST)^*$ is densely defined. From Theorem 2.4, ST is preclosed. The closure $S \hat{\cdot} T$ of ST is the smallest closed extension of ST. If U' is a unitary operator in $\mathcal{R}'$ and $x \in \mathscr{D}(ST)$, then $x \in \mathscr{D}(T)$, $Tx \in \mathscr{D}(S)$, $U'x \in \mathscr{D}(T)$, $TU'x = U'Tx \in \mathscr{D}(S)$, and

$$STU'x = SU'Tx = U'STx.$$

As with $S \hat{+} T$ in (i), $S \hat{\cdot} T \; \eta \; \mathcal{R}$ and $S \hat{\cdot} T$ is the only closed extension of ST affiliated with $\mathcal{R}$. □

Proposition 3.8. *Suppose that operators A, B and C are affiliated with $\mathcal{R}$, then*

$$(A \hat{\cdot} B) \hat{\cdot} C = A \hat{\cdot} (B \hat{\cdot} C),$$

that is, the associative law holds under the multiplication $\hat{\cdot}$ described in Proposition 3.7.

Proof. First, we note that $(A \hat{\cdot} B) \hat{\cdot} C$ is a closed extension of $(A \cdot B) \cdot C$ and $A \hat{\cdot} (B \hat{\cdot} C)$ is a closed extension of $A \cdot (B \cdot C)$, that is,

$$(A \cdot B) \cdot C \subseteq (A \hat{\cdot} B) \hat{\cdot} C \quad \text{and} \quad A \cdot (B \cdot C) \subseteq A \hat{\cdot} (B \hat{\cdot} C),$$

where " $\cdot$ " is the usual multiplication of operators. As a matter of fact,

$$(A \cdot B) \cdot C = A \cdot (B \cdot C)$$

since $\mathscr{D}((A \cdot B) \cdot C) = \mathscr{D}(A \cdot (B \cdot C))$ and $(A \cdot B) \cdot Cx = A \cdot (B \cdot C)x$. Now we let

$$(A \cdot B) \cdot C (= A \cdot (B \cdot C)) = A \cdot B \cdot C.$$

If we can show that the operator $A \cdot B \cdot C$ is densely defined, preclosed and its closure, say, $\overline{A \cdot B \cdot C}$, is affiliated with $\mathcal{R}$, then from Proposition 3.6,

$$\overline{A \cdot B \cdot C} \subseteq (A \hat{\cdot} B) \hat{\cdot} C \quad \text{and} \quad \overline{A \cdot B \cdot C} \subseteq A \hat{\cdot} (B \hat{\cdot} C)$$

will imply that $\overline{A \cdot B \cdot C} = (A \hat{\cdot} B) \hat{\cdot} C = A \hat{\cdot} (B \hat{\cdot} C)$.

Let V_1H_1, V_2H_2 and V_3H_3 be the polar decompositions of A, B and C, respectively. Let E_n, F_n and G_n be the spectral projections for H_1, H_2 and H_3, respectively, corresponding to the interval $[-n, n]$ for each positive integer n. By choice of G_n, H_3G_n is a bounded, everywhere-defined, self-adjoint operator in $\mathcal{R}$. It follows that the operator $CG_n (= V_3H_3G_n)$, denoted by C_n, is a bounded, everywhere-defined operator. Let J_n be the projection with range $G_n(\mathcal{H}) \cap C_n^{-1}(F_n(\mathcal{H}))$. As in the preceding proposition, $\{J_n\}$ is an increasing sequence with strong-operator limit I. Thus $\bigcup_{n=1}^{\infty} J_n(\mathcal{H})$ is dense in $\mathcal{H}$. If $x \in J_n(\mathcal{H})$, then $C_nx \in F_n(\mathcal{H})$ so that $C_nx \in \mathscr{D}(H_2) = \mathscr{D}(B)$. At the same time, $x \in G_n(\mathcal{H})$ so that $x \in \mathscr{D}(H_3) = \mathscr{D}(C)$ and $Cx = CG_nx = C_nx$. Thus $x \in \mathscr{D}(BC)$. Let $B_n = (BC)J_n$. By our definition of J_n (the projection on the range $G_n(\mathcal{H}) \cap C_n^{-1}(F_n(\mathcal{H}))$), $J_n \leqslant G_n$ so that CJ_n is a bounded, everywhere-defined operator in $\mathcal{R}$ and

$$CJ_n(\mathcal{H}) = (CG_n)J_n(\mathcal{H}) = C_nJ_n(\mathcal{H}) \subseteq F_n(\mathcal{H}).$$

It follows that $B_n = (BC)J_n = B(CJ_n)$ is a bounded, everywhere-defined operator in $\mathcal{R}$. Let K_n be the projection with range $J_n(\mathcal{H}) \cap B_n^{-1}(E_n)$. Similarly, $\{K_n\}$ is an increasing sequence with strong-operator limit I. Thus $\bigcup_{n=1}^{\infty} K_n(\mathcal{H})$ is dense in $\mathcal{H}$. If $x \in K_n(\mathcal{H})$, that is

$$x \in J_n(\mathcal{H}) \cap B_n^{-1}(E_n) = G_n(\mathcal{H}) \cap C_n^{-1}(F_n(\mathcal{H})) \cap B_n^{-1}(E_n),$$

then $B_nx \in E_n(\mathcal{H})$ so that $B_nx \in \mathscr{D}(H_1) = \mathscr{D}(A)$. At the same time, $x \in J_n(\mathcal{H})$ so that $x \in \mathscr{D}(BC)$ and $BCx = BCJ_nx = B_nx$. Thus $x \in \mathscr{D}(A \cdot B \cdot C)$. It follows that $A \cdot B \cdot C$ has a dense domain.

Since A, B and C are affiliated with $\mathcal{R}$, A^*, B^* and C^* are affiliated with $\mathcal{R}$. From the preceding conclusion, $C^* \cdot B^* \cdot A^*$ is densely defined. Since $B^* \cdot A^* \subseteq (A \cdot B)^*$, $C^* \cdot B^* \cdot A^* = C^* \cdot (B^* \cdot A^*) \subseteq C^* \cdot (A \cdot B)^* \subseteq (A \cdot B \cdot C)^*$ so that $(A \cdot B \cdot C)^*$ is densely defined. It follows that $A \cdot B \cdot C$ is preclosed. Next, we shall show that the closure $\overline{A \cdot B \cdot C}$ of $A \cdot B \cdot C$ is affiliated with $\mathcal{R}$.

If U' is a unitary operator in $\mathcal{R}'$ and $x \in \mathscr{D}(A \cdot B \cdot C)$, then

$$A \cdot B \cdot C \cdot U'x = A \cdot B \cdot U' \cdot Cx \qquad (x,\ U'x \in \mathscr{D}(C);\ Cx,\ U'Cx \in \mathscr{D}(B))$$

$$= A \cdot U' \cdot B \cdot Cx \qquad (BCx,\ U'BCx \in \mathscr{D}(A))$$
$$= U' \cdot A \cdot B \cdot Cx.$$

From Remark 2.9, $\overline{A \cdot B \cdot C}\ \eta\ \mathcal{R}$ since $\mathscr{D}(A \cdot B \cdot C)$ is a core for $\overline{A \cdot B \cdot C}$. □

Proposition 3.9. *Suppose that operators A, B and C are affiliated with $\mathcal{R}$, then*

$$(A \hat{+} B) \hat{\cdot} C = A \hat{\cdot} C \hat{+} (B \hat{\cdot} C) \quad \text{and} \quad C \hat{\cdot} (A \hat{+} B) = C \hat{\cdot} A \hat{+} (C \hat{\cdot} B),$$

that is, the distributive laws hold under the addition $\hat{+}$ and multiplication $\hat{\cdot}$ described in Proposition 3.7.

