No aids are permitted, except an approved basic scientific calculator. **Show all your work.** Correct answers with little or no supporting work will not be given credit. Write legibly.

For the equation $x'' + x' + x = 0$,

(1) rewrite the equation as a planar system;
(2) find the general solution (as linear combinations of real-valued functions) of the system.

(1) $\begin{cases} y = x' \\ y' + y + x = 0 \end{cases} \Rightarrow \boxed{\begin{cases} x' = y \\ y' = -x - y \end{cases}}$ $(+3)$

(2) $A = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}$

$\det(A - \lambda I) = \det \begin{pmatrix} -\lambda & 1 \\ -1 & -1-\lambda \end{pmatrix} = \lambda^2 + \lambda + 1 = 0$ $\boxed{\lambda = \dfrac{-1 \pm \sqrt{3}\, i}{2}}$ $(+4)$

For $\lambda_1 = \dfrac{-1+\sqrt{3}\,i}{2}$, $\begin{pmatrix} \frac{1-\sqrt{3}\,i}{2} & 1 \\ -1 & \frac{-1-\sqrt{3}\,i}{2} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$\dfrac{1-\sqrt{3}\,i}{2} \cdot x + y = 0 \Rightarrow y = -\dfrac{1-\sqrt{3}\,i}{2}$

Choose $V_1 = \begin{pmatrix} 1 \\ -\frac{1-\sqrt{3}\,i}{2} \end{pmatrix} = \boxed{\begin{pmatrix} 1 \\ -\frac{1}{2} + \frac{\sqrt{3}}{2}i \end{pmatrix}}$ $(+3)$

$e^{\frac{-1+\sqrt{3}\,i}{2}t} \begin{pmatrix} 1 \\ -\frac{1}{2} + \frac{\sqrt{3}}{2}i \end{pmatrix} = e^{-\frac{1}{2}t} \cdot \left(\cos \frac{\sqrt{3}}{2}t + i \sin \frac{\sqrt{3}}{2}t\right) \begin{pmatrix} 1 \\ -\frac{1}{2} + \frac{\sqrt{3}}{2}i \end{pmatrix}$

$= e^{-\frac{1}{2}t} \begin{pmatrix} \cos \frac{\sqrt{3}}{2}t + i \sin \frac{\sqrt{3}}{2}t \\ \left(-\frac{1}{2}\cos\frac{\sqrt{3}}{2}t - \frac{\sqrt{3}}{2}\sin\frac{\sqrt{3}}{2}t\right) + i\left(\frac{\sqrt{3}}{2}\cos\frac{\sqrt{3}}{2}t - \frac{1}{2}\sin\frac{\sqrt{3}}{2}t\right) \end{pmatrix}$

$X_{re}(t) = e^{-\frac{1}{2}t} \begin{pmatrix} \cos \frac{\sqrt{3}}{2}t \\ -\frac{1}{2}\cos\frac{\sqrt{3}}{2}t - \frac{\sqrt{3}}{2}\sin\frac{\sqrt{3}}{2}t \end{pmatrix}$

$X_{im}(t) = e^{-\frac{1}{2}t} \begin{pmatrix} \sin \frac{\sqrt{3}}{2}t \\ \frac{\sqrt{3}}{2}\cos\frac{\sqrt{3}}{2}t - \frac{1}{2}\sin\frac{\sqrt{3}}{2}t \end{pmatrix}$ $(+5)$

General solution: $X(t) = C_1 X_{re}(t) + C_2 X_{im}(t)$