QUIZ 4 MAS 3106 (1) Spring 2014 NAME(print) Solution

NO AIDS are permitted for this quiz. **Show all your work.** Correct answers with little or no supporting work will not be given credit. Write legibly.

1. (6 pts) For the following linear transformations T, determine whether T is invertible and justify your answer.

(1) $T : \mathbb{R}^2 \to \mathbb{R}^3$ defined by $T(a_1, a_2) = (3a_1 - a_2, a_2, 4a_1)$.

Since $\dim(\mathbb{R}^2) = 2 \neq 3 = \dim(\mathbb{R}^3)$, T cannot be invertible.

(Otherwise, $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$ being linear and intertible would imply that $\dim(\mathbb{R}^2) = \dim(\mathbb{R}^3)$.)

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(2) $T : \mathrm{M}_{2\times2}(\mathbb{R}) \to \mathrm{M}_{2\times2}(\mathbb{R})$ defined by $T\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a+b & a \\ c & c+d \end{pmatrix}$.

If $T\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a+b & a \\ c & c+d \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$, then

$$\left.\begin{aligned} a+b &= 0 \\ a &= 0 \\ c &= 0 \\ c+d &= 0 \end{aligned}\right\} \Rightarrow \begin{aligned} a &= 0 \\ b &= 0 \\ c &= 0 \\ d &= 0 \end{aligned}, \quad \text{so } N(T) = \left\{\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}\right\} = \{\vec{0}\}$$

Hence T is one-to-one, which implies that T is onto

(since $\dim(\mathrm{M}_{2\times2}(\mathbb{R})) = \dim(\mathrm{M}_{2\times2}(\mathbb{R})) < \infty$)

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So T is invertible.

2. (6 pts) Let $\sim$ mean "is isomorphic to." Prove that $\sim$ is an equivalence relation on the class of vector spaces over F. That is, for vector spaces V, W, and Z over F, prove

(1) $V \sim V$.

The identity transformation $I_V : V \longrightarrow V$ is linear and invertible (with $I_V^{-1} = I_V$), I_V is an isomorphism.

So $V \sim V$.

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(2) If $V \sim W$, then $W \sim V$.

Suppose $T : V \to W$ is an isomorphism. Then $T^{-1} : W \to V$ is linear and invertible (with $(T^{-1})^{-1} = T$). T^{-1} is an isomorphism

So $W \sim V$.

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(3) If $V \sim W$ and $W \sim Z$, then $V \sim Z$.

Suppose $T : V \to W$ and $U : W \to Z$ are isomorphisms

Then $UT : V \to Z$ is linear and invertible (with $(UT)^{-1} = T^{-1}U^{-1}$). UT is an isomorphism.

So $V \sim Z$.

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