Proof. First, we note the following

$$(A+B)C \subseteq (A \hat{+} B) \hat{\cdot} C, \quad AC + BC \subseteq A \hat{\cdot} C \hat{+} (B \hat{\cdot} C),$$

$$C(A+B) \subseteq C \hat{\cdot} (A \hat{+} B), \quad CA + AB \subseteq C \hat{\cdot} A \hat{+} (C \hat{\cdot} B),$$

and

$$(A+B)C = AC + BC, \quad CA + CB \subseteq C(A+B).$$

As in the proof of Proposition 3.7, we shall show that $(A+B)C$ and $CA+CB$ are densely defined, preclosed and their closures $\overline{(A+B)C}$ and $\overline{CA+CB}$ are affiliated with the finite von Neumann algebra $\mathcal{R}$, respectively. Then from Proposition 3.6,

$$\overline{(A+B)C} = (A \hat{+} B) \hat{\cdot} C = A \hat{\cdot} C \hat{+} (B \hat{\cdot} C)$$

and

$$\overline{CA+CB} = C \hat{\cdot} A \hat{+} (C \hat{\cdot} B) = C \hat{\cdot} (A \hat{+} B).$$

As in the proof of Proposition 3.8, we let V_1H_1, V_2H_2 and V_3H_3 be the polar decompositions of A, B and C, respectively. Let E_n, F_n and G_n be the spectral projections for H_1, H_2 and H_3, respectively, corresponding to the interval $[-n, n]$ for each positive integer n. Define $C_n = CG_n = V_3H_3G_n$. By choice of G_n, C_n is a bounded, everywhere-defined operator. Let J_n be the projection on the range $G_n(\mathcal{H}) \cap C_n^{-1}((E_n \wedge F_n)(\mathcal{H}))$. Then $\bigcup_{n=1}^{\infty} J_n(\mathcal{H})$ is dense in $\mathcal{H}$ since $\{J_n\}$ is an increasing sequence with strong-operator limit I. If $x \in J_n(\mathcal{H})$, then $C_nx \in (E_n \wedge F_n)(\mathcal{H})$ so that $C_nx \in \mathscr{D}(A+B)$. At the same time, $x \in G_n(\mathcal{H})$ so that $x \in \mathscr{D}(H_3) = \mathscr{D}(C)$ and $Cx = CG_nx = C_nx$. Thus $x \in \mathscr{D}((A+B)C)$. It follows that $(A+B)C$ is densely defined.

Let $A_n = AE_n$ and $B_n = BF_n$. Then A_n and B_n are bounded, everywhere-defined operators in $\mathcal{R}$. Let K_n be the projection on the range

$$(E_n(\mathcal{H}) \cap A_n^{-1}(G_n(\mathcal{H}))) \cap (F_n(\mathcal{H}) \cap B_n^{-1}(G_n(\mathcal{H}))).$$

Again, $\{K_n\}$ is an increasing sequence with strong-operator limit I so that $\bigcup_{n=1}^{\infty} K_n(\mathcal{H})$ is dense in $\mathcal{H}$. If $x \in K_n(\mathcal{H})$, then $A_nx \in G_n(\mathcal{H})$ and $B_nx \in G_n(\mathcal{H})$ so that $A_nx \in \mathscr{D}(C)$ and $B_nx \in \mathscr{D}(C)$. At the same time, $x \in E_n(\mathcal{H})$ and $x \in F_n(\mathcal{H})$ so that $x \in \mathscr{D}(A)$, $x \in \mathscr{D}(B)$ and $Ax = AE_nx = A_nx$, $Bx = BF_nx = B_nx$. Thus $x \in \mathscr{D}(CA+CB)$. It follows that $CA+CB$ has a dense domain.

Now, we proceed to show that $(A+B)C$ and $CA+CB$ are preclosed by showing that $((A+B)C)^*$ and $(CA+CB)^*$ are densely defined. Note, again, that if $A,\ B,\ C\ \eta\ \mathcal{R}$, then $A^*,\ B^*,\ C^*\ \eta\ \mathcal{R}$. From the preceding, $C^*A^* + C^*B^*$ and $(A^*+B^*)C^*$ are densely defined. Since

$$C^*A^* + C^*B^* \subseteq C^*(A^*+B^*) \subseteq C^*(A+B)^* \subseteq ((A+B)C)^*$$

and

$$(A^*+B^*)C^* = A^*C^* + B^*C^* \subseteq (CA)^* + (CB)^* \subseteq (CA+CB)^*,$$

$\mathscr{D}(((A+B)C)^*)$ and $\mathscr{D}((CA+CB)^*)$ are dense in $\mathcal{H}$.

It remains to show that the closures $\overline{(A+B)C}$ and $\overline{CA+CB}$ are affiliated with $\mathcal{R}$. If U' is a unitary operator in $\mathcal{R}'$, for $x \in \mathscr{D}((A+B)C)$,

$$\begin{aligned}(A+B)CU'x &= (A+B)U'Cx \\ &= AU'Cx + BU'Cx = U'ACx + U'BCx \\ &= U'(ACx + BCx) = U'(A+B)Cx\end{aligned}$$

and for $x \in \mathscr{D}(CA+CB)$,

$$\begin{aligned}(CA+CB)U'x &= CAU'x + CBU'x = CU'Ax + CU'Bx \\ &= U'CAx + U'CBx = U'(CA+CB)x.\end{aligned}$$

From Remark 2.9, $\overline{(A+B)C}\ \eta\ \mathcal{R}$ and $\overline{CA+CB}\ \eta\ \mathcal{R}$. □

Proposition 3.10. *Suppose that operators A and B are affiliated with $\mathcal{R}$, then*

$$(aA \hat{+} bB)^* = \bar{a}A^* \hat{+} \bar{b}B^* \quad \textit{and} \quad (A \hat{\cdot} B)^* = B^* \hat{\cdot} A^*, \quad a, b \in \mathbb{C},$$

where $*$ is the usual adjoint operation on operators (possibly unbounded).

Proof. From Proposition 3.7, $aA+bB$ and AB are densely defined and preclosed with closures $aA \hat{+} bB$ and $A \hat{\cdot} B$ (affiliated with $\mathcal{R}$), respectively. Then from Theorem 2.4,

$$(aA+bB)^* = (aA \hat{+} bB)^*, \quad (AB)^* = (A \hat{\cdot} B)^*. \tag{3.2}$$

At the same time,

$$\bar{a}A^* + \bar{b}B^* \subseteq (aA+bB)^*, \quad B^*A^* \subseteq (AB)^*; \tag{3.3}$$

and both $(aA+bB)^*$ and $(AB)^*$ are closed (Remark 2.3). We also have $\bar{a}A^* \hat{+} \bar{b}B^*$ and $B^* \hat{\cdot} A^*$ as the closures (smallest closed extensions) of $\bar{a}A^* + \bar{b}B^*$ and B^*A^*, respectively. It follows that

$$\bar{a}A^* + \bar{b}B^* \subseteq \bar{a}A^* \hat{+} \bar{b}B^* \subseteq (aA+bB)^*, \quad B^*A^* \subseteq B^* \hat{\cdot} A^* \subseteq (AB)^*. \tag{3.4}$$

Now, (3.2) together with (3.4),

$$\bar{a}A^* \hat{+} \bar{b}B^* \subseteq (aA \hat{+} bB)^*, \quad B^* \hat{\cdot} A^* \subseteq (A \hat{\cdot} B)^*.$$

Since $\bar{a}A^* \hat{+} \bar{b}B^*$, $(aA \hat{+} bB)^*$, $B^* \hat{\cdot} A^*$ and $(A \hat{\cdot} B)^*$ are all affiliated with $\mathcal{R}$, from Proposition 3.6, $\bar{a}A^* \hat{+} \bar{b}B^* = (aA \hat{+} bB)^*$ and $B^* \hat{\cdot} A^* = (A \hat{\cdot} B)^*$. □

Theorem 3.11. *$\mathscr{A}(\mathcal{R})$ is a * algebra (with unit I) under the operations of addition $\hat{+}$ and multiplication $\hat{\cdot}$.*

4 Representations of the Heisenberg relation

4.1 In $\mathcal{B}(\mathcal{H})$

We are convinced that the relation $QP - PQ = i\hbar I$ cannot be realized in terms of finite matrices. The natural extension of this attempt is to wonder if infinite-dimensional Hilbert spaces might not "support" such a representation with bounded operators. Even this is not possible as we shall now show:

Proposition 4.1. *If A and B are elements of a Banach algebra $\mathfrak{A}$ with unit I, then* $\mathrm{sp}(AB) \cup \{0\} = \mathrm{sp}(BA) \cup \{0\}$.

Proof. If $\lambda \neq 0$ and $\lambda \in \mathrm{sp}(AB)$, then $AB - \lambda I$ and, hence $(\lambda^{-1}A)B - I$ are not invertible. On the other hand, if $\lambda \notin \mathrm{sp}(BA)$, then $BA - \lambda I$ and, hence, $B(\lambda^{-1}A) - I$ are invertible. Our task, then, is to show that $I - AB$ is invertible in $\mathfrak{A}$ if and only if $I - BA$ is invertible in $\mathfrak{A}$, for arbitrary elements A and B of $\mathfrak{A}$.

