MAP 4303(1) Fall 2014 Exam 3 Review Problems

1. Write out the first five terms $(u_0, ..., u_4)$ of the Picard iteration scheme for the initial value problem and find the explicit solution.

(1) $x' = x; \quad x(0) = 3$

(2) $x' = x + 2; \quad x(0) = 2$

2. Consider a pair of undamped harmonic oscillators with equations

$$x_1'' = -\omega_1^2 x_1, \quad x_2'' = -\omega_2^2 x_2.$$

If we introduce new variables $y_j = x_j'$ for $j = 1, 2$, the equations may be written as a system

$$x_1' = y_1, \quad y_1' = -\omega_1^2 x_1, \quad x_2' = y_2, \quad y_2' = -\omega_2^2 x_2.$$

In matrix form, this system is $X' = AX$, where

$$X = \begin{pmatrix} x_1 \\ y_1 \\ x_2 \\ y_2 \end{pmatrix} \quad \text{and} \quad A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -\omega_1^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -\omega_2^2 & 0 \end{pmatrix}$$

Find the general solution of the system.

3. Find the general solution of the system $X' = AX$, where

$$A = \begin{pmatrix} 0 & 0 & -2 \\ 0 & 2 & 0 \\ 1 & 0 & 2 \end{pmatrix}.$$

4. Consider the system $X' = AX$, where

$$A = \begin{pmatrix} 0 & 0 & a \\ 0 & b & 0 \\ a & 0 & 0 \end{pmatrix} \quad \text{with} \quad a > 0, \ b < 0.$$

Find the general solution. Give a basis for the stable subspace and the unstable subspace.

5. Let

$$A = \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{pmatrix}.$$

Calculate $(tA)^2$, $(tA)^3$, $(tA)^4$, and find a formula for $(tA)^k$. Determine $\exp(tA)$. Use $\exp(tA)$ to solve the initial value problem $X' = AX$, $X(0) = X_0$.