Let us argue informally for the moment. The following argument leads us to the correct formula for the inverse of $I - BA$, and gives us a proof that holds in any ring with a unit.

$$(I - AB)^{-1} = \sum_{n=0}^{\infty}(AB)^n = I + AB + ABAB + \cdots$$

and

$$B(I - AB)^{-1}A = BA + BABA + BABABA + \cdots = (I - BA)^{-1} - I.$$

Thus if $I - AB$ has an inverse, we may hope that $B(I - AB)^{-1}A + I$ is an inverse to $I - BA$. Multiplying, we have

$$\begin{aligned}&(I - BA)[B(I - AB)^{-1}A + I]\\ &\quad = B(I - AB)^{-1}A + I - BAB(I - AB)^{-1}A - BA\\ &\quad = B[(I - AB)^{-1} - AB(I - AB)^{-1}]A + I - BA = I,\end{aligned}$$

and similarly for right multiplication by $I - BA$. □

Finally, $\mathrm{sp}(A + I) = \{1 + a : a \in \mathrm{sp}(A)\}$, together with the proposition, yields the fact that the unit element I of a Banach algebra is not the *commutator* $AB - BA$ of two elements A and B. (If $I = AB - BA$, then $\mathrm{sp}(AB) = 1 + \mathrm{sp}(BA)$, which is not consistent with $\mathrm{sp}(AB) \cup \{0\} = \mathrm{sp}(BA) \cup \{0\}$.) Therefore, in quantum theory, the commutation relations (in particular, the Heisenberg relation) are not representable in terms of bounded operators.

In our search for ways to "represent" the Heisenberg relation in some (algebraic) mathematical structure, we have, thus far, eliminated finite matrices, bounded operators on an infinite-dimensional Hilbert space, and even elements of more general complex Banach algebras with a unit element. It becomes clear that unbounded operators would be essential for dealing with the non-commutativity that the Heisenberg relation carries. The following example gives a specific representation of the relation with one of the representing operators bounded and the other unbounded.

Example 4.2. Let $\mathcal{H}$ be the Hilbert space L_2, corresponding to Lebesgue measure on the unit interval $[0, 1]$, and let $\mathscr{D}_0$ be the subspace consisting of all complex-valued functions f that have a continuous derivative f' on $[0, 1]$ and satisfy $f(0) = f(1) = 0$. Let D_0 be the operator with domain $\mathscr{D}_0$ and with range in $\mathcal{H}$ defined by $D_0 f = f'$. We shall show that iD_0 is a densely defined symmetric operator and that

$$(iD_0)M - M(iD_0) = iI|\mathscr{D}_0,$$

where M is the bounded linear operator defined by $(Mf)(s) = sf(s)$ ($f \in L_2;\ 0 \leqslant s \leqslant 1$).

Proof. Each element f of $\mathcal{H}$ can be approximated (in L_2 norm) by a continuous function f_1. In turn, f_1 can be approximated (in the uniform norm, hence in the L_2 norm) by a polynomial f_2. Finally, f_2 can be approximated (in L_2 norm) by an element f_3 of $\mathscr{D}_0$; indeed, it suffices to take $f_3 = gf_2$, where $g : [0, 1] \to [0, 1]$ is continuously differentiable, vanishes at the endpoint 0 and 1, and takes the value 1 except at points very close to $0, 1$.

The preceding argument shows that $\mathscr{D}_0$ is dense in $\mathcal{H}$, so D_0 is a densely defined linear operator. When $f, g \in \mathscr{D}_0$, the function $\bar{g}$ has a continuous derivative $\bar{g}'$, and we have

$$\begin{aligned}\langle D_0 f, g\rangle &= \int_0^1 f'(s)\overline{g(s)}\,ds = \Big[f(s)\overline{g(s)}\Big]_0^1 - \int_0^1 f(s)\overline{g'(s)}\,ds\\ &= -\int_0^1 f(s)\overline{g'(s)}\,ds = -\langle f, D_0 g\rangle.\end{aligned}$$

Thus $\langle iD_0 f, g\rangle = \langle f, iD_0 g\rangle$, for all f and g in $\mathscr{D}_0$; and iD_0 is symmetric.

When $f \in \mathscr{D}_0$, $Mf \in \mathscr{D}_0$ and

$$(D_0Mf)(s) = \frac{d}{ds}(sf(s)) = f(s) + sf'(s) = f(s) + (MD_0f)(s).$$

Thus $D_0Mf - MD_0f = f \quad (f \in \mathscr{D}_0)$. □

One can press this example further to show that iD_0 has a self-adjoint extension.

4.2 The classic representation

Given the discussion and results to this point, what are we to understand by a "representation of the Heisenberg relation", $QP - PQ = i\hbar I$? Having proved that this representation cannot be achieved with finite matrices in place of Q and P and I, nor even with bounded operators on a Hilbert space, nor elements Q, P, I in a complex Banach algebra, we begin to examine the possibility that this representation can be effected with unbounded operators for Q and P. It is "rumored", loosely, that Q, which is associated with the physical observable "position" on $\mathbb{R}$, and P, which is associated with the (conjugate) "momentum" observable, will provide such a representation. The observable Q is modeled, nicely, by the self-adjoint operator, multiplication by x on $L_2(\mathbb{R})$, with domain those f in $L_2(\mathbb{R})$ such that xf is in $L_2(\mathbb{R})$. The observable P is modeled by $i\frac{d}{dt}$, differentiation on some appropriate domain of differentiable functions with derivatives in $L_2(\mathbb{R})$. But $QP - PQ$ certainly cannot equal $i\hbar I$, since its domain is contained in $\mathscr{D}(Q) \cap \mathscr{D}(P)$, which is not $\mathcal{H}$. The domain of P must be chosen so that P is self-adjoint and $\mathscr{D}(QP - PQ)$ is dense in $\mathcal{H}$ and $QP - PQ$ agrees with $i\hbar I$ on this dense domain. In particular, $QP - PQ \subseteq i\hbar I$. Since $i\hbar I$ is bounded, it is closed, and $QP - PQ$ is closable with closure $i\hbar I$. We cannot insist that, with the chosen domains for Q and P, $QP - PQ$ be skew-adjoint, for then it would be closed, bounded, and densely defined, hence, everywhere defined. In the end, we shall mean by "a representation of the Heisenberg relation $QP - PQ = i\hbar I$ on the Hilbert space $\mathcal{H}$" a choice of self-adjoint operators Q and P on $\mathcal{H}$ such that $QP - PQ$ has closure $i\hbar I$.

As mentioned above, the classic way to represent the Heisenberg relation $QP - PQ = i\hbar I$ with unbounded self-adjoint operators Q and P on a Hilbert space $\mathcal{H}$ is to realize $\mathcal{H}$ as $L_2(\mathbb{R})$, the space of square-integrable, complex-valued functions on $\mathbb{R}$ and Q and P as, respectively, the operator Q corresponding to multiplication by x, the identity transform on $\mathbb{R}$, and the operator P corresponding to $i\frac{d}{dt}$, where $\frac{d}{dt}$ denotes differentiation, each of Q and P with a suitable domain in $L_2(\mathbb{R})$. The domain of Q consists of those f in $L_2(\mathbb{R})$ such that xf is in $L_2(\mathbb{R})$. The operator $\frac{d}{dt}$ is intended to be differentiation on $L_2(\mathbb{R})$, where that differentiation makes sense — certainly, on every differentiable functions with derivative in $L_2(\mathbb{R})$. However, specifying a dense domain, precisely, including such functions, on which "differentiation" is a self-adjoint operator is not so simple. A step function, a function on $\mathbb{R}$ that is constant on each connected component of an open dense subset of $\mathbb{R}$ (those components being open intervals) has a derivative almost everywhere (at all but the set of endpoints of the intervals — a countable set), and that derivative is 0. The set of such step functions in $L_2(\mathbb{R})$ is dense in $L_2(\mathbb{R})$, as is their linear span. To include that linear span in a proposed domain for our differentiation operator condemns any closed operator extending our differentiation operator to be the everywhere-defined operator 0. Of course, that is not what we are aiming for. Another problem that we face in this discussion is that of "mixing" measure theory with differentiation. We speak, loosely, of elements of our Hilbert space $L_2(\mathbb{R})$ as "functions". We have learned to work quickly and accurately with the mathematical convenience that this looseness provides us avoiding such pitfalls as taking the union of "too many" sets of measure 0 in the process. The elements of $L_2(\mathbb{R})$ are, in fact, equivalence classes of functions differing from one another on sets of measure 0. On the other hand, differentiation is a process that focuses on points, each point being a set of Lebesgue measure zero. When we speak of the L_2-norm of a *function* in $L_2(\mathbb{R})$ it does not matter which function in the *class* in question we work with; they all have the same norm. It is not the same with differentiability. Not each function in the class of an everywhere differentiable function is everywhere differentiable. There are functions in such classes that are nowhere differentiable, indeed, nowhere continuous (at each point of differentiability a function is continuous). The measure class of each function on $\mathbb{R}$ contains a function that is nowhere continuous. To see this, choose two disjoint, countable, everywhere-dense subsets, for example, the rationals $\mathbb{Q}$ in $\mathbb{R}$ and $\mathbb{Q}+\sqrt{2}$. With f a given function on $\mathbb{R}$, the function g that agrees with f, except on $\mathbb{Q}$ where it takes the value 0 and on $\mathbb{Q}+\sqrt{2}$ where it takes the value 1 is in the measure class of f and is continuous nowhere (since each non-null open set in $\mathbb{R}$ contains a point at which g takes the value 0 and a point at which it takes the value 1). These are some of the problems that arise in dealing with an appropriate domain for $\frac{d}{dt}$.

There is an elegant way to approach the problem of finding precisely the self-adjoint operator and its domain that we are seeking. That approach is through the use of "Stone's theorem" (from the very beginning of the theory of unitary representations of infinite groups). We start with a clear statement of the theorem. Particular attention should be paid to the description of the domain of the generator iH in this statement.

Theorem 4.3. (*Stone's theorem*) *If H is a* (*possibly unbounded*) *self-adjoint operator on the Hilbert space $\mathcal{H}$, then $t\to \exp itH$ is a one-parameter unitary group on $\mathcal{H}$. Conversely, if $t\to U_t$ is a one-parameter unitary group on $\mathcal{H}$, there is a* (*possibly unbounded*) *self-adjoint operator H on $\mathcal{H}$ such that $U_t=\exp itH$ for each real t. The domain of H consists of precisely those vectors x in $\mathcal{H}$ for which $t^{-1}(U_tx-x)$ tends to a limit as t tends to* 0*, in which case this limit is iHx.*

The relevance of Stone's theorem emerges from the basic case of the one-parameter unitary group $t\to U_t$ on $L_2(\mathbb{R})$, where $(U_tf)(s)=f(s-t)$. That is, U_t is "translation by t". In this case, $U_t=\exp itH$, with H a self-adjoint operator acting on $L_2(\mathbb{R})$. The domain of H consists of those f in $L_2(\mathbb{R})$ such that $t^{-1}(U_tf-f)$ tends to a limit g in $L_2(\mathbb{R})$, as t tends to 0, in which case, $iHf=g$. We treat $\frac{d}{dt}$ as the infinitesimal generator of this one-parameter unitary group. An easy measure-theoretic argument shows that this one-parameter unitary group is strong-operator continuous on $\mathcal{H}$. That is, $U_tf\to U_{t'}f$, in the norm topology of $\mathcal{H}$, as $t\to t'$, for each f in $\mathcal{H}$, or what amounts to the same thing, since $t\to U_t$ is a one-parameter group, if $U_{t''}f=U_{t-t'}f\to f$, when $(t-t')=t''\to 0$ for each f in $L_2(\mathbb{R})$. From Stone's theorem, there is a skew-adjoint (unbounded) operator (iH) we denote by $\frac{d}{dt}$ on $\mathcal{H}$ such that $U_t=\exp t\frac{d}{dt}$ for each real t. The domain of $\frac{d}{dt}$ consists of those f in $L_2(\mathbb{R})$ such that $t^{-1}(U_tf-f)$ tends to some g in $L_2(\mathbb{R})$ as t tends to 0, in which case $g=\frac{d}{dt}f$.

Now, let us make some observations to see how Stone's theorem works in our situation. Our aim, at this point, is to study just which functions are and are not in the domain of $\frac{d}{dt}$. (This study will make clear how apt the notation $\frac{d}{dt}$ is for the infinitesimal generator of the group of real translations of $\mathbb{R}$.) To begin with, Stone's theorem requires us to study the convergence behavior of $t^{-1}(U_tf-f)$ as t tends to 0. This requirement is to study the convergence behavior in the Hilbert space metric (in the "mean of order 2", in the terminology of classical analysis), but there is no harm in examining how $t^{-1}(U_tf-f)$ varies pointwise with t at points s in $\mathbb{R}$. For this, note that

$$(t^{-1}(U_tf-f))(s)=\frac{f(s-t)-f(s)}{t}\to f'(s)\qquad \text{as } t\to 0,$$

which suggests f' as the limit of $t^{-1}(U_tf-f)$ when f is differentiable with f' in $L_2(\mathbb{R})$ (and motivates the use of the notation "$\frac{d}{dt}$" for the infinitesimal generator of $t\to U_t$). However, the "instructions" of Stone's theorem tell us to find g in $L_2(\mathbb{R})$ such that

$$\int\left|\frac{f(s-t)-f(s)}{t}-g(s)\right|^2 d\mu(s)\to 0$$

as $t\to 0$, where μ is Lebesgue measure on $\mathbb{R}$. Our first observation is that if f fails to have a derivative at some point s_0 in $\mathbb{R}$ *in an essential way*, then f is not in the domain of $\frac{d}{dt}$. This may be surprising, at first, for the behavior of a function at a point rarely has (Lebesgue) measure-theoretic consequences. In the present circumstances, we shall see that the "local" nature of differentiation can result in exclusion from the domain of an unbounded differentiation operator because of non-differentiability at a single point.

We begin with a definition of "jump in a function" that is suitable for our measure-theoretic situation.

Definition 4.4. *We say that f has jump $a\,(\geqslant 0)$ for width $\delta\,(>0)$ at s_0 in $\mathbb{R}$ when $\inf\{f(s)\}$ with s in one of the intervals $[s_0-\delta,s_0)$ or $(s_0,s_0+\delta]$ is $a+\sup\{f(s)\}$ with s in the other of those intervals.*

Typically, one speaks of a "jump discontinuity" when $\lim_{s\to s_0^-} f(s)$ and $\lim_{s\to s_0^+} f(s)$ exist and are distinct. In the strictly measure-theoretic situation with which we are concerned, the concept of "jump", as just defined, seems more appropriate.

Remark 4.5. If f has a jump a for width δ at some point s_0 in $\mathbb{R}$, then $U_{s_0}f$ has a jump a for width δ at 0, and $bU_{s_0}f$ has jump ba for width δ at 0 when $0 < b$. Letting f_r be the function whose value at s is $f(rs)$, one has that f_r has a jump a at $r^{-1}s_0$ for width $r^{-1}\delta$. Thus $a^{-1}(U_{s_0}f)_\delta$ has jump 1 at 0 for width 1.

Theorem 4.6. *If f has a positive jump, then $f \notin \mathscr{D}(\frac{d}{dt})$.*

Proof. We shall show that $\|t^{-1}(U_tf - f)\|$ is unbounded for t in each open interval in $\mathbb{R}$ containing 0. Of course, this is so if and only if $\|t^{-1}bU_s(U_tf - f)\|$ is unbounded for each given positive b and U_s. Thus, from Remark 4.5, it will suffice to show that $\|t^{-1}(U_tf - f)\|$ is unbounded when f has jump 1 at 0. Noting that $\|g_r\| = r^{-1}\|g\|$ for g in $L_2(\mathbb{R})$, that $(g + h)_r = g_r + h_r$, and that $(U_tf)_r = U_{r^{-1}t}f_r = U_{t'}f_r$, where $t' = r^{-1}t \to 0$ as $t \to 0$, we have

$$\begin{aligned} r^{-1}t^{-1}\|U_tf - f\| &= t^{-1}\|(U_tf - f)_r\| = t^{-1}\|(U_tf)_r - f_r\| \\ &= t^{-1}\|U_{r^{-1}t}f_r - f_r\| = r^{-1}t'^{-1}\|U_{t'}f_r - f_r\|. \end{aligned}$$

Thus $\|t^{-1}(U_tf - f)\| = \|t'^{-1}(U_{t'}f_r - f_r)\|$. It follows that $\|t^{-1}(U_tf - f)\|$ is bounded for t near 0 if and only if $\|t'^{-1}(U_{t'}f_r - f_r)\|$ is. This holds for each positive r, in particular, when r is δ, where f has jump 1 at 0 for width δ. Since f_δ has jump 1 at 0 for width 1 $(= \delta^{-1}\delta)$, from Remark 4.5, it will suffice to show that $\|t^{-1}(U_tf - f)\|$ is unbounded for t near 0, when f has jump 1 at 0 for width 1. We shall do this by finding a sequence $t_2, t_3, \dots$ of positive numbers t_j tending to 0 such that $\|t_j^{-1}(U_{t_j}f - f)\| \to \infty$ as $j \to \infty$. We assume that f has jump 1 at 0 for width 1. In this case, $|f(s') - f(s'')| \geqslant 1$ when $s' \in [-1, 0)$ and $s'' \in (0, 1]$. Thus, when $t_n = \frac{1}{n-1}$,

$$\begin{aligned} \|t_n^{-1}(U_{t_n}f - f)\|^2 &= \int_{\mathbb{R}} |t_n^{-1}(U_{t_n}f - f)|^2 \, d\mu(s) \\ &\geqslant \int_{(0,\frac{1}{n}]} |(n-1)(f(s - t_n) - f(s))|^2 \, d\mu(s) \\ &\geqslant \frac{1}{n}(n-1)^2 = n - 2 + \frac{1}{n}. \end{aligned}$$

It follows that $\|(n - 1)(U_{(n-1)^{-1}}f - f)\| \to \infty$ as $n \to \infty$. Hence $t^{-1}(U_tf - f)$ has no limit in $L_2(\mathbb{R})$ as $t \to 0$ and $f \notin \mathscr{D}(\frac{d}{dt})$. □

Theorem 4.7. *If f_1 is a continuously differentiable function on $\mathbb{R}$ such that f_1 and f_1' are in $L_2(\mathbb{R})$, then $f_1 \in \mathscr{D}(\frac{d}{dt})$; and $\frac{d}{dt}(f_1) = f_1'$.*

Proof. We prove, first, that if f, in $L_2(\mathbb{R})$, vanishes outside some interval $[-n, n]$, with n a positive integer, and f is continuously differentiable on $\mathbb{R}$ with derivative f' in $L_2(\mathbb{R})$, then $f \in \mathscr{D}(\frac{d}{dt})$ and $\frac{d}{dt}(f) = f'$.

From Stone's theorem, we must show that $\|t^{-1}(U_tf - f) - f'\|_2 \to 0$ as $t \to 0$. Now,

$$\begin{aligned} \|t^{-1}(U_tf - f) - f'\|_2^2 &= \int_{[-n,n]} |[t^{-1}(U_tf - f) - f'](s)|^2 \, d\mu(s) \\ &= \int_{[-n,n]} \left| \frac{f(s - t) - f(s)}{t} - f'(s) \right|^2 d\mu(s). \end{aligned}$$

Note that $t^{-1}(U_tf - f) - f'$ tends to 0 (pointwise) everywhere on $\mathbb{R}$ as t tends to 0. Of course, $t^{-1}(U_tf - f)$ and f' vanish outside of $[-(n + 1), n + 1]$ when $|t| < 1$. Since f is differentiable, it is continuous and bounded on $[-(n + 1), n + 1]$. By assumption, f' is continuous, hence bounded on $[-(n + 1), n + 1]$ (on $\mathbb{R}$). Say, $|f'(s)| \leqslant M$, for each s. From the Mean Value theorem, for s in $[-n, n]$,

$$|t^{-1}(U_tf - f)(s)| = \left| \frac{f(s - t) - f(s)}{t} \right| = |f'(s')| \leqslant M,$$

for some s' in the interval with endpoints s and $s-t$. Thus $|t^{-1}(U_tf-f)|$ is bounded by M, on $[-n,n]$ for all t in $(-1,1)$. At the same time, $t^{-1}(U_tf-f)$ tends to f' everywhere (that is, pointwise) on $[-n,n]$. From Egoroff's theorem, $t^{-1}(U_tf-f)$ tends *almost uniformly* to f' on $[-n,n]$ as t tends to 0. Hence, given a positive ε, there is a subset S of $[-n,n]$ of measure less than $\varepsilon/8M^2$ such that $t^{-1}(U_tf-f)$ converges uniformly to f' on $[-n,n]\setminus S$.

We show, now, that $t^{-1}(U_tf-f)$ converges to f' in $L_2(\mathbb{R})$. With ε and S chosen as in the preceding paragraph, by uniform convergence on $[n,-n]\setminus S$, we find a positive δ such that for $0<|t|<\delta$, and s in $[-n,n]\setminus S$,

$$|t^{-1}(f(s-t)-f(s))-f'(s)|^2<\frac{\varepsilon}{4n}.$$

Hence, when $0<|t|<\delta$,

$$\begin{aligned}\|t^{-1}(U_tf-f)-f'\|_2^2&=\int_{[-n,n]\setminus S}\left|\frac{f(s-t)-f(s)}{t}-f'(s)\right|^2d\mu(s)+\int_S\left|\frac{f(s-t)-f(s)}{t}-f'(s)\right|^2d\mu(s)\\&\leqslant 2n\frac{\varepsilon}{4n}+4M^2\frac{\varepsilon}{8M^2}=\varepsilon.\end{aligned}$$

The desired convergence of $t^{-1}(U_tf-f)$ to f' in $L_2(\mathbb{R})$ follows from this.

With f_1 as in the statement of this theorem, suppose that we can find f as in the preceding discussion (that is, vanishing outside a finite interval) such that $\|f_1-f\|_2$ and $\|f_1'-f'\|_2$ are less than a preassigned positive ε. Then (f_1,f_1') is in the closure of the graph of $\frac{d}{dt}$, since each (f,f') is in that closure from what we have proved. But $\frac{d}{dt}$ is skew-adjoint (from Stone's theorem); hence, $\frac{d}{dt}$ is closed. Thus, if we can effect the described approximation of f_1 and f_1' by f and f', it will follow that $f_1\in\mathscr{D}(\frac{d}{dt})$ and $\frac{d}{dt}(f_1)=f_1'$.

Since f_1 and f_1' are continuous and in $L_2(\mathbb{R})$, the same is true for $|f_1|+|f_1^-|+|f_1'|+|f_1'^-|$, where $g^-(s)=g(-s)$ for each s in $\mathbb{R}$ and each complex-valued function g on $\mathbb{R}$. (Note, for this, that $s\to -s$ is a Lebesgue-measure-preserving homeomorphism of $\mathbb{R}$ onto $\mathbb{R}$.) It follows that, for each positive integer n, there is a real s_n such that $n<s_n$ and

$$|f_1(s_n)|+|f_1(-s_n)|+|f_1'(s_n)|+|f_1'(-s_n)|<\frac{1}{n}.$$

(Otherwise, $|f_1(s)|+|f_1(-s)|+|f_1'(s)|+|f_1'(-s)|\geqslant\frac{1}{n}$, for each s outside of $[-n,n]$, contradicting the fact that $|f_1|+|f_1^-|+|f_1'|+|f_1'^-|\in L_2(\mathbb{R})$.) We can choose s_n such that $s_{n-1}<s_n$. Since $n<s_n$, we have that $s_n\to\infty$ as $n\to\infty$, and

$$\int_{[-s_n,s_n]}|h(s)|^2\,d\mu(s)\to\|h\|_2^2\qquad n\to\infty,$$

for each h in $L_2(\mathbb{R})$. Thus $\|h-h^{(n)}\|_2\to 0$ as $n\to\infty$, where $h^{(n)}$ is the function that agrees with h on $[-s_n,s_n]$ and is 0 outside this interval. With ε (<1) positive, there is an n_0 such that, if $n>n_0$, then each of $\|f_1-f_1^{(n)}\|_2$, $\|f_1^--f_1^{-(n)}\|_2$, $\|f_1'-f_1'^{(n)}\|_2$, and $\|f_1'^--f_1'^{-(n)}\|_2$ is less than $\frac{\varepsilon}{2}$. At the same time, we may choose n_0 large enough so that $\frac{1}{n}<\frac{\varepsilon}{4}$ when $n>n_0$. For such an n, a "suitably modified" $f_1^{(n)}$ will serve as the desired f for our approximation. In the paragraphs that follow, we describe that modification.

Our aim is to extend $f_1^{(n)}$ to $\mathbb{R}$ from $[-s_n,s_n]$ so that the extension f remains continuously differentiable with f and f' vanishing outside some finite interval and so that the projected approximations $\|f_1-f\|_2<\varepsilon$ and $\|f_1'-f'\|_2<\varepsilon$ are realized. In effect, we want $\|f_1^{(n)}-f\|_2$ and $\|f_1'^{(n)}-f'\|_2$ to be less than $\frac{\varepsilon}{2}$. Combined, then, with our earlier choice of n_0 such that, for $n>n_0$, $\|f_1-f_1^{(n)}\|_2<\frac{\varepsilon}{2}$ and $\|f_1'-f_1'^{(n)}\|_2<\frac{\varepsilon}{2}$, we have the desired approximation.

To construct f, we add to $f_1^{(n)}$ a function g continuous and continuously differentiable on $(-\infty,-s_n]\cup[s_n,\infty)$ such that $g(s_n)=f_1(s_n)$, $g'(s_n)=f_1'(s_n)$, $g(-s_n)=f_1(-s_n)$, $g'(-s_n)=f_1'(-s_n)$, g vanishes on $(-\infty,-s_n-1]\cup[s_n+1,\infty)$, and $\|g\|_2<\frac{\varepsilon}{2}$, $\|g'\|_2<\frac{\varepsilon}{2}$. With f so defined, $\|f_1^{(n)}-f\|_2=\|g\|_2<\frac{\varepsilon}{2}$ and $\|f_1'^{(n)}-f'\|_2=\|g'\|_2<\frac{\varepsilon}{2}$, as desired. We describe the construction of g on $[s_n,\infty)$. The construction of g on $(-\infty,-s_n]$ follows the same pattern. We present the construction of g geometrically — with reference to the graphs of the functions involved. The graphs are described in an XY plane, where $\mathbb{R}$ is identified with the X-axis. By choice of s_n and n $(> n_0)$, $|f_1(s_n))| < \frac{\varepsilon}{4}$, and $|f_1'(s_n)| < \frac{\varepsilon}{4}$.

Translating s_n to the origin, we see that our task is to construct a function h on $[0,1]$ continuously differentiable, 0 on $[\frac{1}{2},1]$, with given initial data $h(0)$, $h'(0)$ satisfying $|h(0)| < \frac{\varepsilon}{4}$, $|h'(0)| < \frac{\varepsilon}{4}$ such that $\|h\|_2 < \frac{\varepsilon}{2}$ and $\|h'\|_2 < \frac{\varepsilon}{2}$. If $h(0) = h'(0) = 0$, then h, with $h(x) = 0$, for each x in $[0,1]$, will serve as our h. If $h'(0) \neq 0$, we define h, first, on $[0,x_0]$, where $x_0 = \frac{1}{2}h(0)h'(0)$ and

$$(y_0 =)h(x_0) = \frac{1}{2}h(0)[1 + (1 + h'(0)^2)^{\frac{1}{2}}].$$

The restriction of h to $[0,x_0]$ has as its graph the ("upper, smaller") arch of the circle with center $(x_0, \frac{1}{2}h(0)$ and radius $\frac{1}{2}h(0)(1 + h'(0)^2)^{\frac{1}{2}}$ (tangent to the line with slope $h'(0)$ at $(0,h(0))$). Note that $h(0) < y_0 < 2h(0) < \frac{\varepsilon}{2}$ and that the circle described has a horizontal tangent at (x_0,y_0); that is, $h'(x_0) = 0$, as h has been defined.

We complete the definition of h by adjoining to the graph of h over $[0,x_0]$ the graph of $\frac{1}{2}y_0[\cos((\frac{1}{2} - x_0^{-1})\pi(x - x_0)) + 1]$ over $[x_0, \frac{1}{2}]$. Note that this graph passes through (x_0,y_0) and $(\frac{1}{2},0)$. Finally, we define $h(x)$ to be 0 when $x \in [\frac{1}{2},1]$. As constructed, h is continuously differentiable on $[0,1]$. Since $|h(x)| \leqslant 2|h((0)| < \frac{\varepsilon}{2}$ for x in $[0,\frac{1}{2}]$ and h vanishes on $[\frac{1}{2},1]$, $\|h\|_2 < \frac{\varepsilon}{2}$. □

We may ask whether the converse statement to the preceding theorem holds as well. Does a function class in $\mathscr{D}(\frac{d}{dt})$ necessarily contain a continuously differentiable function with derivative in $L_2(\mathbb{R})$? As it turns out, there are more functions, not as well behaved as continuously differentiable functions, in the domain of $\frac{d}{dt}$. We shall give a complete description of that domain in Theorem 4.8. However, the proof of the theorem requires a good deal of analytic-measure theoretic preparation. We shall indicate some of the main lines of the argument, but not include details.

Our notation and terminology has a somewhat "schizophrenic" character to it — much in the style of the way mathematics treats certain topics. In the present instance, we use the notation '$L_2(\mathbb{R})$' to denote, both, the collection (linear space) of measurable functions f such that $|f|^2$ is Lebesgue integrable on $\mathbb{R}$ and the Hilbert space of (measure-theoretic) equivalence classes of such functions equipped with the usual Hilbert space structure associated with L_2 spaces. In most circumstances, there is no danger of serious confusion or misinterpretation. In our present discussion of the domain of $\frac{d}{dt}$, these dangers loom large. We note, earlier in this section, that each measure-theoretic equivalence class of functions contains a function that is continuous at no point of $\mathbb{R}$. It can make no sense to attempt to characterize special elements x of $L_2(\mathbb{R})$ by the "smoothness" properties of *all* the functions in the equivalence class denoted by 'x' (their continuity, differentiability, and so forth). Despite this, our next theorem describes the domain given to us by the generator, which we are denoting by '$\frac{d}{dt}$', of the one-parameter unitary group $t \to U_t$ of translations of the equivalence classes of functions in $L_2(\mathbb{R})$ (to other such classes) in terms of smoothness properties. However, these smoothness properties will be those of a *single* element in the class as we shall see. We note, first, that if an equivalence class contains a continuous function f on $\mathbb{R}$, then f is the unique such function in the class. This is immediate from the fact that $f - g$ vanishes nowhere on some non-null, open interval when f and g are distinct continuous functions, whence f and g differ on a set of positive Lebesgue measure and lie in different measure classes.

The unique continuous function in each measure class of some family of measure classes allows us to distinguish subsets of this family by smoothness properties of that continuous function in the class. In the case of the one-parameter unitary group induced by translations on $\mathbb{R}$, corresponding to an element x in the domain of the Stone generator $\frac{d}{dt}$, the measure class x contains a continuous function (hence, as noted, a unique such function), and this function must be absolutely continuous, in $L_2(\mathbb{R})$, of course, with derivative almost everywhere on $\mathbb{R}$ in $L_2(\mathbb{R})$. Moreover, an absolutely continuous function in $L_2(\mathbb{R})$ with derivative almost everywhere on $\mathbb{R}$ in $L_2(\mathbb{R})$ has measure class an element of the Hilbert space on which the unitary group (corresponding to the translations of $\mathbb{R}$) acts that lies in the domain of $\frac{d}{dt}$. So, this absolute-continuity smoothness, together with the noted L_2 restrictions, characterizes the domain of $\frac{d}{dt}$. It is dangerously misleading to speak of the domain of $\frac{d}{dt}$ as "consisting of absolutely continuous functions in L_2 with almost everywhere derivatives in L_2"; it consists of the measure classes of such functions and each such class contains, as noted, a function which is nowhere continuous.

The preceding discussion raises an associated problem. Is there a measure class containing no continuous function? Our concept of (measure-theoretic) jumps of a function f at a point t_0 (see Definition 4.4) provides us with an answer to this. If f has jump $a(>0)$ at t_0 for width δ (>0) and g is continuous at t_0, then $\inf\{f(t)\}$ either with t in $(t_0-\delta,t_0)$ or t in $(t_0,t_0+\delta)$, but not both, exceeds $\sup\{f(t)\}$, for t in the other of those two intervals, by a. At the same time, from continuity of g at t_0, values of g at points t close to t_0 both in $(t_0-\delta,t_0)$ and in $(t_0,t_0+\delta)$ are close to $g(t_0)$, hence, closer to each other than a, whereas, the absolute value of the difference of the values of f at such points exceeds a. Thus f and g differ at at least one of each pair of points on either side of t_0 and close to t_0 and, hence, on a set of positive measure. It follows that no function continuous at t_0 lies in the measure class of f; the measure class of a function with a positive jump contains no everywhere continuous function.

Theorem 4.8. *The domain of* $\frac{d}{dt}$ *is the linear subspace of measure classes in* $\mathcal{H}(=L_2(\mathbb{R}))$ *corresponding to absolutely continuous functions on* $\mathbb{R}$ *whose almost-everywhere derivatives lie in* $L_2(\mathbb{R})$.

Stone's theorem (Theorem 4.3) provides us with the precise description of the domain of the generator of *each* (strong-operator-continuous) one-parameter, unitary group on a Hilbert space $\mathcal{H}$. In the case of the group arising from the translations of $\mathbb{R}$, we can give a more specific description of that domain (Theorem 4.8). An element x of $\mathcal{H}$ is in that domain if and only if the class of L_2-functions it designates contains a (necessarily, unique) continuous function, and that function is an absolutely continuous function with its (almost everywhere) derivative a function in $L_2(\mathbb{R})$. If x is in the domain and x' is the measure class of the norm limit of $t^{-1}(U_tx-x)$ as $|t|\to 0^+$, we choose some g in the class of x'. By assumption, $g\in L_2(\mathbb{R})$. An "indefinite integral" of g should have the same "derivative" as U_tx, and thus, differ from a function in the class of x by a constant function. Since the indefinite integral plus a constant is absolutely continuous, x contains an absolutely continuous function with (almost everywhere) derivative in $L_2(\mathbb{R})$. On the other hand, if the class of x contains an absolutely continuous function h whose (almost everywhere) derivative g is in $L_2(\mathbb{R})$, then detailed calculations and estimates show that $t^{-1}(U_tx-x)$ tends to x', the class of g. There is much to do to make this sketch a sound and complete mathematical argument.

We now describe a core, for $\frac{d}{dt}$, that is particularly useful for computations.

Theorem 4.9. *The family* $\mathscr{D}_0$ *of functions in* $L_2(\mathbb{R})$ *that vanish outside a finite interval and are continuously differentiable with derivatives in* $L_2(\mathbb{R})$ *determines a core for the generator* $\frac{d}{dt}$ *of the one-parameter, translation, unitary group on* $L_2(\mathbb{R})$.

Proof. Suppose $f\in\mathscr{D}(\frac{d}{dt})$. For any $\varepsilon>0$, there is a positive integer N $(N\geqslant 1)$ such that

$$\|f-f_{[-N,N]}\|_2<\frac{\varepsilon}{2}\quad\text{and}\quad\|f'-f'_{[-N,N]}\|_2<\frac{\varepsilon}{2},$$

where $f_{[-N,N]}$ denotes the function on $\mathbb{R}$ that agrees with f on $[-N,N]$ and is 0 outside $[-N,N]$.

Since f is absolutely continuous on $\mathbb{R}$ ($f\in\mathscr{D}(\frac{d}{dt})$), $f_{[-N,N]}$ is absolutely continuous on $[-N,N]$. Thus, $f_{[-N,N]}$ is differentiable almost everywhere on $[-N,N]$ with derivative $f'_{[-N,N]}$ (in $L_2([-N,N])$) and

$$f_{[-N,N]}(x)=\int_{-N}^{x}f'_{[-N,N]}(s)ds+f_{[-N,N]}(-N),\quad x\in[-N,N],$$

from the absolute continuity of $f_{[-N,N]}$ on $[-N,N]$.

We approximate $f'_{[-N,N]}$ by a continuous function g'_N on $[-N,N]$ so that

$$\|f'_{[-N,N]}-g'_N\|_2<\frac{\varepsilon}{8N}.$$

Now, comparing the indefinite integrals,

$$f_{[-N,N]}(x)=\int_{-N}^{x}f'_{[-N,N]}(s)ds+f_{[-N,N]}(-N)$$

and

$$g_N(x) = \int_{-N}^{x} g_N'(s)ds + f_{[-N,N]}(-N),$$

we have

$$\begin{aligned}|f_{[-N,N]}(x) - g_N(x)| &= \left| \int_{-N}^{x} [f_{[-N,N]}'(s) - g_N'(s)]ds \right| \\ &\leqslant \left(\int_{-N}^{N} |f_{[-N,N]}'(s) - g_N'(s)|^2 ds \right)^{\frac{1}{2}} \left(\int_{-N}^{N} |1|^2 ds \right)^{\frac{1}{2}} \\ &< \frac{\varepsilon}{8N}\sqrt{2N}\end{aligned}$$

Hence

$$\|f_{[-N,N]} - g_N\|_2 = \left(\int_{-N}^{N} |f_{[-N,N]}(x) - g_N(x)|^2 dx \right)^{\frac{1}{2}} < \frac{\varepsilon}{8N} 2N = \frac{\varepsilon}{4}.$$

Using the technique in the proof of Theorem 4.7, we extend g_N to $\mathbb{R}$ from $[-N, N]$ so that the extension g remains continuously differentiable with g and g' vanishing outside some finite interval and

$$\|g_N - g\|_2 < \frac{\varepsilon}{4} \quad \text{and} \quad \|g_N' - g'\|_2 < \frac{\varepsilon}{4}.$$

Then

$$\begin{aligned}\|f_{[-N,N]} - g\|_2 &\leqslant \|f_{[-N,N]} - g_N\|_2 + \|g_N - g\|_2 < \frac{\varepsilon}{4} + \frac{\varepsilon}{4} = \frac{\varepsilon}{2}, \\ \|f_{[-N,N]}' - g'\|_2 &\leqslant \|f_{[-N,N]}' - g_N'\|_2 + \|g_N' - g'\|_2 < \frac{\varepsilon}{8N} + \frac{\varepsilon}{4} < \frac{\varepsilon}{2}.\end{aligned}$$

Finally,

$$\begin{aligned}\|f - g\|_2 &\leqslant \|f - f_{[-N,N]}\|_2 + \|f_{[-N,N]} - g\|_2 < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon, \\ \|f' - g'\|_2 &\leqslant \|f' - f_{[-N,N]}'\|_2 + \|f_{[-N,N]}' - g'\|_2 < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.\end{aligned}$$

Thus, if $(f, f') \in \mathscr{G}(\frac{d}{dt})$, it can be approximated as closely as we wish by (g, g') with $g \in \mathscr{D}_0$. It follows that $\mathscr{D}_0$ is a core for $\frac{d}{dt}$. □

In the classic representation of the Heisenberg relation, $QP - PQ = i\hbar I$, the operator Q corresponds to multiplication by x, the identity transform on $\mathbb{R}$. The domain of Q consists of functions f in $L_2(\mathbb{R})$ such that xf is in $L_2(\mathbb{R})$. Elementary measure-theoretic considerations establish that $\mathscr{D}_0$ is also a core for Q. Moreover, $\mathscr{D}_0 \subseteq \mathscr{D}(QP) \cap \mathscr{D}(PQ)$, that is, $\mathscr{D}_0$ is contained in the domain of $QP - PQ$. A calculation, similar to the one at the end of Example 4.2, shows that

$$[QP - PQ]|\mathscr{D}_0 = -iI|\mathscr{D}_0.$$

Moreover, for any $f \in \mathscr{D}$ $(= \mathscr{D}(QP - PQ)$, the domain of $QP - PQ)$,

$$((QP - PQ)f)(t) = t\left(i\frac{d}{dt}f\right)(t) - \left(i\frac{d}{dt}\right)(tf(t)) = itf'(t) - (if(t) + itf'(t)) = -if(t).$$

Thus

$$[QP - PQ]|\mathscr{D} = -iI|\mathscr{D}.$$

4.3 In $\mathscr{A}(\mathcal{M})$ with $\mathcal{M}$ a factor of type II$_1$

The following simple lemma will prove useful to us.

Lemma 4.10. *Suppose that T is a closed operator on the Hilbert space $\mathcal{H}$ and $B \in \mathcal{B}(\mathcal{H})$. Then the operator TB is closed.*

Proof. Suppose $(x_n, y_n) \in \mathscr{G}(TB)$ and $x_n \to x$, $y_n = TBx_n \to y$. We show that $(x, y) \in \mathscr{G}(TB)$. By assumption, $Bx_n \in \mathscr{D}(T)$. Since B is bounded (hence, continuous), $Bx_n \to Bx$. Since T is closed and $TBx_n = y_n \to y$, we have that $(Bx, y) \in \mathscr{G}(T)$, so that $Bx \in \mathscr{D}(T)$ and $TBx = y$. Hence $(x, y) \in \mathscr{G}(TB)$ and TB is closed. □

Remark 4.11. With T and B as in the preceding lemma, the operator BT is not necessarily closed in general, even not preclosed (closable).

Consider the following example. Let $\{y_1, y_2, y_3, \dots\}$ be an orthonormal basis for a Hilbert space $\mathcal{H}$, and let

$$\mathscr{D} = \left\{x \in \mathcal{H} : \sum_{n=1}^{\infty} n^4 |\langle x, y_n\rangle|^2 < \infty\right\}, \qquad z = \sum_{n=1}^{\infty} n^{-1} y_n.$$

Define B in $\mathcal{B}(\mathcal{H})$ by $Bx = \langle x, z\rangle z$; and define mapping T with domain $\mathscr{D}$ by

$$Tx = \sum_{n=1}^{\infty} n^2 \langle x, y_n\rangle y_n.$$

Note, first, that T is a closed densely defined operator. To see this, by definition, $\mathscr{D}$ certainly contains the submanifold of all finite linear combinations of the basis elements $y_1, y_2, y_3, \dots$, from which $\mathscr{D}$ is dense in $\mathcal{H}$. Now, suppose $\{u_m\}$ is a sequence in $\mathscr{D}$ tending to u and $\{Tu_m\}$ converges to v. Then, for $y_{n'} \in \{y_1, y_2, y_3, \dots\}$

$$\langle Tu_m, y_{n'}\rangle = \left\langle \sum_{n=1}^{\infty} n^2 \langle u_m, y_n\rangle y_n, y_{n'} \right\rangle = n^2 \langle u_m, y_{n'}\rangle \to n^2 \langle u, y_{n'}\rangle.$$

But $\langle Tu_m, y_{n'}\rangle \to \langle v, y_{n'}\rangle$, so that $\langle v, y_{n'}\rangle = n^2 \langle u, y_{n'}\rangle$; and

$$\sum_{n=1}^{\infty} |n^2 \langle u, y_n\rangle|^2 = \sum_{n=1}^{\infty} |\langle v, y_n\rangle|^2 = \|v\|^2 < \infty.$$

Thus $u \in \mathscr{D}$ and

$$Tu = \sum_{n=1}^{\infty} n^2 \langle u, y_n\rangle y_n = \sum_{n=1}^{\infty} \langle v, y_n\rangle y_n = v,$$

so that $\mathscr{G}(T)$ is closed. Hence T is densely defined and closed. However, BT is not preclosed. If $u_m = m^{-1} y_m$, then $u_m \to 0$, but

$$\begin{aligned} BTu_m = \langle Tu_m, z\rangle z &= \left\langle \sum_{n=1}^{\infty} n^2 \langle u_m, y_n\rangle y_n, \sum_{n=1}^{\infty} n^{-1} y_n \right\rangle z \\ &= \left\langle \sum_{n=1}^{\infty} n^2 \langle m^{-1} y_m, y_n\rangle y_n, \sum_{n=1}^{\infty} n^{-1} y_n \right\rangle z \\ &= \langle m y_m, m^{-1} y_m\rangle z = z \neq 0. \end{aligned}$$

Hence BT is not preclosed. (Recall that an operator S is preclosed, i.e., $\mathscr{G}(S)^-$ is a graph of a linear transformation, if and only if convergence of the sequence $\{x_n\}$ in $\mathscr{D}(S)$ to 0 and $\{Sx_n\}$ to z implies that $z = 0$.)

Lemma 4.12. *If $\mathcal{M}$ is a factor of type II_1, P is a self-adjoint operator affiliated with $\mathcal{M}$, and A is an operator in $\mathcal{M}$, such that $P \hat{\cdot} A \hat{-} A \hat{\cdot} P$ is a bounded operator B, necessarily, affiliated with $\mathcal{M}$ and, hence, in $\mathcal{M}$, then $\tau(B)$, the trace of B, where τ is the trace on $\mathcal{M}$, is 0. In particular, B is not of the form aI with a some non-zero scalar in this case.*

Proof. Let E_n be the spectral projection for P corresponding to the interval $[-n, n]$ for each positive integer n. Then PE_n is an everywhere defined bounded self-adjoint operator as is E_nPE_n, and $E_nPE_n =$

PE_n. Note, for this, that $E_nP \subseteq PE_n$, so, E_nP is bounded and its closure $E_n \hat{\cdot} P = PE_n$. From the (algebraic) properties, established in Section 3, of the algebra $\mathscr{A}(\mathcal{M})$ of operators affiliated with $\mathcal{M}$,

$$E_n \hat{\cdot} (P \hat{\cdot} A) \hat{\cdot} E_n \mathbin{\hat{-}} E_n \hat{\cdot} (A \hat{\cdot} P) \hat{\cdot} E_n = E_nBE_n;$$

and from Lemma 4.10,

$$E_n \hat{\cdot} (P \hat{\cdot} A)E_n \mathbin{\hat{-}} E_n \hat{\cdot} (A \hat{\cdot} P)E_n = E_nBE_n.$$

(Since $P \hat{\cdot} A$ and $A \hat{\cdot} P$ are closed and E_n is bounded, $(P \hat{\cdot} A)E_n$ and $(A \hat{\cdot} P)E_n$ are closed. Hence they are equal to their closures $(P \hat{\cdot} A) \hat{\cdot} E_n$ and $(A \hat{\cdot} P) \hat{\cdot} E_n$, respectively.) Now, since E_n, A and $E_n \hat{\cdot} P = PE_n = E_nPE_n$ are all bounded,

$$E_n \hat{\cdot} (P \hat{\cdot} A)E_n = (E_n \hat{\cdot} P) \hat{\cdot} AE_n = E_nPE_nAE_n = E_nPE_nE_nAE_n$$

and

$$E_n \hat{\cdot} (A \hat{\cdot} P)E_n = (E_n \hat{\cdot} A) \hat{\cdot} (PE_n) = E_nAE_nPE_n = E_nAE_nE_nPE_n.$$

Thus

$$E_nPE_nE_nAE_n - E_nAE_nE_nPE_n = E_nBE_n. \tag{$*$}$$

Since E_nPE_n and E_nAE_n are bounded and in $\mathcal{M}$, the left-hand side of $(*)$ is a commutator in $\mathcal{M}$. Hence $\tau(E_nBE_n) = 0$. As $\|E_nBE_n\| \leqslant \|B\|$, for each n, and $E_n \uparrow I$ in the strong-operator topology, E_nBE_n is strong (hence, weak)-operator convergent to B. From [5, Theorem 8.2.8], τ is ultraweakly continuous on $\mathcal{M}$. Thus $0 = \tau(E_nBE_n) \to \tau(B)$. □

Theorem 4.13. *If $\mathcal{M}$ is a factor of type II_1, P and Q are self-adjoint operators affiliated with $\mathcal{M}$, and $P \hat{\cdot} Q \mathbin{\hat{-}} Q \hat{\cdot} P$ is a bounded operator B, then B has trace 0. In particular, $P \hat{\cdot} Q \mathbin{\hat{-}} Q \hat{\cdot} P$ is not of the form aI for some non-zero scalar a.*

Proof. Since $P \hat{\cdot} Q \mathbin{\hat{-}} Q \hat{\cdot} P$ is affiliated with $\mathcal{M}$, it is, by definition, closed on its dense domain. We are given that B is bounded on this domain. Hence B is everywhere defined. With E_n as in Lemma 4.12, we argue as in Lemma 4.12, with Q in place of A, to conclude that

$$E_n \hat{\cdot} (P \hat{\cdot} Q) \hat{\cdot} E_n \mathbin{\hat{-}} E_n \hat{\cdot} (Q \hat{\cdot} P) \hat{\cdot} E_n = E_nBE_n.$$

In this case,

$$\begin{aligned} E_n \hat{\cdot} (P \hat{\cdot} Q) \hat{\cdot} E_n &= (E_n \hat{\cdot} P) \hat{\cdot} (Q \hat{\cdot} E_n) = E_nPE_n \hat{\cdot} (Q \hat{\cdot} E_n) \\ &= E_nPE_nE_n \hat{\cdot} QE_n \qquad \text{(Lemma 4.10)} \\ &= E_nPE_n \hat{\cdot} (E_n \hat{\cdot} QE_n), \end{aligned}$$

and

$$\begin{aligned} E_n \hat{\cdot} (Q \hat{\cdot} P) \hat{\cdot} E_n &= E_n \hat{\cdot} Q \hat{\cdot} PE_n = E_n \hat{\cdot} QPE_n \qquad \text{(Lemma 4.10)} \\ &= E_n \hat{\cdot} QE_nPE_n \\ &= E_n \hat{\cdot} QE_nE_nPE_n = (E_n \hat{\cdot} QE_n) \hat{\cdot} E_nPE_n. \end{aligned}$$

Thus

$$(E_nPE_n) \hat{\cdot} (E_n \hat{\cdot} QE_n) \mathbin{\hat{-}} (E_n \hat{\cdot} QE_n) \hat{\cdot} (E_nPE_n) = E_nBE_n.$$

Since E_nPE_n and E_nBE_n are bounded operators in $\mathcal{M}$, Lemma 4.12 applies, and $\tau(E_nBE_n) = 0$. Again, $E_n \uparrow I$ and $\tau(B) = 0$. It follows that B cannot be aI with $a \neq 0$. □

Corollary 4.14. *The Heisenberg relation, $QP - PQ = i\hbar I$, cannot be satisfied with self-adjoint operators Q and P in the algebra of operators affiliated with a factor of type II_1.*

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